# HOMEWORK 3 471

HW2 is due by Wednesday October 14th at 11:59 pm CT pm. Please upload your solutions to Canvas.

**Attach the codes/command windows that you use to answer the questions.**

(1) Let $A = \int_0^\infty |\cos(x + x^2)|e^{-2x}dx$ and, given a density $f$, consider estimating $A$ using the following importance sampling procedure: Sample $X_1, \ldots, X_n$ from $f$ and compute the estimator

$$I_n := \frac{1}{n}\sum_{k=1}^{n} \frac{|\cos(X_k + X_k^2)|e^{-2X_k}}{f(X_k)}.$$

- Indicate each of the following examples of $f$ for which the estimator will converge to $A$ as $n \to \infty$:
  (a) $e^{-x}\mathbb{1}_{x\geq 0}$
  (b) $2e^{-2x}\mathbb{1}_{x\geq 0}$
  (c) $\frac{1}{2}e^{-|x|}$
  (d) $\frac{1}{Z}e^{-2x}\mathbb{1}_{x\in[0,2\pi]}$ ; for appropriate normalization constant $Z$.
  (e) $3e^{-3x}\mathbb{1}_{x\geq 0}$
  (f) $\frac{2}{\sqrt{2\pi}}e^{-x^2/2}\mathbb{1}_{x\geq 0}$

  Explain.
- Implement the importance sampling algorithm for those examples of $f$ for which the estimator $I_n$ is guaranteed to converge towards $A$. Use $1000$ samples and produce a $90\%$ confidence interval for the value of $A$. Are these meaningful confidence intervals?

(2) In what follows we consider the function

$$h(x_1, x_2, x_3) = \cos(x_1 + x_2)exp(-(x_1^2 + x_2^2 + x_3^2)).$$

The goal is to estimate the quantities:

$$I^{(1)} := \int_{B_1} h(x)dx, \quad I^{(2)} := \int_{[0,1]^3} h(x)dx, \quad I^{(3)} := \int_{\mathbb{R}^3} h(x)dx, \quad I^{(4)} := \mathrm{Vol}(B_{1.1}\cap[0,1]^3).$$

- Consider $U_1, \ldots, U_n$ i.i.d. samples from $Uniform([-1,1]^3)$. Use the samples to define an estimator for each of the above quantities. If you can not use the samples to estimate one of the quantities, explain why this is so.
- What distribution would you use, to sample points $X_1, \ldots, X_n$, in order to estimate the above quantities? Implement your importance sampling algorithm with $1000$ samples and build a $90\%$ confidence interval for each of the quantities to be estimated.

(3) (Confidence intervals)
  (a) Let $X$ be a random variable with distribution $binomial(n,p)$. Suppose that $n$ is very large. How would you use the central limit theorem to build a $90\%$ confidence interval for the quantity

  $$a_1 := \mathbb{P}(X \in [a,b])?$$

  (b) Suppose that $X_1, \ldots, X_n$ are i.i.d. random variables with distribution $exponential(1)$. Suppose that $n$ is very large. How would you use the central limit theorem to estimate the quantity:

$$a_2 := \frac{1}{n}\mathbb{E}\left(\sum_{k=1}^{n}(X_k)^2\right)?$$

(c) Use the parametric and non-parametric Bootstrap as alternatives to build 90% confidence intervals for both of the quantities mentioned above. More precisely: For (a) let $n = 30$ and $p = 3/4$. Then sample $X_1, \dots, X_{500}$ from $binomial(n, p)$. Pretending you don't know the value of $p$ but that you do know that $n = 30$ use the resulting samples as a starting point to use the parametric and non-parametric bootstrap. For (b): let $n = 500$ and sample $X_1, \dots, X_n$ from $exponetial(1)$. Use these samples to apply the parametric and non-parametric bootstrap assuming you know that the samples were drawn from an exponential distribution with parameter $\lambda$. Compare the confidence intervals obtained using the Bootstrap with the confidence intervals you obtain using the CLT.

(4) Consider the optimization problem

$$\min_{\lambda>0} \int_0^{\infty} \log(\rho_\lambda(x))p_\lambda(x)dx + \int_0^{\infty} xp_\lambda(x)dx,$$

where $p_\lambda(x) = \lambda\exp(-\lambda x)\mathbb{1}_{x>0}$ is the density function of the exponential distribution with parameter $\lambda$.

- Find the explicit solution $\lambda^*$ to the above optimization problem.
- Use the Robins-Monro algorithm to approximate $\lambda^*$. Make plots of the evolutions of the algorithm for different choices of step-size schedules. Comment on your observations.

**Optional Problems:** Implement the consensus dynamics under stochastic perturbations (more details on page 137 in the attached document) described by:

$$x_i(k+1) = x_i(k) + \gamma_k \sum_j \delta_{ij}(k)(x_j(k) - x_i(k) + \nu_{ij}(k)), \quad i = 1, \dots, L$$

where $x_i(t)$ denotes the position in $\mathbb{R}^3$ of a particle $i$ at time $t$, $\nu_{ij}(k)$ is a 3-dimensional Gaussian random variable with mean vector 0 and covariance matrix given by $\sigma^2$ times the identity, and $\delta_{ij} \sim bernoulli(p_{ij})$. Note that once specified the number of particles $L$, these stochastic dynamics depend on the parameters $\sigma$ and $P = (p_{ij})_{ij}$, as well as on the step-size schedule $\gamma_k$.

- How do you interpret each of the terms in the above system of equations?
- Run the dynamics for different choices of $L$, $P$, $\sigma$. Plot the trajectories of the particles. Do you see consensus emerging? What happens if you change the parameters? Are there cases where you expect no consensus to be reached?