HOMEWORK 4 471

HW4 is due by Wednesday November 4th at 11:59 pm CT pm. Please upload your solutions to Canvas.

Attach the codes/command windows that you use to answer the questions.

(1) Compute the quantity:

$$\int_A \cos(x_1 + x_2) \exp(-|x_1| - 2x_2^2) dx_1 dx_2,$$

where A is the set:

$$A; = \{(x_1, x_2) \ : \ x_1^2 + x_2 \geq 0\},$$

using 1) importance sampling and 2) MCMC. Note: for MCMC you may still need to compute a normalization constant before getting A; this normalization constant can be computed by hand.

(2) Implement the Random walk sampler and the MALA where the target distribution ρ satisfies:

$$\rho(x) \propto \exp(-x^2/2\sigma_1^2) + \exp(-(x-\gamma)^2/2\sigma_2^2).$$

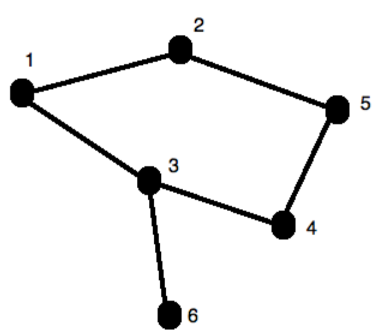
Run the Markov chains for at least 10^5 iterations. Produce trace plots and corresponding histograms for the locations visited by the chains. Experiment with different initializations and different choices of σ_1, σ_2, γ, and step size. How should your histograms look like? Which algorithms seem to be mixing faster?

(3) Consider four binary random variables X_1, X_2, X_3, X_4 , each taking values in $\{1, 2\}$, and having joint pmf:

$$p_{1,2,3,4}(x_1, x_2, x_3, x_4) = \frac{1}{Z} x_1^{x_2} x_2^{x_3} x_3^{x_4} x_4^{x_1}$$

- Compute Z. (You can do this by hand or using software.)
- What minimal graph does (X_1, X_2, X_3, X_4) respect?
- Use the general results about independence and conditioning to determine whether X_1 and X_3 are independent given $X_2 = 1$ and $X_4 = 2$.
- Confirm your answer to the previous question by calculating the exact expression for $\mathbb{P}(X_1 = x_1, X_3 = x_3 | X_2 = 1, X_4 = 2)$ as a function of x_1 and x_3.

(4) Consider the graph G



Assume that $X = (X_1, X_2, X_3, X_4, X_5, X_6)$ is a GRF with respect to this G, and that G is minimal, i.e. if any edge is removed then X no longer respects G.

- Draw the smallest graph that $(X_1, X_2, X_3, X_5, X_6)$ is guaranteed to respect given X_4 (For this it is useful to read about the "marginalization rule" in Section 4.4. in our lecture notes). Use this graph to determine whether X_5 and X_6 are guaranteed to be independent given X_4.
- Draw the smallest graph that $(X_1, X_2, X_4, X_5, X_6)$ is guaranteed to respect given X_3. Use this graph to determine whether X_5 and X_6 are guaranteed to be independent given X_3.
- Draw the smallest graph that $(X_1, X_3, X_4, X_5, X_6)$ is guaranteed to respect given X_2. Use this graph to determine whether X_4 and X_5 are guaranteed to be dependent given X_2.
- Draw the smallest graph that (X_1, X_4, X_5, X_6) is guaranteed to respect given X_2 and X_3. Use this graph to determine whether X_1 and X_4 are guaranteed to be independent given X_2 and X_3.

(5) Let $X = (X_1, X_2, X_3, X_4)$. This time, the goal is to draw a graph G with as many edges as possible (not as few as possible), such that whenever X respects G we can conclude that : 1) X_2 and X_3 are conditionally independent given X_4 and X_1; and 2), that X_2 and X_4 are conditionally independent given X_3 and X_1.

Optional problems:

(1) (Based on Section 3.4) In class we considered $S_0, S_1, S_2, \ldots$, a symmetric random walk in 1d defined by $S_0 = 0$ and

$$S_{k+1} = S_k + X_{k+1}.$$

The random variables X_k are all i.i.d and satisfy

$$X_k = \begin{cases} 1 & \text{with prob. } 1/2 \\ -1 & \text{with prob. } 1/2. \end{cases}$$

We extended the above random walk in discrete time to a random walk in continuous time by linearly interpolating the discrete walk. Namely, for $t \in [k, k+1)$ we defined

$$S_t := S_k + (t-k)X_{k+1}.$$

Then we chose $n \in \mathbb{N}$ and rescaled the random walk to obtain the walk:

$$Y_t^{(n)} := \frac{S_{nt}}{\sqrt{n}}, \quad t \in [0, \infty)$$

- For a fixed value $t \in (0, \infty)$, compute the expectation and variance of the random variable $Y_t^{(n)}$. Also, as $n \to \infty$, what is the limiting distribution of $Y_t^{(n)}$?
- For $s < t$, what is the limiting distribution of $Y_t^{(n)} - Y_s^{(n)}$ (as $n \to \infty$)?
- Given $0 < r < s < t$, explain why $Y_t^{(n)} - Y_s^{(n)}$ is independent of $Y_r^{(n)}$ for large enough n.

(2) Let $P \in \mathbb{R}^{L \times L}$ be a transition probability matrix for a Markov chain $S_0, S_1, \ldots$ on $X = \{a_1, \ldots, a_L\}$. Suppose that

$$P_{ij} > 0, \quad \forall i, j.$$

Show that every state is recurrent. In other words prove that for every i,

$$R_i := \mathbb{P}(\exists n \geq 1 \text{ s.t. } S_n = a_i | S_0 = a_i) = 1.$$

Moreover, let τ_i be the first time the chain returns to its original state a_i:

$$\tau_i := \min\{n \geq 1 \text{ s.t. } S_n = a_i\}.$$

Show that a_i is a *positive recurrent* state by proving that

$$\mathbb{E}(\tau_i) < \infty.$$

(3)
- What does the Perron-Frobenius theorem state, and how is it related to the existence and uniqueness of invariant measures of time homogeneous Markov chains on finite state spaces?
- What does the Brouwer's fixed point theorem state, and how is it related to the existence and uniqueness of invariant measures of time homogeneous Markov chains on finite state spaces?
- What does the Gershgorin's disk theorem state, and how is it related to the existence and uniqueness of invariant measures of time homogeneous Markov chains on finite state spaces?

(4) The rooted binary tree T is an infinite graph (infinitely many vertices) with one distinguished vertex A from which it comes a single edge; at every other vertex,there are three edges connected to it and there are no loops in the graph..

We consider a random walk on T as follows. At time 0 the walk starts at node A and then jumps from a vertex along each available edge with equal probability. What is the probability of ever returning to A?