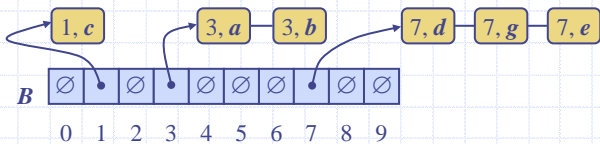# Bucket-Sort and Radix-Sort

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Bucket-Sort

- Let be S be a sequence of n (key, element) entries with keys in the range $[0, N-1]$
- Bucket-sort uses the keys as indices into an auxiliary array B of sequences (buckets)
  - Phase 1: Empty sequence S by moving each entry (k, o) into its bucket $B[k]$
  - Phase 2: For $i = 0, \ldots, N-1$, move the entries of bucket $B[i]$ to the end of sequence S
- Analysis:
  - Phase 1 takes $O(n)$ time
  - Phase 2 takes $O(n + N)$ time
  Bucket-sort takes $O(n + N)$ time

Algorithm *bucketSort(S, N)*
  Input sequence S of (key, element) items with keys in the range $[0, N-1]$
  Output sequence S sorted by increasing keys
  $B \leftarrow$ array of N empty sequences
  while $\neg S.empty()$
    $(k, o) \leftarrow S.front()$
    $S.eraseFront()$
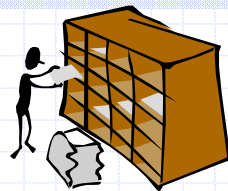 $B[k].insertBack((k, o))$
  for $i \leftarrow 0$ **to** $N-1$
    while $\neg B[i].empty()$
      $(k, o) \leftarrow B[i].front()$
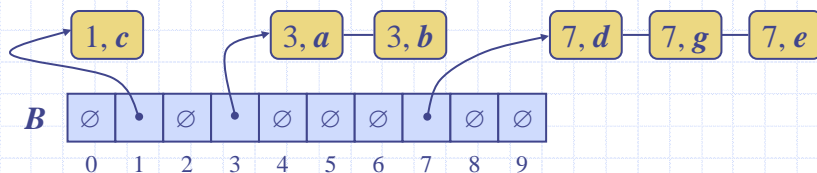 $B[i].eraseFront()$
      $S.insertBack((k, o))$

Example

- Key range $[0, 9]$

$7, d$ — $1, c$ — $3, a$ — $7, g$ — $3, b$ — $7, e$

Phase 1

$1, c$ $3, a$ — $3, b$ $7, d$ — $7, g$ — $7, e$

B: $\varnothing$ | | $\varnothing$ | | $\varnothing$ $\varnothing$ $\varnothing$ | | $\varnothing$ $\varnothing$
   0   1   2   3   4   5   6   7   8   9

Phase 2

$1, c$ — $3, a$ — $3, b$ — $7, d$ — $7, g$ — $7, e$

Properties and Extensions

- **Key-type Property**
  - The keys are used as indices into an array and cannot be arbitrary objects
  - No external comparator
- **Stable Sort Property**
  - The relative order of any two items with the same key is preserved after the execution of the algorithm

Extensions

- Integer keys in the range $[a, b]$
  - Put entry (k, o) into bucket $B[k - a]$
- String keys from a set D of possible strings, where D has constant size (e.g., names of the 50 U.S. states)
  - Sort D and compute the rank $r(k)$ of each string k of D in the sorted sequence
  - Put entry (k, o) into bucket $B[r(k)]$

Lexicographic Order

- A d-tuple is a sequence of d keys $(k_1, k_2, \ldots, k_d)$, where key k_i is said to be the i-th dimension of the tuple
- Example:
  - The Cartesian coordinates of a point in space are a 3-tuple
- The lexicographic order of two d-tuples is recursively defined as follows

$$(x_1, x_2, \ldots, x_d) < (y_1, y_2, \ldots, y_d)$$
$$\Leftrightarrow$$
$$x_1 < y_1 \ \lor \ x_1 = y_1 \land (x_2, \ldots, x_d) < (y_2, \ldots, y_d)$$

I.e., the tuples are compared by the first dimension, then by the second dimension, etc.

Lexicographic-Sort

- Let C_i be the comparator that compares two tuples by their i-th dimension
- Let $stableSort(S, C)$ be a stable sorting algorithm that uses comparator C
- Lexicographic-sort sorts a sequence of d-tuples in lexicographic order by executing d times algorithm $stableSort$, one per dimension
- Lexicographic-sort runs in $O(dT(n))$ time, where $T(n)$ is the running time of $stableSort$

Algorithm *lexicographicSort*(S)
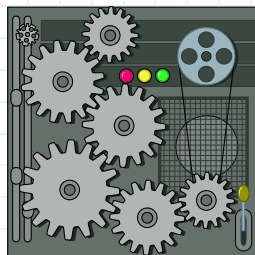 **Input** sequence S of d-tuples
  Output sequence S sorted in lexicographic order

  for $i \leftarrow d$ **downto** 1
    $stableSort(S, C_i)$

Example:

$(7,4,6) \ (5,1,5) \ (2,4,6) \ (2, 1, 4) \ (3, 2, 4)$

$(2, 1, 4) \ (3, 2, 4) \ (5,1,5) \ (7,4,6) \ (2,4,6)$

$(2, 1, 4) \ (5,1,5) \ (3, 2, 4) \ (7,4,6) \ (2,4,6)$

$(2, 1, 4) \ (2,4,6) \ (3, 2, 4) \ (5,1,5) \ (7,4,6)$

Radix-Sort

- Radix-sort is a specialization of lexicographic-sort that uses bucket-sort as the stable sorting algorithm in each dimension
- Radix-sort is applicable to tuples where the keys in each dimension i are integers in the range $[0, N-1]$
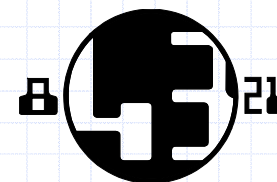- Radix-sort runs in time $O(d(\,n + N))$

Algorithm *radixSort*(S, N)
  Input sequence S of d-tuples such that $(0, \ldots, 0) \leq (x_1, \ldots, x_d)$ and $(x_1, \ldots, x_d) \leq (N-1, \ldots, N-1)$ for each tuple $(x_1, \ldots, x_d)$ in S
  Output sequence S sorted in lexicographic order
  for $i \leftarrow d$ **downto** 1
    $bucketSort(S, N)$
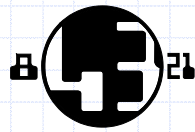

Radix-Sort for Binary Numbers

- Consider a sequence of n b-bit integers
$$x = x_{b-1} \ldots x_1 x_0$$
- We represent each element as a b-tuple of integers in the range $[0, 1]$ and apply radix-sort with $N = 2$
- This application of the radix-sort algorithm runs in $O(bn)$ time
- For example, we can sort a sequence of 32-bit integers in linear time

Algorithm *binaryRadixSort*(S)
  Input sequence S of b-bit integers
  Output sequence S sorted
  replace each element x of S with the item $(0, x)$
  for $i \leftarrow 0$ **to** $b-1$
    replace the key k of each item (k, x) of S with bit x_i of x
    $bucketSort(S, 2)$

Example

◆ Sorting a sequence of 4-bit integers

1001		0010		1001		1001		0001
0010		1110		1101		0001		0010
1101	⇒	1001	⇒	0001	⇒	0010	⇒	1001
0001		1101		0010		1101		1101
1110		0001		1110		1110		1110