Complex Analysis

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1 Cauchy-Riemann equation

A function $f(z) = u(x,y) + iv(x,y)$, $z = x + iy$ is complex differentiable at $z_0 = x_0 + iy_0$ if and only if u and v are real differentiable at (x_0, y_0).
也就是說 $u_x = v_y$ 且 $u_y = -v_x$ 等價於 $f(z)$ 在 z_0 可微分。

2 Entire

$f(x)$ is called entire if it is differentiable in all $\mathbb{C}$ fields.

differentiable 是對點 (z_0)，analytic 是對區域 (open set)，entire 是對整個 $\mathbb{C}$

Ex1. show e^z is entire.

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x(\cos y + i \sin y) \\ &= e^x \cos y + ie^x \sin y \end{aligned}$$

check by Cauchy-Riemann equation $u_x = v_y$ and $u_y = -v_x$ is hold on all $\mathbb{R}$ fields. thus e^z is entire.

Ex2. show e^z is only entire function satisfied

$$\begin{aligned} f(z_1) + f(z_2) &= f(z_1)f(z_2) \\ f(x) &= e^x, x \in \mathbb{R} \end{aligned}$$

let $f(z) = g(x,y) + ih(x,y)$

$\because f(z)$ is entire, $f'(0) = e^0 = 1$
$\therefore g'(0) + ih'(0) = 1$
then $g'(0) = 1, h'(0) = 0$

thus $g(x, y) = x, h(x, y) = 0$

反向用 Cauchy-Riemann equation，得到:

$$g_x = h_y$$
$$g_y = -h_x$$

we get

$$f''(x) + f(x) = 0$$

所以 $f(x) = c_1 sin(x) + c_2 cos(x)$，(微分兩次等於自己相反也只有 sin,cos 的產物)
帶入 $f'(0) = 1$，得到 $c_1 = 1, c_2 = i \longrightarrow f(z) = e^z$

3 Convergence of complex series

Thm1
$\sum a_n z^n = A(z)$ is Convergence at radius R_1
$\sum b_n z^n = B(z)$ is Convergence at radius R_2
$\Rightarrow \sum_{n=0}^{\infty} (\sum_{k=0}^{n} a_n \cdot b_{n-k}) z^n = A(z) \cdot B(z)$ is Convergence at radius $min(R_1, R_2)$
proof: 爆破斜線排在一起

Thm2
power series 微分後收斂半徑不變，且無限可微
hint: 展開微分用原本的 R 作 Convergence test

Thm3 (uniqueness)
當 $f(z) = \sum_{n=0}^{\infty} c_n z^n$ 滿足

1. 存在非 0 數列 $\{z_n\}$, 且 $lim_{n\to\infty} z_n = 0$

2. 對於所有 z_n, $f(z_n) = 0$

則 $f(z) \equiv 0$ (identical zero) ($\{c_n\} = 0$ 對所有 n)
* 推廣：到任何常數 c 也可以 $(g(z) = f(z) + c)$

*** 他告訴了我們如果兩個 analytic function 在一個 accumulated point 相等，則兩個 function 相等

Thm3.1: uniqueness of analytic function
如果兩個 analytic function 在一個收斂數列處處相等，則兩個 function 相等。

4 Accumulated point

A point z_0 is called an accumulated point of a set E if every neighborhood of z_0 contains at least one point of E different from z_0 itself.
(可透過收縮 open set 找到一串收斂數列即 $\{d_n\}$,d_{n+1} 在半徑為 $d(d_n, z_0)$ 的 openball 裡面找)

5 M-L formula

if $f(z)$ is continuous on a curve γ with length L, and if $|f(z)| \leq M$ on γ, then

$$|\int_\gamma f(z)dz| \leq ML$$

6 analytic function

幾個 analytic function 的性質

1. $f(z)$ is analytic at z_0 if and only if $f(z)$ can be represented by a convergent power series in some neighborhood of z_0.
2. 如果 $f(z)$ 在一個區域 z_0 內解析，則 $f(z)$ 在 z_0 內無限可微。(power series 微分後收斂半徑不變，且無限可微)

所以解析函數 == 無限可微函數 = power series (複數空間的微分超強特性)

7