Complex Analysis

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1 Cauchy-Riemann equation

A function $f(z) = u(x, y) + iv(x, y)$, $z = x + iy$ is complex differentiable at $z_0 = x_0 + iy_0$ if and only if u and v are real differentiable at (x_0, y_0).
也就是說 $u_x = v_y$ 且 $u_y = -v_x$ 等價於 $f(z)$ 在 z_0 可微分。

2 Entire

$f(x)$ is called entire if it is differentiable in all $\mathbb{C}$ fields.

differentiable 是對點 (z_0)，analytic 是對區域 (open set)，entire 是對整個 $\mathbb{C}$

Ex1. show e^z is entire.

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x(\cos y + i \sin y) \\ &= e^x \cos y + ie^x \sin y \end{aligned}$$

check by Cauchy-Riemann equation $u_x = v_y$ and $u_y = -v_x$ is hold on all $\mathbb{R}$ fields. thus e^z is entire.

Ex2. show e^z is only entire function satisfied

$$\begin{aligned} f(z_1) + f(z_2) = f(z_1)f(z_2) \\ f(x) = e^x, x \in \mathbb{R} \end{aligned}$$

let $f(z) = g(x, y) + ih(x, y)$

$\because f(z)$ is entire, $f'(0) = e^0 = 1$
$\therefore g'(0) + ih'(0) = 1$
then $g'(0) = 1, h'(0) = 0$

thus $g(x, y) = x, h(x, y) = 0$

反向用 Cauchy-Riemann equation，得到:

$$
\begin{aligned}
g_x &= h_y \\
g_y &= -h_x
\end{aligned}
$$

we get

$$f''(x) + f(x) = 0$$

所以 $f(x) = c_1 sin(x) + c_2 cos(x)$，(微分兩次等於自己相反也只有 sin,cos 的產物)
帶入 $f'(0) = 1$，得到 $c_1 = 1, c_2 = i \longrightarrow f(z) = e^z$

3 Convergence of complex series

Thm1
$\sum a_n z^n = A(z)$ is Convergence at radius R_1
$\sum b_n z^n = B(z)$ is Convergence at radius R_2
$\Rightarrow \sum_{n=0}^{\infty} (\sum_{k=0}^{n} a_n \cdot b_{n-k}) z^n = A(z) \cdot B(z)$ is Convergence at radius $min(R_1, R_2)$
proof: 爆破斜線排在一起

Thm2
power series 微分後收斂半徑不變，且無限可微
hint: 展開微分用原本的 R 作 Convergence test

Thm3 (uniqueness)
當 $f(z) = \sum_{n=0}^{\infty} c_n z^n$ 滿足

1. 存在非 0 數列 $\{z_n\}$, 且 $lim_{n\to\infty} z_n = 0$

2. 對於所有 z_n, $f(z_n) = 0$

則 $f(z) \equiv 0$ (identical zero) ($\{c_n\} = 0$ 對所有 n)
* 推廣：到任何常數 c 也可以 $(g(z) = f(z) + c)$

*** 他告訴了我們如果兩個 analytic function 在一個 accumulated point 相等，則兩個 function 相等

Thm3.1: uniqueness of analytic function
如果兩個 analytic function 在一個收斂數列處處相等，則兩個 function 相等。

4 Accumulated point

A point z_0 is called an accumulated point of a set E if every neighborhood of z_0 contains at least one point of E different from z_0 itself.
(可透過收縮 open set 找到一串收斂數列即 $\{d_n\}$,d_{n+1} 在半徑為 $d(d_n, z_0)$ 的 openball 裡面找)

5 M-L formula

if $f(z)$ is continuous on a curve γ with length L, and if $|f(z)| \leq M$ on γ, then

$$|\int_\gamma f(z)dz| \leq ML$$

6 analytic function

幾個 analytic function 的性質

1. $f(z)$ is analytic at z_0 if and only if $f(z)$ can be represented by a convergent power series in some neighborhood of z_0.
2. 如果 $f(z)$ 在一個區域 z_0 內解析，則 $f(z)$ 在 z_0 內無限可微。(power series 微分後收斂半徑不變，且無限可微)

所以解析函數 == 無限可微函數 = power series (複數空間的微分超強特性)

7 Cauchy integral formula

if $f(z)$ is analytic inside and on a simple closed circle $C = Re^{i\theta}$ 半徑 R 的封閉圓圈, then

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz$$

proof:

by closed curve integral thm

$$\int_C \frac{f(z)-f(a)}{z-a}dz = 0$$

thus

$$\int_C \frac{f(z)}{z-a}dz - \int_C \frac{f(a)}{z-a}dz = 0$$

let $z-a = re^{i\theta}$, $dz = ire^{i\theta}d\theta$, C_p 是圓心 a 的圓半徑為 p 的圓
where $0 < p < r$
then

$$\int_{C_p} \frac{d_z}{z-a} = \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}}d\theta = 2\pi i$$

所以我們帶入上面的結果

$$\begin{aligned}\int_C \frac{f(a)}{z-a}dz &= f(a)\int_0^{2\pi} \frac{1}{re^{i\theta}}ire^{i\theta}d\theta \\ &= f(a)\int_0^{2\pi} id\theta \\ &= 2\pi i f(a)\end{aligned}$$

8 liouville's theorem

如果 $f(z)$ 在整個複數平面上解析且有界，則 $f(z)$ 為常數函數。
*entire + bounded $\Rightarrow$ 常數函數
proof:
假設 $|f(z)| \leq M$，則對任意 $z_0 \in \mathbb{C}$, $f(z)$ 在 z_0 的 power series 展開為

$$f(z) = \sum_{n=0}^{\infty} c_n(z-z_0)^n$$

其中

$$c_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i}\int_{|z-z_0|=r} \frac{f(z)}{(z-z_0)^{n+1}}dz$$

由 M-L 公式可知

$$|c_n| \leq \frac{M}{r^n}$$

因為 r 可任意大，所以 $c_n = 0$ 對所有 $n \geq 1$，所以 $f(z) = c_0$ 為常數函數。