Complex Analysis

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1 Cauchy-Riemann equation

A function $f(z) = u(x, y) + iv(x, y)$, $z = x + iy$ is complex differentiable at $z_0 = x_0 + iy_0$ if and only if u and v are real differentiable at (x_0, y_0).
也就是說 $u_x = v_y$ 且 $u_y = -v_x$ 等價於 $f(z)$ 在 z_0 可微分。

2 Entire

$f(x)$ is called entire if it is differentiable in all $\mathbb{C}$ fields.

differentiable 是對點 (z_0)，analytic 是對區域 (open set)，entire 是對整個 $\mathbb{C}$

Ex1. show e^z is entire.

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x(\cos y + i \sin y) \\ &= e^x \cos y + ie^x \sin y \end{aligned}$$

check by Cauchy-Riemann equation $u_x = v_y$ and $u_y = -v_x$ is hold on all $\mathbb{R}$ fields. thus e^z is entire.

Ex2. show e^z is only entire function satisfied

$$\begin{aligned} f(z_1) + f(z_2) &= f(z_1)f(z_2) \\ f(x) &= e^x, x \in \mathbb{R} \end{aligned}$$

let $f(z) = g(x, y) + ih(x, y)$

$\because f(z)$ is entire, $f'(0) = e^0 = 1$
$\therefore g'(0) + ih'(0) = 1$
then $g'(0) = 1, h'(0) = 0$

thus $g(x,y)=x, h(x,y)=0$

反向用 Cauchy-Riemann equation，得到:

$$
\begin{aligned}
g_x &= h_y \\
g_y &= -h_x
\end{aligned}
$$

we get

$$f''(x)+f(x)=0$$

所以 $f(x)=c_1 sin(x)+c_2 cos(x)$，(微分兩次等於自己相反也只有 sin,cos 的產物)
帶入 $f'(0)=1$，得到 $c_1=1, c_2=i \longrightarrow f(z)=e^z$

3 Convergence of complex series

Thm1
$\sum a_n z^n = A(z)$ is Convergence at radius R_1
$\sum b_n z^n = B(z)$ is Convergence at radius R_2
$\Rightarrow \sum_{n=0}^{\infty}(\sum_{k=0}^{n} a_n \cdot b_{n-k})z^n = A(z)\cdot B(z)$ is Convergence at radius $min(R_1, R_2)$
proof: 爆破斜線排在一起

Thm2
power series 微分後收斂半徑不變，且無限可微
hint: 展開微分用原本的 R 作 Convergence test

Thm3 (uniqueness)
當 $f(z)=\sum_{n=0}^{\infty} c_n z^n$ 滿足

1. 存在非 0 數列 $\{z_n\}$, 且 $lim_{n\to\infty} z_n = 0$

2. 對於所有 z_n, $f(z_n)=0$

則 $f(z)\equiv 0$ (identical zero) ($\{c_n\}=0$ 對所有 n)
* 推廣：到任何常數 c 也可以 ($g(z)=f(z)+c$)
*** 他告訴了我們如果兩個 analytic function 在一個 accumulated point 相等，則兩個 function 相等

Thm3.1: uniqueness of analytic function
如果兩個 analytic function 在一個收斂數列處處相等，則兩個 function 相等。

4 Accumulated point

A point z_0 is called an accumulated point of a set E if every neighborhood of z_0 contains at least one point of E different from z_0 itself.
(可透過收縮 open set 找到一串收斂數列即 $\{d_n\}$,d_{n+1} 在半徑為 $d(d_n, z_0)$ 的 openball 裡面找)

5 M-L formula

if $f(z)$ is continuous on a curve γ with length L, and if $|f(z)| \leq M$ on γ, then

$$|\int_\gamma f(z)dz| \leq ML$$

6 analytic function

幾個 analytic function 的性質

1. $f(z)$ is analytic at z_0 if and only if $f(z)$ can be represented by a convergent power series in some neighborhood of z_0.

2. 如果 $f(z)$ 在一個區域 z_0 內解析，則 $f(z)$ 在 z_0 內無限可微。(power series 微分後收斂半徑不變，且無限可微)

所以解析函數 == 無限可微函數 = power series (複數空間的微分超強特性)

7 Cauchy integral formula

if $f(z)$ is analytic inside and on a simple closed circle $C = Re^{i\theta}$ 半徑 R 的封閉圓圈, then

$$f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz$$

proof:
by closed curve integral thm

$$\int_C \frac{f(z)-f(a)}{z-a}dz = 0$$

thus

$$\int_C \frac{f(z)}{z-a}dz - \int_C \frac{f(a)}{z-a}dz = 0$$

let $z - a = re^{i\theta}$, $dz = ire^{i\theta}d\theta$, C_p 是圓心 a 的圓半徑為 p 的圓

where $0 < p < r$
then

$$\int_{C_p} \frac{d_z}{z-a} = \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}} d\theta = 2\pi i$$

所以我們帶入上面的結果

$$\begin{aligned}\int_C \frac{f(a)}{z-a} dz &= f(a) \int_0^{2\pi} \frac{1}{re^{i\theta}} ire^{i\theta} d\theta \\ &= f(a) \int_0^{2\pi} i d\theta \\ &= 2\pi i f(a)\end{aligned}$$

8 Liouville's theorem

如果 $f(z)$entire 且有界，則 $f(z)$ 為常數函數。
*entire + bounded $\Rightarrow$ 常數函數
proof:
假設 $|f(z)| \leq M$，我們要證明對於任意 $a, b \in \mathbb{C}$, $f(a) = f(b)$
其中引用 Cauchy integral formula 可得

$$\begin{aligned}f(a) - f(b) &= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)} dz - \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-b)} dz \\ &= \frac{1}{2\pi i} \int_C \frac{f(z)(b-a)}{(z-a)(z-b)} dz\end{aligned}$$

又因為 $f(z) < M$, $R = |z| > |a|, |b|$ R 我們可以挑的任意大

$$\frac{1}{2\pi i} \int_C \frac{f(z)(b-a)}{(z-a)(z-b)} dz < \frac{1}{2\pi i} \frac{M(b-a)}{R^2} \cdot 2\pi R = \frac{M(b-a)}{R}$$

所以當 R 挑的足夠大，M 也是個常數下

$$f(a) - f(b) < \frac{M(b-a)}{R} = 0$$

因此 $f(z)$ 為常數函數

Ex1.
show that is no nonconstant entire function can satisfied

- $f(z+1) = f(z)$
- $f(z+i) = f(z)$

for all $z \in \mathbb{C}$
proof:
用 liouville's theorem
Assume $f(z)$ is a nonconstant entire function
then $\forall z \in \mathbb{C}$, $\exists \zeta = x - \lfloor x \rfloor + i\,(y - \lfloor y \rfloor)$, where $z = x + iy$

we pick a set $S = \{x + iy | 0 \leq |x| \leq 1, 0 \leq |y| \leq 1\}$, $(\zeta \in S)$
as we konwn $f(z)$ is continuous on S and S is closed and bounded on $\mathbb{R}^2$, by heine-borel $f(z)$ is compact
(沒想到吧，是我高微 closed + bdd == compact)
since $f(S)$ is bounded , $f(z)$ is entire
by liouville's theorem, $f(z)$ is constant function $(\rightarrow\leftarrow)$

Thus there is no nonconstant entire function can satisfied the above two conditions.

9 Maximum modulus theorem

if $f(z)$ is analytic and nonconstant in a region D, 且無 local maximum 內點

他還有一個學生姊妹 thm：minimum modulus theorem
證明方法就前面的 cauchy integral formula 所帶出的的 $f(z)$ 會是那個小環的平均值 (Cauchy Mean-value thm)。你不是平面那你就會一邊高一邊低的感覺
也就是說 analytic nonconstantfunction 在 bdd 區域內不會有 local max or min == 極值都在邊邊上

10 Automorphism of unit disk

後面常用到的 automorphism of unit disk

$$B_a(z) = \frac{z - a}{1 - \overline{a}z}, |a| < 1$$

11 Schwarz lemma

if $f(z)$ is analytic in the unit disk ($|z| < 1$) and satisfies $f(0) = 0$ and $|f| < 1$, then

1. $|f(z)| \leq |z|$ for all z in the unit disk.
2. $|f'(0)| \leq 1$
3. 只要 (1.) or (2.) 的等號成立 then iff $f(z) = e^{i\theta}z$.(即換角 function)

proof:
用 Maximum modulus theorem
let

$$g(z) = \begin{cases} \frac{f(z)}{z}, 0 < |z| < 1 \\ f'(0) = g(0) \end{cases}$$

then $g(z)$ is analytic(課本 6.7) in the unit disk.
且 $|g(z)| \leq \frac{1}{r}$ 把 r 限制在 unit disk 內即 $r = 1$
by Maximum modulus theorem, $|g(z)|$ has no local maximum in the unit disk thus $|g(z)| \leq 1$ for all z in the unit disk.
hence $|f(z)| \leq |z|$ for all z in the unit disk.
白話：我們不是造出 g(z) 嗎？我們來想一下他在邊邊的最大值也就是 ($r = 1$ $f(z) \leq 1$)
是不是就小於 1 ，又因為他沒有 local maximum，所以他在邊邊的值的最大值一定大於其他內點
也就是整個 region 都小於 1
講一下等號，$f'(0) = 1$ or $f(z) = |z|$ 等價於 $g(z)$ 是 constant

Ex1.
找出對 unit disk fixed point α 來說，$max_f|f'(\alpha)|$
最大的 alanlytic function f

proof:
let

$$h(z) = \begin{cases} \frac{f(z)-f(\alpha)}{z-\alpha} \cdot \frac{1-z\overline{\alpha}}{1-f(z)\overline{f(\alpha)}}, z \neq \alpha \\ f'(\alpha) \cdot \frac{1-|\alpha|^2}{1-f(\alpha)}|^2, z = \alpha \end{cases}$$

then $h(z)$ is analytic in the unit disk and $|h(z)| \leq 1$
by Schwarz lemma, $|h(\alpha)| \leq 1$
因此 $|f'(\alpha)| \leq \frac{1-f(\alpha)}{1-|\alpha|^2}$

12 homotopy theorem

if $f(z)$ is analytic in a region D and if two closed curves γ_1 and γ_2 in D are homotopic in D, then

$$\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$$

好像要畫圖才比較好理解，兩條 homotopy 就是想像可以用連續函數慢慢拉直到第一條 curve 的形狀變成第二條 curve (請想像一條固定端點的繩子被拉成兩種不同樣貌)
因為這兩個東西可以透過簡單的變數變換，所以積分值相等 (用分析幾何的想法)

Ex1.
證明在 punch plane 中 unit cirle cannot homotopic to a constant curve

proof:
首先觀察上就不可能了，punch plane 就是 $\mathbb{C} - \{0\}$ 把 0 點挖掉的複數空間
我們可以考慮函數 $f(z) = \frac{1}{z}$
因為 $f(z)$ 在 punch plane 內解析
所以我們可以用 homotopy thm 驗證

$$\frac{1}{2\pi i} = \int_{C_{unit}} \frac{1}{z}dz \neq \int_{\alpha_0} \frac{1}{z}dz = 0$$

前面是我們的老朋友了
後面是 constant curve (constant curve 也就是單個點) 積分當然是 0
橡皮筋中釘了一個釘子 → 橡皮筋無法縮成一個點

13 Cauchy closed curve theorem

if $f(z)$ is analytic in a region D and if γ is a smooth closed curve in D, then

$$\int_{\gamma} f(z)dz = 0$$

讓複變變輕鬆的定理之一閉環積一圈等於 0
(內部必須無奇點奇點的話就要用 residue thm 那等等再講)
那你就可以計算一些以前積分積不出來的東西透過構造多個閉環疊加
堆出你想要的積分區域 (讓數學變畫圖遊戲)

proof:
我們前面提到 analytic function 可等價於 power series
所以簡單的一次積分是可以積上去的
那積一圈就會變成簡單的 $F(\gamma(0)) - F(\gamma(1))$

起終同點即 $\gamma(0) = \gamma(1)$ 所以

$$\int_C f(z)dz = 0$$

14 Laurent expansion

if $f(z)$ is analytic in the annulus $r < R < |z - z_0| < R_2$, then $f(z)$ can be represented by a convergent power series of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

這樣講可能不好理解，他其實就是強行進行 Taylor 展開
由於整個函數有部分無法對某特定點微分
即對於 a 點展開時由於函數內部擁有 $\frac{1}{(z-a)^n}$ 這樣因式所以無法展開
所以我們就把函數分成兩個部分

$$\begin{aligned}
(\text{原函式}) &= (\text{principal part（不可解析）}) \times (\text{analytic part（可解析）}) \\
&= (\text{principal part（不可解析）}) \times (\text{analytic part (Taylor 展開式)}) \\
&= (\text{Laurent 展開式})
\end{aligned}$$

15 Cauchy residue theorem

Suppose f is analytic in a simply connected domain D except for isolated singularities at

$$z_1, z_2, \ldots, z_m.$$

Let γ be a closed curve not intersecting any of the singularities. Then

$$\int_\gamma f(z)\, dz = 2\pi i \sum_{k=1}^{m} n(\gamma, z_k)\, \mathrm{Res}(f; z_k),$$

where $n(\gamma, z_k)$ denotes the winding number of the curve γ around the point z_k, and $\mathrm{Res}(f; z_k)$ is the residue of f at z_k.

簡單來說對於閉環中有奇點的積分 $= 2\pi i$ 乘上所有奇點的 residue 和
(這個定理可以說是複變中最強大的定理了，因為他可以把你積不出來的東西變成一個個 residue 相加)

proof:
對於每個奇點我們生成一個小閉環 C_k 包住奇點 z_k
由於閉環的特性我們可以把原本的閉環積分拆成多個閉環積分相加 (closed curve integration theorem)
(請好好畫出那些環再用拓譜的形式一個一個拉到相互抵消掉你就會看懂了)

如果用畫面來想的話就是在閉環內數個點開始長泡泡泡泡貼泡泡會併成一個大泡泡
由於每個泡泡都是順時針方向 (你開心也是可以全部改逆時針) 所以接觸的共線段方向相反會抵消
只留下大泡泡的部分

至於為什麼是 $2\pi i$ 呢？
回到我們的老朋友 Cauchy integral formula

$$\int_{C_p} \frac{d_z}{z-a} = \int_0^{2\pi} \frac{ire^{i\theta}}{re^{i\theta}} d\theta = 2\pi i$$

對於繞著不可解析點的閉環積分積出來就是 $2\pi i$ (重根記得算兩個)

16 log function

The complex logarithm is a multi-valued function defined as follows:

$$\log(z) = \ln|z| + i\arg(z) + 2\pi i k, \quad k \in \mathbb{Z}$$

log function 在複數空間中是多值的，因為 arg function 本來就是多值的
我們回頭去想 log 的反函數 e^z
$e^{i\theta}$ 函數可以把整個實數域壓到 $[0, 2\pi)$
那反過來我們會把 $[0, 2\pi)$ 送回整個實數域
他會有點像螺旋洋芋片那樣一圈一圈的往上升
所以我們會需要一個 branch cut 來把他切開

所以我們定義了兩個東西

- 0 旁邊絕對不要靠近，是一對無限多
- branch cut 每次都從 $-\pi$ 到 π

總而言之，在使用 log function 時，請務必注意所選擇的 branch cut(Codomain).