Complex Analysis

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1 Cauchy-Riemann equation

A function $f(z) = u(x, y) + iv(x, y)$, $z = x + iy$ is complex differentiable at $z_0 = x_0 + iy_0$ if and only if u and v are real differentiable at (x_0, y_0).
$u_x = v_y u_y = -v_x$ $f(z)$ z_0

2 Entire

$f(x)$ is called entire if it is differentiable in all $\mathbb{C}$ fields.

differentiable (z_0)analytic (open set)entire $\mathbb{C}$

Ex1. show e^z is entire.

$$\begin{aligned} e^z &= e^{x+iy} \\ &= e^x(\cos y + i \sin y) \\ &= e^x \cos y + ie^x \sin y \end{aligned}$$

check by Cauchy-Riemann equation $u_x = v_y$ and $u_y = -v_x$ is hold on all $\mathbb{R}$ fields. thus e^z is entire.

Ex2. show e^z is only entire function satisfied

$$\begin{aligned} f(z_1) + f(z_2) &= f(z_1)f(z_2) \\ f(x) &= e^x, x \in \mathbb{R} \end{aligned}$$

let $f(z) = g(x, y) + ih(x, y)$

$\because f(z)$ is entire, $f'(0) = e^0 = 1$
$\therefore g'(0) + ih'(0) = 1$
then $g'(0) = 1, h'(0) = 0$

thus $g(x, y) = x, h(x, y) = 0$

Cauchy-Riemann equation:

$$g_x = h_y$$
$$g_y = -h_x$$

we get

$$f''(x) + f(x) = 0$$

$f(x) = c_1 sin(x) + c_2 cos(x)$(sin,cos)
$f'(0) = 1 c_1 = 1, c_2 = i \longrightarrow f(z) = e^z$

3 Convergence of complex series

Thm1
$\sum a_n z^n = A(z)$ is Convergence at radius R_1
$\sum b_n z^n = B(z)$ is Convergence at radius R_2
$\Rightarrow \sum_{n=0}^{\infty} (\sum_{k=0}^{n} a_n \cdot b_{n-k}) z^n = A(z) \cdot B(z)$ is Convergence at radius $min(R_1, R_2)$
proof:

Thm2
power series
hint: R Convergence test

Thm3 (uniqueness)
$f(z) = \sum_{n=0}^{\infty} c_n z^n$

1. $0\{z_n\}$, $lim_{n \to \infty} z_n = 0$

2. z_n, $f(z_n) = 0$

$f(z) \equiv 0$ (identical zero) ($\{c_n\} = 0n$)
* c $(g(z) = f(z) + c)$