Independence of Two Random Variables

Random variables X and Y are independent if

$$P(X = a,\ Y = b) = P(X = a)P(Y = b) \quad \text{for all values } a \text{ and } b.$$

If this equation does not hold for at least one pair of a and b, then X and Y are not independent.

When X and Y are independent, we have that

$$E(XY) = (EX)(EY), \qquad \operatorname{var}(X + Y) = \operatorname{var} X + \operatorname{var} Y.$$

The two equations above are not true in general. They are true when X and Y are independent. The following equations, however, are true in general:

$$E(X + Y) = EX + EY \qquad \text{(linearity of expectation)},$$
$$E(aX + b) = a(EX) + b, \qquad \operatorname{var}(aX + b) = a^2 \operatorname{var} X.$$

Example 1. Suppose that X and Y have the following joint distribution.

	$X=-1$	$X=2$	
$Y=1$	0.06	0.24	0.3
$Y=3$	0.14	0.56	0.7
	0.2	0.8	

Here X and Y are independent because

$$P(X = -1, Y = 1) = 0.06 = 0.2 \times 0.3 = P(X = -1)P(Y = 1)$$
$$P(X = -1, Y = 3) = 0.14 = 0.2 \times 0.7 = P(X = -1)P(Y = 3)$$
$$P(X = 2, Y = 1) = 0.24 = 0.8 \times 0.3 = P(X = 2)P(Y = 1)$$
$$P(X = 2, Y = 3) = 0.56 = 0.8 \times 0.7 = P(X = 2)P(Y = 3).$$

Let's compute $E(XY)$.

Method 1: using independence. Since X and Y are independent, we know that $E(XY) = (EX)(EY)$. So we compute EX and EY first:

$$EX = (-1)(0.2) + 2(0.8) = 1.4. \qquad EY = 1(0.3) + 3(0.7) = 2.4.$$

Therefore, $E(XY) = (EX)(EY) = 1.4 \times 2.4 = 3.36$.

Method 2: directly. XY has four possible values: -1, 2, -3 and 6, corresponding to four possible pairs of values of X and Y. (For example, $XY = 2$ when $X = 2$ and $Y = 1$. Then $P(XY = 2) = P(X = 2, Y = 1) = 0.24$.) So

$$E(XY) = (-1)(0.06) + 2(0.24) + (-3)(0.14) + 6(0.56) = 3.36.$$

As an exercise, compute $\operatorname{var}(X + Y)$ in two ways: using independence and using the direct method.

Example 2. Toss three fair coins. Let X be the number of coins that show heads, and Y be the number of coins that show tails. Then $P(X = 3) = (\frac{1}{2})^3 = \frac{1}{8}$ and $P(Y = 3) = (\frac{1}{2})^3 = \frac{1}{8}$. However, $P(X = 3, Y = 3) = 0$ (because it is impossible to have three coins showing heads *and* three coins showing tails.) So $P(X = 3, Y = 3) \neq P(X = 3)P(Y = 3)$. So X and Y are *not* independent.

As an exercise, show that $\operatorname{var}(X + Y) = 0$.

Binomial Distribution

When we repeat a trial a specific number of times and count the number of times "success" happens, that number of times we have success is a random variable and it has binomial distribution.

Let n be the number of times we repeat the trial, p be the probability of success in each trial, and X be the number of times we have success. Then $X = 0, 1, \dots$ or n, and

$$P(X = k) = \binom{n}{k} p^k (1 - p)^{n-k}, \qquad k = 0, 1, \dots, n.$$

(See problem 3 of the exercises from week 3.) The mean and the variance of X are

$$EX = np, \qquad \operatorname{var} X = np(1 - p).$$

Example 3. The forecast says that in the next five days the chance of rain for each day is 25%. Suppose that the weather on each day does not depend on the weather on the other days. What is the probability that it will rain for at least two days in the next five days? For how many days on average will it rain in the next five days?

Solution. Let X be the number of days of rain. Then X has binomial distribution with parameters $n = 5$ and $p = 0.25$. We want to find $P(X \geq 2)$ and EX.

$$\begin{aligned} P(X \geq 2) &= 1 - P(X < 2) = 1 - P(X = 0) - P(X = 1) \\ &= 1 - (0.75)^5 - \binom{5}{1}(0.25)(0.75)^4. \end{aligned}$$

Finally, $EX = np = 5(0.25) = 1.25$ days. ■

Solutions for Mock Midterm 1

1. **X is a random variable that takes on only two possible values: 4 and -3. Suppose that $EX=-1$. Compute $\operatorname{var} X$.**

   Let $p = P(X = 4)$. Since -3 is the only other possible value, we have $P(X=-3) = 1-p$. Then

   $$-1 = EX = 4P(X=4) + (-3)P(X=-3) = 4p - 3(1-p) = 7p - 3.$$

   So $2 = 7p$. That is, $p = \frac{2}{7}$. Then

   $$EX^2 = 4^2P(X=4) + (-3)^2P(X=-3) = 16p + 9(1-p) = 16\cdot\tfrac{2}{7} + 9\cdot\tfrac{5}{7} = 11.$$

   Therefore, $\operatorname{var} X = EX^2 - (EX)^2 = 11 - (-1)^2 = \boxed{10}$.

2. **X and Y are two random variables that both take on two possible values: 1 and 2. Suppose that $P(Y=1) = \frac{2}{5}$, $P(X=1|Y=1) = \frac{1}{2}$, and $P(X=2|Y=2) = \frac{2}{3}$.**

   (a) **Compute $P(Y=2|X=1)$.**

   (See the tree diagram at the end.)

   $$\begin{aligned} P(Y=2|X=1) &= \frac{P(Y=2, X=1)}{P(X=1)} \\ &= \frac{P(Y=2,X=1)}{P(Y=1,X=1) + P(Y=2,X=1)} \\ &= \frac{\frac{3}{5}\cdot\frac{1}{3}}{\frac{2}{5}\cdot\frac{1}{2} + \frac{3}{5}\cdot\frac{1}{3}} = \boxed{\frac{1}{2}}. \end{aligned}$$

   (b) **Compute EX and EY.**

   $$EY = 1P(Y=1) + 2P(Y=2) = 1\cdot\tfrac{2}{5} + 2\cdot\tfrac{3}{5} = \boxed{\frac{8}{5}}.$$

   $$EX = 1P(X=1) + 2P(X=2) = 1(\tfrac{2}{5}\cdot\tfrac{1}{2} + \tfrac{3}{5}\cdot\tfrac{1}{3}) + 2(\tfrac{2}{5}\cdot\tfrac{1}{2} + \tfrac{3}{5}\cdot\tfrac{2}{3}) = \boxed{\frac{8}{5}}.$$

   (c) **Are X and Y independent? Explain why.**

   $\boxed{\text{No}}$, they are not because from part (a), $P(Y=2|X=1) \neq P(Y=2)$. That is, $P(Y=2, X=1) \neq P(Y=2)P(X=1)$.

   (d) **Compute $E(XY)$.**

   Since X and Y equal 1 or 2, we have that $XY = 1$, 2, or 4.
   $P(XY=1) = P(X=1, Y=1) = \frac{2}{5}\cdot\frac{1}{2} = \frac{1}{5}$.
   $P(XY=2) = P(X=1, Y=2) + P(X=2, Y=1) = \frac{3}{5}\cdot\frac{1}{3} + \frac{2}{5}\cdot\frac{1}{2} = \frac{2}{5}$.
   $P(XY=4) = P(X=2, Y=2) = \frac{3}{5}\cdot\frac{2}{3} = \frac{2}{5}$.

   Therefore,
   $$\begin{aligned} E(XY) &= 1P(XY=1) + 2P(XY=2) + 4P(XY=4) \\ &= 1\cdot\tfrac{1}{5} + 2\cdot\tfrac{2}{5} + 4\cdot\tfrac{2}{5} = \boxed{\tfrac{13}{5}}. \end{aligned}$$

   Note that $E(XY) \neq (EX)(EY)$, another evidence that X and Y are not independent.

3. **A teacher is randomly dividing six students into two groups of three for two different topics of research: topics A and B. (You may think of the process as if the teacher randomly chooses three students for topic A, and the unchosen students then get topic B.) Suppose that Jeff and Kate are two of the six students.**

   (a) **What is the probability that Jeff will be assigned topic A?**

   $$\boxed{\frac{\binom{5}{2}}{\binom{6}{3}} = \frac{1}{2}.}$$

   Out of $\binom{6}{3}$ ways of dividing students, there are $\binom{5}{2}$ to choose Jeff and two other students for topic A.

   (b) **What is the probability that Jeff and Kate are assigned the same topic?**

   $$\boxed{\frac{\binom{4}{1} + \binom{4}{1}}{\binom{6}{3}} = \frac{2}{5}.}$$

   Out of $\binom{6}{3}$ ways of dividing students, there are $\binom{4}{1}$ to choose Jeff, Kate, and another student for topic A, and there are $\binom{4}{1}$ to choose Jeff, Kate, and another student for topic B.