Fixed Income Derivatives - The Ho-Lee Model

In this problem, we will consider the Ho-Lee model, in which the short rate r_t is assumed to have the following dynamics under the risk-neutral measure $\mathbb{Q}$

$$dr_t = \Theta(t)dt + \sigma dW_t \tag{1}$$

where $\Theta(t)$ and σ are model parameters.

Problem 1 - Zero coupon bond prices in the Ho-Lee model

Let us recall that the Ho-Lee model possesses an affine term structure and that zero coupon bond prices are of the form

$$\begin{aligned} P(t,T) &= e^{A(t,T)-B(t,T)r_t}, \\ A(t,T) &= \frac{\sigma^2}{2}\frac{(T-t)^3}{3} + \int_t^T \Theta(s)(s-T)ds, \\ B(t,T) &= T-t. \end{aligned} \tag{2}$$

Let us denote observed prices at time 0 of maturity T zero coupon bonds by $p^*(0,T)$ and observed forward rates by $f^*(0,T)$.

a) We have previously shown that $\Theta(t) = \frac{\partial f^*(0,t)}{\partial T} + \sigma^2 t$ where $\frac{\partial}{\partial T}$ refers to derivative with respect to the second argument of $f^*(0,t)$. Use this and integration by parts to show that ZCB prices in the Ho-Lee model are given by

$$p(t,T) = \frac{p^*(0,T)}{p^*(0,t)} \exp\Big\{(T-t)f^*(0,t) - \frac{\sigma^2}{2}t(T-t)^2 - (T-t)r_t\Big\} \tag{3}$$

b) Use the expression for ZCB prices in (2) to show that the dynamics of ZCB prices in the Ho-Lee model are

$$dP(t,T) = rP(t,T)dt + \sigma(T-t)P(t,T)dW_t \tag{4}$$

c) For u such that $t < u < T$ find a solution to (4).

d) Show that $\ln P(u,T)|\mathcal{F}_t$ follows a log-normal distribution and find the mean and variance of $\ln P(u,T)$.

e) Use integration by parts to express the solution for r_T given $\mathcal{F}_t$ in terms of f^* and find the distribution of $r_T|\mathcal{F}_t$.

Problem 2 - Option prices in the Ho-Lee model

In this problem, we will consider a European call option with time t price denoted $\Pi(t;T_1,T_2)$, strike K and exercise at T_1 on a maturity T_2 zero coupon bond where of course $t < T_1 < T_2$. It is tempting to try to use the results from Problem 1 to find the price of the European call option and though it can be done, there is a much simpler solution that we will pursue.

a) Argue that $\Pi(t,T)$ can be found using the following version of our general option pricing formula

$$\Pi(t;T_1,T_2) = P(t,T_2)\mathbb{Q}^{T_2}\big(P(T_1,T_2) \geq K\big) - Kp(t,T_1)\mathbb{Q}^{T_1}\big(P(T_1,T_2) \geq K\big) \tag{5}$$

b) Find the dynamics of $Y(t) = \frac{P(t,T_1)}{P(t,T_2)}$ and use the fact that $Y(t)$ is martingale under $\mathbb{Q}^{T_2}$ with same diffusion coefficient as under $\mathbb{Q}$ to solve for $Y(T_1)$. Next, use this solution to find $\mathbb{Q}^{T_2}\big(P(T_1,T_2) \geq K\big)$.

c) Find the dynamics of $Z(t) = \frac{P(t,T_2)}{P(t,T_1)}$ and use the fact that $Z(t)$ is martingale under $\mathbb{Q}^{T_1}$ with same diffusion coefficient as under $\mathbb{Q}$ to solve for $Z(T_1)$. Next, use this solution to find $\mathbb{Q}^{T_1}\big(P(T_1,T_2) \geq K\big)$.

d) Finally, use your results to conclude that

$$\begin{aligned} \Pi(t;T_1,T_2) &= P(t,T_2)\Phi(d) - Kp(t,T_1)\Phi(d-\sigma_p), \\ d = \frac{1}{\sigma_p}\ln\left(\frac{p(t,T_2)}{p(t,T_1)K}\right) &+ \frac{1}{2}\sigma_p, \quad \sigma_p = \sigma(T_2-T_1)\sqrt{T_1-t} \end{aligned} \tag{6}$$

Problem 3 - Fitting the yield curve using a Vasicek Model

For this problem, we assume you have the following market data available and that the interest rate swaps pay 6M EURIBOR semiannually against a fixed rate paid annually.

EURIBOR	Fixing	FRA	Midquote	IRS	Midquote
6M	0.0430136	1X7	0.0455066	2Y	0.0558702
		2X8	0.0477436	3Y	0.058811
		3X9	0.0497492	4Y	0.0600937
		4X10	0.0515456	5Y	0.0605263
		5X11	0.0531529	7Y	0.0601899
		6X12	0.0545893	10Y	0.0586669
		7X13	0.0558712	15Y	0.0562267
		8X14	0.0570135	20Y	0.0547351
		9X15	0.0580298	30Y	0.0535523

a) Fit a ZCB spot rate curve to the market data and plot spot and forward rates from your choice of interpolation method. Also discuss if the spot and forward rates from your fit have the properties that we would like fitted spot- and forward rates to have,

b) Using the initial values $r_0 = 0.035$, $a = 0.5$, $b = 0.025$, $\sigma = 0.03$, fit a Vasicek model to the ZCB spot rates you have calibrated from market data.

c) Plot the fitted spot- and forward rates from your Vasicek model in a plot that also contains the spot and forward rates you found in a).

d) You are likely to have found that the Vasicek model does not fit the market data very well. Try to explain why the Vasicek model never really stood a chance considering the spot rate curve you got in a).

Problem 4 - Fitting a Ho-Lee model to the data

In this problem, we will fit a Ho-Lee model to our market data and you can assume that $\sigma = 0.005$.

a) Use the function '*interpolate.py*' to first find $\frac{\partial f^*(0,t)}{\partial T}$ and then find $\Theta(t)$.

b) Argue that the Ho-Lee model will by construction fit the initial term structure by appealing to (3).

Problem 5 - Pricing an interest rate cap in the Ho-Lee model

We will now consider the pricing of an interest rate cap on a 10Y receiver swap in which the holder receives a an annual fixed rate in exchange for paying 6M Euribor on the floating leg. As in the previous question, you can assume that $\sigma = 0.005$.

a) Explain the relationship between a caplet on an individual Euribor rate payment and also explain, how a caplet can be seen as a type of European option on a specific underlying asset.

b) Derive an explicit expression for the price of a European put option with the ZCB as the underlying asset.

c) Compute the prices of all caplets on 6M forward Euribor with a strike of $K = 0.06$ corresponding to the floating rate payments on the 10Y receiver swap and use these prices to find.

d) Compute the price of a 10Y interest rate cap with strike $K = 0.06$ on 6M Euribor and express the price of this cap both in terms of an upfront payment as well as a premium to be paid semi-annually for 10 years.

e) Investigate how the price of the cap depends on σ and compute the DV01 of changing σ by 0.001 both up and down.

Problem 6 - Simulation of r_t and confidence intervals in the Ho-Lee model

Now we will simulate short rates in the Ho-Lee model using the usual first order Euler scheme on a grid of mesh δ that runs from initial time t_0 to terminal time T. Denote by M, the number of steps in your simulation. The time points in your simulation will be numbered $m = 0, 1, 2, ..., M-1, M$, the time points will be $\big[t_0, t_1, ..., t_{M-1}, t_M\big] = \big[0, \delta, 2\delta, ..., T-\delta, T\big]$ and $\delta = \frac{T}{M}$. Again, you can assume that $\sigma = 0.005$.

a) Develop both an exact an Euler scheme to simulate the short rate.

b) Run the simulation of both the two schemes for $T = 10$ and plot the trajectories for both schemes. Try to use the same randomness by setting the same 'seed' for both schemes and access if the two schemes should and also do so in practice, result in the same trajectories.

c) Use the explicit solution you have found for $r_T|r_t$ to construct upper, lower and two sided confidence intervals for r_t and include a two-sided 95 per cent confidence interval in your plot.

d) Evaluate what happens to the distribution of the short rate in the Ho-Lee model when t grows large. Does the short rate mean-revert in this model and does it make sense to talk about the "asymptotic" properties of r_t in the Ho-Lee model?

Problem 7 - Pricing a swaption in the Ho-Lee model

Finally, we will find the price of a 2Y46 receiver swaption with a strike of $K = 0.06$. The receiver swaption gives the owner the right but not obligation to enter into an 4Y receiver swap with semiannual fixed payments at exercise in $T_n = 1$ years. To compute the price of this swaption, you can simulate the short rate.

a) Argue that the payoff function $\chi(T_n)$ and the discounted payoff function $\tilde{\chi}(T_n)$ of the payer swaption are.

$$\begin{aligned}
\chi(T_n) &= S_n^N(T_n)\big(K - R_n^N(T_n)\big)_+ \\
\tilde{\chi}(T_n) &= \exp\Big\{-\int_0^{T_n} r_t dt\Big\} S_n^N(T_n)\big(K - R_n^N(T_n)\big)_+
\end{aligned} \tag{7}$$

b) Find a method to compute the price at $t = 0$ of the swaption by simulating at least $L = 1000$ trajectories and having at least $M = 1000$ steps in your simulation.

c) Investigate if the price you have computed is accurate by plotting the value of the derivative for various choices of L.

d) Explain how the price of the swaption depends on σ, T_n, T_N and of course K.

e) Discuss which types of financial institutions might need a receiver swaption to manage their interest rate risk.

f) It is also possible to compute the price of a swaption explicitly in the Ho-Lee model. Explain which method can be used and apply the method in practice to insure that the explicit solution and the simulated estimate for the swaption price agree.