Fixed Income Derivatives E2025 - Problem Set Week 4

Problem 1

Let W_t be a Bownian motion, assume $s < t < u < v$ and solve the problems below. In doing so, you will need to use that W_t is Markov and has stationary independent increments. That is, for $0 < s < t$ we know that $W_t - W_s|\mathcal{F}_s = W_t - W_s|W_s = w_s \sim N\big(0, t-s\big)$.

a) Find the conditional distribution of W_t given $\mathcal{F}_s$.

c) Find $\mathbb{E}\big[W_s W_t\big]$, $\mathrm{Cov}\big[W_s, W_t\big]$ and $\mathrm{Cor}\big[X_t, Z_t\big]$.

c) Show that $W_t^2 - t$ is a Martingale.

d) Find $\mathbb{E}\big[W_s W_t W_u\big]$.

e) Find $\mathbb{E}\big[W_s W_t W_u W_v\big]$.

Problem 2

Let X_t and Y_t be independent Brownian motions for $t \geq 0$. Define $Z_t = \rho X_t + \sqrt{1-\rho^2} Y_t$.

a) Show that Z_t is a Brownian motion

b) Find $\mathrm{Cor}\big[X_t, Z_t\big]$.

c) Find $\mathbb{E}\big[Z_t \big| X_t = x\big]$ and $\mathrm{Var}\big[Z_t \big| X_t = x\big]$.

Let $W_t^{(1)}, W_t^{(2)}, ..., W_t^{(N)}$ be independent Brownian motions and let $\mathbf{\Sigma}$ be an $M \times N$-dimensional matrix where row i, $\mathbf{\Sigma}_{i\cdot}$, satisfies $\|\mathbf{\Sigma}_{i\cdot}\| = \Sigma_{i1}^2 + \Sigma_{i2}^2 + ... + \Sigma_{iN}^2 = 1$. Define the M-dimensional vector $\mathbf{Y}_t = \mathbf{\Sigma}\mathbf{W}_t$

d) Find the covariance matrix of the random vector $\mathbf{Y}_t$. Show that the covariance matrix is positive definite?

e) What is the correlation matrix of $\mathbf{Y}_t$?

f) What is the distribution of $Y_t^{(i)}$ and what is the joint distribution of $\mathbf{Y}_t$?

g) Is $\mathbf{Y}_t$ a multivariate Brownian motion?

Problem 3

Consider a stochastic process r_t for $t \geq 0$ with dynamics

$$dr_t = (b - ar_t)dt + \sigma dW_t, \quad b > 0$$

a) Show that the solution $r(T)$ corresponding to these dynamics are

$$r_T = e^{-aT} r_0 + \frac{b}{a}\big(1 - e^{-aT}\big) + \sigma \int_0^t e^{-a(T-t)} dW_t$$

by performing the following steps

i) Apply Ito's formula to $f(t, r) = e^{at} r$.

ii) Simplify to get an expression for $d\big(e^{at} r\big)$ that does not depend on r_t.

iii) Integrate from 0 to T and solve the time-integral.

b) Use Ito isometry to show that $r_T | r_t \sim N\Big(e^{-aT} r_0 + \frac{b}{a}\big(1 - e^{-aT}\big), \frac{\sigma^2}{2a}\big[1 - e^{-2aT}\big]\Big)$.

c) Find the limiting distribution of r_T as $T \nearrow \infty$.

d) If you had to guess, what is your best guess of the r in the long run? How does the limiting distribution of r_T depend on r_0 and what is the implication?

Problem 4

Suppose that the stochastic process S_t follows a Geometric Brownian motion and has dynamics

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma S_t dW_t \\ S_0 &= s_0 \end{aligned}$$

a) Show that the solution $S(T)$ corresponding to these dynamics is $S(T) = s_0 e^{(\mu - \frac{1}{2}\sigma^2)T + \sigma W_T}$.

b) Find $\mathbb{E}[S(T)]$ in terms of s_0, μ and σ.

c) Find the dynamics of $Z_t = S_t^m$ and show that Z_t also follows a geometric Brownian motion.

d) Use these results to find $\mathbb{E}\big[S^m(T)\big]$.

Problem 5

Let $\sigma(t)$ be a given *deterministic* function of time and define the process X_t by

$$X(t) = \int_0^t \sigma(s) dW_s$$

Also define $Z(t) = e^{i\omega X(t)}$ where i is the complex unit and thus a constant and ω is also a constant.

a) Find the dynamics of X_t.

b) Find the dynamics of Z_t and show that Z_t has dynamics

$$\begin{aligned} dZ_t &= -\frac{1}{2}\omega^2\sigma^2(t)Z(t)dt + i\omega\sigma(t)Z_t dW_t \\ Z_0 &= 1 \end{aligned}$$

c) Integrate dZ_t and take expectations to find an expression for $\mathrm{E}\big[Z(t)\big]$.

d) Define $m(t) = \mathrm{E}\big[Z(t)\big]$ and show that $m(t)$ satisfies the ODE.

$$\begin{aligned} m'(t) &= -\frac{1}{2}\omega^2\sigma^2(t)m(t) \\ m(0) &= 1 \end{aligned}$$

e) Argue that $\mathrm{E}\big[e^{i\omega X(t)}\big] = \exp\Big(-\frac{1}{2}\omega^2\int_0^t \sigma^2(s)ds\Big)$ and why we can say that $X(t) \sim N\Big(0, \int_0^t \sigma^2(s)ds\Big)$.