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Coursework & Homework demo

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(a) Note that $\theta_2 = \theta_1 + \theta_2 - \theta_1 = \theta_1 + \phi_2, \theta_3 = \theta_2 + \theta_3 - \theta_2 = \theta_1 + \sum_{i=2}^{3} \phi_i$. Then it is clear that $\theta_k = \theta_1 + \sum_{i=2}^{k} \phi_i$ with $k > 2$.

(b) From the hint, the partial derivative of representation (2) with respect to θ_1 is:

$$\frac{\partial}{\partial \theta_1} \left(\sum_{i=1}^{n} (y_i - \theta_i)^2 + \lambda \sum_{i=2}^{n} |\phi_i| \right) = 2\theta_1 - 2y_1 - 2\lambda\phi_2 = -2(y_1 - \theta_1 + \lambda\phi_2)$$

Hence, the equation is minimized with $\hat{\theta}_i = \theta_1 + \lambda \sum_{j=2}^{i} \phi_j$, which gives $\sum_{i=1}^{n} (y_i - \hat{\theta}_i) = 0$.

(c) Want to show $|y_i - \hat{\theta}_i| \leq \lambda$, note that for all i, the sub-gradient of $\{\lambda \sum_{i=2}^{n} |\theta_i - \theta_{i-1}|\}$, in other word: the sub-gradient of:

$$\lambda|\theta_{i+1} - \theta_i| + \lambda|\theta_i - \theta_{i-1}|$$

is

$$\lambda[-1, 1] + \lambda[-1, 1] = 2\lambda[-1, 1] = [-2\lambda, 2\lambda]$$

hence the sub-gradient for $\theta_1, ..., \theta_n, \partial\{\lambda \sum_{i=2}^{n} |\theta_i - \theta_{i-1}|\} \subset [-2\lambda, 2\lambda]^n$
Note that $\frac{\partial}{\partial \theta_1} \left(\sum_{i=1}^{n} (y_i - \theta_i)^2 + \lambda \sum_{i=2}^{n} |\phi_i| \right)$ 2 equals $2\theta_1 - 2y_1 - 2\lambda\phi_2$:
Note that for $i = 1$:

$$2\theta_1 - 2y_1 - 2\lambda\phi_2 + \partial\{\lambda \sum_{i=2}^{n} |\theta_i - \theta_{i-1}|\} = 0$$

$$-2(y_1 - \theta_1 + \lambda\phi_2) = \partial\{\lambda \sum_{i=2}^{n} |\theta_i - \theta_{i-1}|\} = 0 \subset [-2\lambda, 2\lambda]$$

$$|(y_1 - \hat{\theta}_1)| \leq \lambda$$

And finally, we have $|y_i - \hat{\theta}_i| \leq \lambda$ for all i.

(d) Note that by taking the sub-gradient of (2) with respect to ϕ_j:

$$-2 \sum_{i=j}^{n} (y_i - \theta_1 - \sum_{k=2, k\neq j}^{i} \phi_k) + \text{sgn}(\lambda) = 0$$

$$-2 \sum_{i=j}^{n} (y_i - \theta_1 - \sum_{k=2, k\neq j}^{i} \phi_k) = -\text{sgn}(\lambda)$$

$$2 \sum_{i=j}^{n} (y_i - \theta_1 - \sum_{k=2, k\neq j}^{i} \phi_k) = \text{sgn}(\lambda)$$

Note that if $|\lambda| = 2 \sum_{i=1}^{n} (y_i - \theta_1 - \sum_{k=2, k\neq j}^{i} \phi_k)$, we will have $\hat{\theta}_1 = ... = \hat{\theta}_n = \bar{y}$.

2

(a) Want to find $E_\lambda(M)$. Note that

$$\begin{aligned}
E_\lambda(M) &= \sum_{m=0}^{\infty} m\binom{n+m-1}{m}(1-\kappa_\lambda(r))^m \kappa_\lambda(r)^n \\
&= \sum_{m=1}^{\infty} \frac{(n+m-1)!}{(m-1)!(n-1)!}(1-\kappa_\lambda(r))^m \kappa_\lambda(r)^n \\
&= \sum_{m=1}^{\infty} \frac{n(1-\kappa_\lambda(r))}{\kappa_\lambda}\binom{n+m-1}{m-1}\kappa_\lambda(r)^{n+1}(1-\kappa_\lambda(r))^{m-1} \\
&= \frac{n(1-\kappa_\lambda(r))}{\kappa_\lambda(r)}\sum_{z=0}^{\infty}\binom{n+1+z-1}{z}\kappa_\lambda(r)^{n+1}(1-\kappa_\lambda(r))^{z} \qquad \text{With } z=m-1 \\
&= n\frac{1-\kappa_\lambda(r)}{\kappa_\lambda(r)}
\end{aligned}$$

(b) Note that since $X_{n+1},...,X_{n+M}$ independent of $X_1,...,X_n$, then:

$$\begin{aligned}
E_\lambda(\sum_{i=n+1}^{n+M} X_i|X_1=x_1,...,X_n=x_n) &= E_\lambda(\sum_{i=n+1}^{n+M} X_i) \\
&= E_\lambda(E_\lambda(\sum_{i=n+1}^{n+M} X_i|M=m)) \qquad \text{with } X_{n+1},...,X_{n+M}\leq r \\
&= E_\lambda(m(E_\lambda(X_i|X_i<r))) \\
&= E_\lambda(M)E_\lambda(X_i|X_i<r)
\end{aligned}$$

(c) The truncated model is useful in this case, and as the code shown below, the estimated M is reasonable as M converges to around 2730.

```
> lambda = (1317+2*239+3*42+4*14+5*4+6*4+7)/(1317+239+42+14+4+4+1)
> lambda
[1] 1.25108
> lambda = (1317+2*239+3*42+4*14+5*4+6*4+7)/(1317+239+42+14+4+4+1)
> for (i in 1:100){
+ k = 1-exp(-lambda)
+ n = 1621
+ M = n*(1-k)/k
+ lambda = ( 1317 + 2 * 239 + 3 * 42 + 4 * 14 + 5 * 4 + 6 * 4 + 7)/(1317+239+42+14+4+4+1+M)
+ i = i + 1
+ }
> M
[1] 2730.148
> lambda
[1] 0.4660839
```