

### Solutions to homework 4:

1. Recall that we say  $n \in \mathbb{Z}$  is a perfect square when  $n = k^2$  for some  $k \in \mathbb{Z}$ . Let  $n \in \mathbb{Z}$ .

(a) Prove that if  $n \equiv 2 \pmod{4}$  or  $n \equiv 3 \pmod{4}$ , then  $n$  is not a perfect square.

**Proof by Contrapositive:** If  $n$  is a perfect square then  $n \equiv 2 \pmod{4}$  or  $n \not\equiv 3 \pmod{4}$ . Assume that  $n$  is a perfect square  $n = k^2$ .

- **Case 1:** Assume that  $k$  is even such that  $k = 2\ell$ ,  $\ell \in \mathbb{Z}$ , and consider  $n = 4\ell^2$ . By definition of divisibility by 4,  $4\ell^2 \equiv 0 \pmod{4}$  which implies that  $n \equiv 0 \pmod{4}$
- **Case 2:** Assume that  $k$  is odd such that  $k = 2\ell + 1$ ,  $\ell \in \mathbb{Z}$ , and consider  $n = 4\ell^2 + 4\ell + 1 = 4(\ell^2 + \ell) + 1$ . Since  $\ell$  is an integer we  $\ell^2 + \ell = m$ ,  $m \in \mathbb{Z}$ . By definition of divisibility by definition of 1 mod 4,  $4m + 1 \equiv 1 \pmod{4}$  which implies that  $n \equiv 1 \pmod{4}$

Hence in both cases and the contrapositive, we can conclude that if  $n \equiv 2 \pmod{4}$  or  $n \equiv 3 \pmod{4}$

(b) Prove that if  $n^2 + 1$  is a perfect square then  $n = 0$ . Hint: Factor a difference of squares and think about divisors.

**Proof:** Assume that  $n^2 + 1$  is a perfect square such that  $n^2 + 1 = k^2$ . We can say that  $n^2 = k^2 - 1$ .

$$k^2 - n^2 = (k + n)(k - n)$$

$$1 = (k + n)(k - n)$$

Now consider the following cases, since we know that the only divisors of 1 are 1 and  $-1$ , we can conclude that either both  $(k + n)$  and  $(k - n)$  are equal to 1 or they are equal  $-1$ . Consider the following case work.

- **Case 1:** Assume that  $(k + n) = 1$  and  $(k - n) = 1$ .

$$k = 1 - n$$

$$\text{We rearrange } (k + 1) = -1(1 - n) - n = 1$$

$$-2n = 0n \quad \quad \quad = 0$$

- **Case 2:** Assume that  $(k + n) = -1$  and  $(k - n) = -1$ .

$$k = -1 - n$$

$$\text{We rearrange } (k + 1) = -1(-1 - n) - n = -1$$

$$-2n = 0n \quad \quad \quad = 0$$

Hence, in either case that  $(k + n)$  or  $(k - n)$  are equal to 1 or  $-1$  we can conclude that  $n = 0$ , thus if  $n^2 + 1$  is a perfect square then  $n = 0$ . ■

2. Recall Bézout's identity: Let  $a, b \in \mathbb{Z}$  such that  $a$  and  $b$  are not both zero. Then there exists  $x, y \in \mathbb{Z}$  such that  $ax + by = \gcd(a, b)$ . Use this result to prove the following result: Let  $a, b, c \in \mathbb{Z}$  such that  $\gcd(a, b) = 1$ . Then  $(a \mid bc) \implies (a \mid c)$ .

**Proof:** Assume that  $(a \mid bc)$  such that  $bc = ak$ , for some  $k \in \mathbb{Z}$ . Now consider the fact that  $\gcd(a, b) = 1$  and by Bézout's identity we know that  $ax + by = 1$ . Multiplying both sides by  $c$  we find  $acx + bcy = c$  and using our assumption  $a \mid bc$  we know  $acx + ak y = c$ . Factoring  $a$  out of the result yields,  $a(cx + by) = c$ , and since we know that the result of  $(cx + by) \in \mathbb{Z}$  we can conclude that  $a \mid c$ . Hence  $a \mid c$ . ■

3. Let  $P \subseteq \mathbb{N}$  be the set of prime numbers  $P = \{2, 3, 5, 7, 11, \dots\}$ . Determine whether the following statements are true or false. Prove your answers ("true" or "false" is not sufficient).

1.  $\forall x \in P, \forall y \in P, x + y \in P$ . False we prove negation.
  - Choose  $x = 3, y = 5, x + y = 8$ , and  $8 \notin P$ .
2.  $\forall x \in P, \exists y \in P$  such that  $x + y \notin P$ . This statement is true.

**Proof:** Assume  $x \in P$  and choose  $y = 23$ . Consider the following case work.

- **Case 1:** Assume that  $x$  is even, such that  $x = 2$ , since 2 is the only even number in  $P$ .  $23 + 2 = 25$ , thus if  $x$  is even then  $x + y \notin P$ .
- **Case 2:** Assume that  $x$  is odd, such that  $x = 2k + 1$ . We can write  $x + y$  as  $(2k + 1) + 2(10) + 3 = 2k + 2(10) + 4$ . Factoring 2 out of our expression yields  $2(\ell)$ , since we know that  $x + y > 2$ , since  $x + y$  is even and greater than two, we can conclude that  $(x + y) \notin P$ .

Hence by both cases we can conclude that  $\forall x \in P, \exists y \in P$  such that  $x + y \notin P$ . ■

3.  $\exists x \in P$  such that,  $\forall y \in P, x + y \in P$ . This statement is false, we can prove the negations.

**Proof:**  $\forall x \in P, \exists y \in P$  such that  $x + y \notin P$ . Assume  $x \in P$  and choose  $y = 7$ . Consider the following case work.

- **Case 1:** Assume that  $x$  is even, such that  $x = 2$ , since 2 is the only even number in  $P$ .  $7 + 2 = 9$ , thus if  $x$  is even then  $x + y \notin P$ .
- **Case 2:** Assume that  $x$  is odd, such that  $x = 2k + 1$ . We can write  $x + y$  as  $(2k + 1) + 2(4) + 1 = 2k + 2(10) + 2$ . Factoring 2 out of our expression yields  $2(\ell)$ , since we know that  $x + y > 2$ , since  $x + y$  is even and greater than two, we can conclude that  $(x + y) \notin P$ .

Hence by proving both cases of the negation we can conclude that our original statement is false. ■

4. Prove the following statement: For every positive number  $\varepsilon$  there is a positive number  $M$  such that

$$\left| \frac{2x^2}{x^2 + 1} - 2 \right| < \varepsilon$$

whenever  $x \geq M$ .

**Proof:** Assume that  $\varepsilon > 0$ , let  $M = \sqrt{\frac{2}{\varepsilon}}$  and assume  $x \geq M$ . Consider the following.

$$x \geq M$$

$$x \geq \sqrt{\frac{2}{\varepsilon}}$$

$$x^2 \geq \frac{2}{\varepsilon}$$

$$\varepsilon \geq \frac{2}{x^2}$$

$$\varepsilon > \frac{2}{x^2 + 1}$$

Hence

$$\left| \frac{2x^2}{x^2 + 1} - 2 \right| = \left| \frac{2x^2 - 2x^2 - 2}{x^2 + 1} \right| = \frac{2}{x^2 + 1} < \varepsilon$$

■

5. We say that a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous at  $a \in \mathbb{R}$  if  $\lim_{x \rightarrow a} f(x) = f(a)$ . Let

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Is  $f$  continuous at  $x = 0$ ? (You may use the result:  $\forall x \in \mathbb{R}, |\sin x| \leq 1$ .)

Note: Make sure to use the definition of a limit to justify your answer, namely: Let  $a, L \in \mathbb{R}$ . The limit of a function  $f$  as  $x$  approaches  $a$  is  $L$ , when (for  $\varepsilon, \delta, x \in \mathbb{R}$ )

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } (0 < |x - a| < \delta) \implies (|f(x) - L| < \varepsilon).$$

Hint: You can make use of the fact that  $|f(x)| \leq x^2$ .

**Proof:** Assume that  $\varepsilon > 0$ , let  $\delta = \sqrt{\varepsilon}$  and assume that  $|0 < x - 0 < \delta|, x \in \mathbb{R}$ . Consider the following.

$$0 < x < \sqrt{\varepsilon}$$

$$0 < x^2 < \varepsilon$$

Hence

$$|f(x)| \leq x^2 < \varepsilon \text{ and } |f(x) - 0| < \varepsilon$$

Thus, we can conclude that since  $\lim_{x \rightarrow 0} f(x) = 0$  and  $f(0) = 0$  it must be true that  $f(x)$  is continuous. ■

6. We say that a sequence  $(x_n)$  converges to  $L$  if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, (n > N) \implies |x_n - L| < \varepsilon.$$

Using the definition, prove that the sequence  $(x_n)$  with

$$x_n = (-1)^n + \frac{1}{n}$$

does not converge to any  $L \in \mathbb{R}$ .

**Proof:** Consider the definition of non-convergence.  $\exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists n \in \mathbb{N}, (n > N) \wedge |x_n - L| \geq \varepsilon$ . Choose  $\varepsilon = \frac{1}{2}$  and consider the following case work.

• **Case 1:** Assume  $L \geq 0$

Choose  $n > N \geq 2$  such that  $n$  is odd.

Then:

$$x_n = (-1)^n + \frac{1}{n} = -1 + \frac{1}{n} < -1 + \frac{1}{2} = -\frac{1}{2}$$

Since  $L \geq 0$  and  $x_n < -\frac{1}{2}$ :

$$|x_n - L| = L - x_n > 0 - \left(-\frac{1}{2}\right) = \frac{1}{2}$$

Therefore  $|x_n - L| \geq \frac{1}{2}$ .

• **Case 2:** Assume  $L < 0$

Choose  $n > N \geq 2$  such that  $n$  is even.

Then:

$$x_n = (-1)^n + \frac{1}{n} = 1 + \frac{1}{n} > 1 > \frac{1}{2}$$

Since  $L < 0$  and  $x_n > \frac{1}{2}$ :

$$|x_n - L| = x_n - L > \frac{1}{2} - L > \frac{1}{2} - 0 = \frac{1}{2}$$

Therefore  $|x_n - L| \geq \frac{1}{2}$ .

Hence, for every  $L \in \mathbb{R}$  and every  $N \in \mathbb{N}$ , we found  $n > N$  such that  $|x_n - L| \geq \frac{1}{2}$ .

Therefore, the sequence  $(x_n)$  does not converge to any  $L \in \mathbb{R}$ .

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