Solutions to homework 4:

1. Recall that we say $n \in \mathbb{Z}$ is a perfect square when $n = k^2$ for some $k \in \mathbb{Z}$. Let $n \in \mathbb{Z}$.

(a) Prove that if $n \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$, then n is not a perfect square.

Proof by Contrapositive: If n is a perfect square then $n \equiv 2.\neg \pmod 4$ or $n \not\equiv 3 \pmod 4$. Assume that n is a perfect square $n = k^2$.

- **Case 1:** Assume that k is even such that $k = 2\ell$, $\ell \in \mathbb{Z}$, and consider $n = 4\ell^2$. By definition of divisibility by 4, $4\ell^2 \equiv 0(\operatorname{mod} 4)$ which implies that $n \equiv 0(\text{mod})4$
- **Case 2:** Assume that k is odd such that $k = 2\ell + 1$, $\ell \in \mathbb{Z}$, and consider $n = 4\ell^2 + 4\ell + 1 = 4(\ell^2 + \ell) + 1$. Since ℓ is an integer we $\ell^2 + \ell = m, m \in \mathbb{Z}$. By definition of divisibility by definition of $1 \bmod 4$, $4m + 1 \equiv 1(\operatorname{mod} 4)$ which implies that $n \equiv 1(\text{mod})4$

Hence in both cases and the contrapositive, we can conclude that if $n \equiv 2 \pmod 4$ or $n \equiv 3 \pmod 4$

(b) Prove that if $n^2 + 1$ is a perfect square then $n = 0$. Hint: Factor a difference of squares and think about divisors.

Proof: Assume that $n^2 + 1$ is a perfect square such that $n^2 + 1 = k^2$. We can say that $n^2 = k^2 - 1$.

$$\begin{aligned} k^2 - n^2 &= (k+n)(k-n) \\ 1 &= (k+n)(k-n) \end{aligned}$$

Now consider the following cases, since we know that the only divisors of 1 are 1 and -1, we can conclude that either both $(k + n)$ and $(k - n)$ are equal to 1 or they are equal -1. Consider the following case work.

- **Case 1:** Assume that $(k + n) = 1$ and $(k - n) = 1$.

$$\begin{aligned} k &= 1 - n \\ & \text{We rearrange } (k+1) = -1(1-n) - n = 1 \\ -2n &= 0n \qquad\qquad = 0 \end{aligned}$$

- **Case 2:** Assume that $(k + n) = -1$ and $(k - n) = -1$.

$$\begin{aligned} k &= -1 - n \\ & \text{We rearrange } (k+1) = -1(-1-n) - n = -1 \\ -2n &= 0n \qquad\qquad = 0 \end{aligned}$$

Hence, in either case that $(k + n)$ or $(k - n)$ are equal to 1 or -1 we can conclude that $n = 0$, thus if $n^2 + 1$ is a perfect square then $n = 0$.

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2. Recall Bézout's identity: Let $a, b \in \mathbb{Z}$ such that a and b are not both zero. Then there exists $x, y \in \mathbb{Z}$ such that $ax + by = \gcd(a, b)$. Use this result to prove the following result: Let $a, b, c \in \mathbb{Z}$ such that $\gcd(a, b) = 1$. Then $(a \mid bc) \implies (a \mid c)$.

   Proof: Assume that $(a \mid bc)$ such that $bc = ak$, for some $k \in \mathbb{Z}$. Now consider the fact that $\gcd(a, b) = 1$ and by Bézout's identity we know that $ax + by = 1$. Multiplying both sides by c we find $acx + bcy = c$ and using our assumption $a \mid bc$ we know $acx + aky = c$. Factoring a out of the result yields, $a(cx + by) = c$, and since we know that the result of $(cx + by) \in \mathbb{Z}$ we can conclude that $a\ell = c$. Hence $a \mid c$.

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3. Let $P \subseteq \mathbb{N}$ be the set of prime numbers $P = \{2, 3, 5, 7, 11, ...\}$. Determine whether the following statements are true or false. Prove your answers ("true" or "false" is not sufficient).

   1. $\forall x \in P, \forall y \in P, x + y \in P$. False we prove negation.
      - Choose $x = 3, y = 5, x + y = 8$, and $8 \notin P$.
   2. $\forall x \in P, \exists y \in P$ such that $x + y \notin P$. This statement is true.

      Proof: Assume $x \in P$ and choose $y = 23$. Consider the following case work.

      - **Case 1:** Assume that x is even, such that $x = 2$, since 2 is the only even number in P. $23 + 2 = 25$, thus if x is even then $x + y \notin P$.
      - **Case 2:** Assume that x is odd, such that $x = 2k + 1$. We can write $x + y$ as $(2k + 1) + 2(10) + 3 = 2k + 2(10) + 4$. Factoring 2 out of our expression yields $2(\ell)$, since we know that $x + y > 2$, since $x + y$ is even and and greater than two, we can conclude that $(x + y) \notin P$.

      Hence by both cases we can conclude that $\forall x \in P, \exists y \in P$ such that $x + y \notin P$.

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   3. $\exists x \in P$ such that, $\forall y \in P, x + y \in P$. This statement is false, we can prove the negations.

      Proof: $\forall x \in P, \exists y \in P$ such that $x + y \notin P$. Assume $x \in P$ and choose $y = 7$. Consider the following case work.

      - **Case 1:** Assume that x is even, such that $x = 2$, since 2 is the only even number in P. $7 + 2 = 9$, thus if x is even then $x + y \notin P$.
      - **Case 2:** Assume that x is odd, such that $x = 2k + 1$. We can write $x + y$ as $(2k + 1) + 2(4) + 1 = 2k + 2(10) + 2$. Factoring 2 out of our expression yields $2(\ell)$, since we know that $x + y > 2$, since $x + y$ is even and and greater than two, we can conclude that $(x + y) \notin P$.

Hence by proving both cases of the negation we can conclude that our original statement is false.

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4. Prove the following statement: For every positive number ε there is a positive number M such that

$$\left|\frac{2x^2}{x^2+1}-2\right| < \varepsilon$$

whenever $x \geq M$.

Proof: Assume that $\varepsilon > 0$, let $M = \sqrt{\frac{2}{\varepsilon}}$ and assume $x \geq M$. Consider the following.

$$\begin{aligned} x &\geq M \\ x &\geq \sqrt{\frac{2}{\varepsilon}} \\ x^2 &\geq \frac{2}{\varepsilon} \\ \varepsilon &\geq \frac{2}{x^2} \\ \varepsilon &> \frac{2}{x^2+1} \end{aligned}$$

Hence

$$\left|\frac{2x^2}{x^2+1}-2\right| = \left|\frac{2x^2-2x^2-2}{x^2+1}\right| = \frac{2}{x^2+1} < \varepsilon$$

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5. We say that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R}$ if $\lim_{x\to a} f(x) = f(a)$. Let

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), \text{if } x \neq 0 \\ 0, \text{if } x = 0 \end{cases}$$

Is f continuous at $x = 0$? (You may use the result: $\forall x \in \mathbb{R}, |\sin x| \leq 1$.)

Note: Make sure to use the definition of a limit to justify your answer, namely: Let $a, L \in \mathbb{R}$. The limit of a function f as x approaches a is L, when (for $\varepsilon, \delta, x \in \mathbb{R}$)

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } (0 < |x-a| < \delta) \implies (|f(x) - L| < \varepsilon).$$

Hint: You can make use of the fact that $|f(x)| \leq x^2$.

Proof: Assume that $\varepsilon > 0$, let $\delta = \sqrt{\varepsilon}$ and assume that $|0 < x - 0 < \delta|, x \in \mathbb{R}$. Consider the following.

$$0 < x < \sqrt{\varepsilon}$$
$$0 < x^2 < \varepsilon$$

Hence

$$|f(x)| \leq x^2 < \varepsilon \text{ and } |f(x) - 0| < \varepsilon$$

Thus, we can conclude that since $\lim_{x\to 0} f(x) = 0$ and $f(0) = 0$ it must be true that $f(x)$ is continuous.

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6. We say that a sequence (x_n) converges to L if

$$\forall \varepsilon > 0, \exists N \in \mathbb{N}, \forall n \in \mathbb{N}, (n > N) \implies |x_n - L| < \varepsilon.$$

Using the definition, prove that the sequence (x_n) with

$$x_n = (-1)^n + \frac{1}{n}$$

does not converge to any $L \in \mathbb{R}$.

Proof: Consider the definition of non-convergence. $\exists \varepsilon > 0, \forall N \in \mathbb{N}, \exists n \in \mathbb{N}, (n > N) \wedge |x_n - L| \geq \varepsilon$. Choose $\varepsilon = \frac{1}{2}$ and consider the following case work.

- **Case 1:** Assume $L \geq 0$

  Choose $n > N \geq 2$ such that n is odd.

  Then:

  $$x_n = (-1)^n + \frac{1}{n} = -1 + \frac{1}{n} < -1 + \frac{1}{2} = -\frac{1}{2}$$

  Since $L \geq 0$ and $x_n < -\frac{1}{2}$:

  $$|x_n - L| = L - x_n > 0 - \left(-\frac{1}{2}\right) = \frac{1}{2}$$

  Therefore $|x_n - L| \geq \frac{1}{2}$.

- **Case 2:** Assume $L < 0$

  Choose $n > N \geq 2$ such that n is even.

  Then:

$$x_n = (-1)^n + \frac{1}{n} = 1 + \frac{1}{n} > 1 > \frac{1}{2}$$

Since $L < 0$ and $x_n > \frac{1}{2}$:

$$|x_n - L| = x_n - L > \frac{1}{2} - L > \frac{1}{2} - 0 = \frac{1}{2}$$

Therefore $|x_n - L| \geq \frac{1}{2}$.

Hence, for every $L \in \mathbb{R}$ and every $N \in \mathbb{N}$, we found $n > N$ such that $|x_n - L| \geq \frac{1}{2}$. Therefore, the sequence (x_n) does not converge to any $L \in \mathbb{R}$.

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