Solutions to homework 5:

1. **Solutions**

- (a) $u_1 = \frac{4}{3}, u_2 = \frac{10}{7}, u_3 = \frac{24}{17}$
- (b) *Proof*

  Proof by Induction:

  Base Case: $n = 0$, $u_0 = 2$ so $1 \leq u_0 \leq 2$. Base case holds.

  I.H. for $k \geq 0$, $1 \leq u_k \leq 2$

  I.S.

  - Lower Bound

$$\begin{aligned} u_k &\geq 1 && \text{By I.H.} \\ u_k + 2 &\geq u_k + 1 \\ \frac{u_k + 2}{u_k + 1} &\geq 1 && \text{By I.H. } u_k + 1 > 0 \\ u_{k+1} &\geq 1 \end{aligned}$$

  - Upper Bound

$$\begin{aligned} u_k &\leq 2 && \text{By I.H.} \\ u_k + 2 &\leq 2u_k + 2 \\ \frac{u_k + 2}{u_k + 1} &\leq 2 && \text{By I.H. } u_k + 1 > 0 \\ u_{k+1} &\leq 2 \end{aligned}$$

  Hence, by induction $1 \leq u_k \leq 2$.

2. **Solution**

- **Proof by Induction:**
  - **Base Case:** $n = 1$, $1\frac{1+1}{2} = 1^3$, $1 = 1$. Base case holds.
  - **I.H.**

$$\sum_{k=1}^{\ell} k^3 = \left(\frac{\ell(\ell+1)}{2}\right)^2$$

  - **I.S.**

$$\begin{aligned}
\sum_{k=1}^{\ell} k^3 &= \left(\frac{\ell(\ell+1)}{2}\right)^2 \\
\sum_{k=1}^{\ell} k^3 + (\ell+1)^3 &= \left(\frac{\ell(\ell+1)}{2}\right)^2 + (\ell+1)^3 \\
\sum_{k=1}^{\ell+1} k^3 &= \frac{\ell^2(\ell+1)^2 + 4(\ell+1)^3}{4} \\
&= \frac{(\ell+1)^2(\ell^2+4\ell+4)}{4} \\
&= \frac{(\ell+1)^2(\ell+2)^2}{4} \\
&= \left(\frac{(\ell+1)(\ell+2)}{2}\right)^2
\end{aligned}$$

Hence by induction $\forall n \in \mathbb{N}, \sum_{k=1}^{\ell} k^3 = \left(\frac{\ell(\ell+1)}{2}\right)^2$.

3. - **Proof by Induction:**
     - **Base Case:** $n = 1$, $\frac{1}{1} \leq 2 - 1$, $1 \leq 1$. Base case holds.
     - **I.H.**

$$\sum_{i=1}^{k} \frac{1}{i^2} \leq 2 - \frac{1}{k}$$

     - **I.S.**

$$\sum_{i=1}^{k} \frac{1}{i^2} = 2 - \frac{1}{k}$$

$$\sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} \leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

$$\sum_{i=1}^{k+1} \frac{1}{i^2} \leq 2 - \frac{1}{k} + \frac{1}{k+1}$$

$$\sum_{i=1}^{k+1} \frac{1}{i^2} < 2 - \frac{1}{k} + \frac{2}{k+1}$$

$$< 2 + \frac{-(k+1) + 2k}{k(k+1)}$$

$$< 2 + \frac{k}{k(k+1)} - \frac{1}{k(k+1)}$$

$$< 2 + \frac{k}{k(k+1)}$$

Hence by induction $\forall n \in \mathbb{N}, \sum_{i=1}^{n} \frac{1}{i^2} \leq 1 - \frac{1}{n}$.