Solutions to homework 5:

1. **Solutions**

   - (a) $u_1 = \frac{4}{3}, u_2 = \frac{10}{7}, u_3 = \frac{24}{17}$

   - (b) *Proof*

     Proof by Induction:

     Base Case: $n = 0$, $u_0 = 2$ so $1 \leq u_0 \leq 2$. Base case holds.

     I.H. for $k \geq 0$, $1 \leq u_k \leq 2$

     I.S.

     ▸ Lower Bound

     $$u_k \geq 1 \qquad\qquad \text{By I.H.}$$
     $$u_k + 2 \geq u_k + 1$$
     $$\frac{u_k + 2}{u_k + 1} \geq 1 \qquad \text{By I.H. } u_k + 1 > 0$$
     $$u_{k+1} \geq 1$$

     ▸ Upper Bound

     $$u_k \leq 2 \qquad\qquad \text{By I.H.}$$
     $$u_k + 2 \leq 2u_k + 2$$
     $$\frac{u_k + 2}{u_k + 1} \leq 2 \qquad \text{By I.H. } u_k + 1 > 0$$
     $$u_{k+1} \leq 2$$

     Hence, by induction $1 \leq u_k \leq 2$.

2. **Solution**

   - **Proof by Induction:**

     ▸ **Base Case:** $n = 1$, $1\frac{1+1}{2} = 1^3$, $1 = 1$. Base case holds.

     ▸ **I.H.**

     $$\sum_{k=1}^{\ell} k^3 = \left(\frac{\ell(\ell + 1)}{2} \right)^2$$

     ▸ **I.S.**

$$\sum_{k=1}^{\ell} k^3 = \left(\frac{\ell(\ell+1)}{2}\right)^2$$

$$\sum_{k=1}^{\ell} k^3 + (\ell+1)^3 = \left(\frac{\ell(\ell+1)}{2}\right)^2 + (\ell+1)^3$$

$$\sum_{k=1}^{\ell+1} k^3 = \frac{\ell^2(\ell+1)^2 + 4(\ell+1)^3}{4}$$

$$= \frac{(\ell+1)^2(\ell^2 + 4\ell + 4)}{4}$$

$$= \frac{(\ell+1)^2(\ell+2)^2}{4}$$

$$= \left(\frac{(\ell+1)(\ell+2)}{2}\right)^2$$

Hence by induction $\forall n \in \mathbb{N}, \sum_{k=1}^{\ell} k^3 = \left(\frac{\ell(\ell+1)}{2}\right)^2$.

3. • **Proof by Induction:**

   ‣ **Base Case:** $n = 1$, $\frac{1}{1} \leq 2 - 1$, $1 \leq 1$. Base case holds.

   ‣ **I.H.**

   $$\sum_{i=1}^{k} \frac{1}{i^2} < 2 - \frac{1}{k}$$

   ‣ **I.S.**

$$\sum_{i=1}^{k} \frac{1}{i^2} < 2 - \frac{1}{k}$$

$$\sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

$$\sum_{i=1}^{k+1} \frac{1}{i^2} < 2 - \frac{1}{k} + \frac{1}{k+1}$$

$$\sum_{i=1}^{k+1} \frac{1}{i^2} < 2 - \frac{1}{k} + \frac{2}{k+1}$$

$$< 2 + \frac{-(k+1)+2k}{k(k+1)}$$

$$< 2 + \frac{k}{k(k+1)} - \frac{1}{k(k+1)}$$

$$< 2 + \frac{k}{k(k+1)}$$

$$S_{k+1} < 2 + \frac{1}{k+1}$$

Hence by induction $\forall n \in \mathbb{N}, \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$

4. • Solutions

(a) $u_{n+1} - 5^2 \cdot u_n = -17 \cdot 2^{3n+1}$

(b) *Proof*

Proof by Induction

Base Case $n = 0, 3 \cdot 5^{0+1} + 2^{0+1} = 17$ and $17 (\text{mod } 17) = 0$. Base case holds.

I.H. $\forall k \in \mathbb{Z}, k \geq 0, u_k = 3 \cdot 5^{2k+1} + 2^{3k+1}$ s.t. $17 \mid u_k$.

Lemma: $\forall k \in \mathbb{N}, u_{n+1} = -17 \cdot 2^{3n+1} + 5^2 \cdot u_n$

I.S.

$$
\begin{aligned}
u_{k+1} &= -17 \cdot 2^{3n+1} + 5^2 \cdot u_k && \text{By Lemma} \\
&= -17 \cdot 2^{3n+1} + 5^2 \cdot 17i && \text{By I.H. and } i \in \mathbb{Z} \\
&= 17(-2^{3n+1}) + 17(5^2 \cdot i) \\
&= 17(-2^{3n+1} + 5^2 \cdot i) \\
&= 17\ell && \ell \in \mathbb{Z}
\end{aligned}
$$

Hence by induction, $17 \mid u_n$.

5. **Proof by Induction**

   Base Case $n = 1, 2^{1^2} > 1!$ so $2 > 1$. Base case holds.

   I.H. $\forall k \in \mathbb{N}, 2^{k^2} > k!$.

   - **Sub Induction:** $\forall t \in \mathbb{N}, 2^{2t+1} > (t+1)$

     Base Case: $t = 1, 2^{2(1)+1} > 2$. Base case holds.

     I.H. $\forall s \in \mathbb{N}, 2^{2s+1} > (s+1)$

     I.S.

     $$2^{2s+1} > (s+1)$$
     $$2 \cdot 2^{2s+1} > 2(s+1)$$
     $$2^{2s+2} > 2s + 2$$
     $$2^{2s+2} > s + 2$$

   Thus if $P(s)$ then $P(s+1)$.

   Hence by induction, $\forall t \in \mathbb{N}, 2^{2t+1} > t+1$

   Lemma: $\forall k \in \mathbb{N}, 2^{2k+1} > k+1$

   I.S.

   $$2^{k^2} > k!$$
   $$2^{k^2} \cdot (k+1) > k! \cdot (k+1) \qquad\qquad k \geq 1$$
   $$2^{k^2} \cdot 2^{2k+1} > 2^{k^2} \cdot (k+1) > (k+1)! \quad \text{By Lemma}$$
   $$2^{k^2+2k+1} > (k+1)!$$
   $$2^{(k+1)^2} > (k+1)!$$

   Thus if $P(k)$ then $P(k+1)$

   Hence by induction, $\forall n \in \mathbb{N}, 2^{n^2} > n!$