Solutions to homework 1:

1. (a) The sentence, "2 is even and 11 is prime" is a true statement and can be translated to "$P \wedge Q$", where P: "2 is even", and Q : "11 is prime".

   (b) The sentence, "if n is a multiple of 7 and 4, then it is a multiple of 14" is a false statement and can be translated to "$(P(n) \wedge Q(n)) \Rightarrow R(n)$", where $P(n)$: "n is a multiple of 7", $Q(n)$: "n is a multiple of 4", and $R(n)$: "n is a multiple of 14".

   (c) The sentence if "$5 \leq x \leq 17$" is an open statement, thus it has no truth value. It can be translated to $P(x)$, where $P(x)$: "$5 \leq x \leq 17$".

   (d) The sentence, "a real number x is less than -3 or greater than 3 if its square is greater than or equal to 9" is a true statement and can be translated to "$P(n) \Rightarrow (Q(n) \vee R(n)$", where $P(n)$: "the square of x is greater than 9", $Q(n)$: "x is less than -3", and $R(n)$: "x is greater than 3".

2. Let $a \in \mathbb{Z}$, if $5a + 11$ is odd then $9a + 3$ is odd.

   Proof: Assume that $5a + 11$ is odd, which implies that $5a + 11 = 2k + 1$, for some integer $k \in \mathbb{Z}$.

$$\begin{aligned} 5a + 11 &= 2k + 1 \\ a + 2(2a + 5) &= 2k \\ a + 2i &= 2k, \qquad i \in \mathbb{Z} \\ a &= 2(k - i) \\ a &= 2j, \qquad j \in \mathbb{Z} \end{aligned}$$

   Thus, it follows that a is an even integer. Now, consider the expression $9a + 3$. It follows that, $9a + 3 = 9(2k) + 3 = 2(9k + 1) + 1$. Knowing that $9k + 1$ is an integer, we can conclude that $9a + 3 = 2\ell + 1, \ell \in \mathbb{Z}$. Hence, $9a + 3$ is an odd integer.

   $\square$

3. Let $n \in \mathbb{R}$. If $3 \mid n - 2$ then $3 \mid n^2 + 2n + 8$.

   Proof: Assume that $3 \mid n - 2$ such that $n - 2 = 3\ell$ and $\ell \in \mathbb{Z}$. Now consider the expression $n^2 + 2n + 8$.

$$\begin{aligned} n^2 + 2n + 8 &= n^2 + 2n - 8 + 16 \\ n^2 + 2n - 8 + 16 &= (n - 2)(n + 4) + 16 \\ (n - 2)(n + 4) + 16 &= (3\ell)(n + 4) + 15 + 1 \\ (3\ell)(n + 4) + 16 &= 3(\ell(n + 4) + 5) + 1 \\ 3(\ell(n + 4) + 5) + 1 &= 3t \qquad\qquad t \in \mathbb{Z} \end{aligned}$$

Thus, we can see that the expression $n^2 + 2n + 8$ is of the form $3t$ under the assumption that $3 \mid n - 2$. Hence, $3 \mid n^2 + 2n + 8$.

4. Let $x, y \in \mathbb{R}$. Show that $xy \leq \frac{1}{2}(x^2 + y^2)$.

   Proof: Consider the fact that $q^2 \geq 0, q \in \mathbb{R}$. Now consider the expression $(x - y)^2$ such that $x, y \in \mathbb{R}$. Meaning $(x - y)^2$ is of the form q^2, thus $(x - y)^2 \geq 0$. Expanding the expression, we find $x^2 - 2xy + y^2 \geq 0$ and after adding $2xy$ and divding both sides by 2 we find the expression $\frac{1}{2}(x^2 + y^2) \geq xy$. Hence, $xy \leq \frac{1}{2}(x^2 + y^2)$.

5. Let $n, a, b, c, d \in \mathbb{Z}$, with n > 0. If $n|a$ and $n|c$, then $n|(ab + cd + ac)$.

   Proof: Assume $n|a$ and $n|c$ and consider the expression $(ab + cd + ac)$ such that $n, a, b, c, d \in \mathbb{Z}$. It follows that $a = nk$ and $c = n\ell$ for some integers $k, \ell \in \mathbb{Z}$. Consider the following.

$$
\begin{aligned}
(ab + cd + ac) &= ((nk)b + (n\ell)d + (nk)(n\ell)) \\
((nk)b + (n\ell)d + (nk)(n\ell)) &= n(kb + \ell d + k\ell) \\
n(kb + \ell d + k\ell) &= nq, & q \in \mathbb{Z}
\end{aligned}
$$

   Thus, we can see that $(ab + cd + ac)$ under the assumption that $n|a$ and $n|c$ is of the form nq for some $q \in \mathbb{Z}$. Hence, $n|(ab + cd + ac)$.

6. Let $a \in \mathbb{Z}$. If $3a + 1$ is odd then $5a + 2$ is even.

   Answer: No, our friend's solution is incorrect under a false conclusion they made regarding the expression $2\left(\frac{5k}{3} + 1\right)$. In their solution they are able to conclude that $5a + 2$ is even under the assumption that $3a + 1$ is odd. However in their process, they made a critical mistake by assuming that $\frac{5k}{3} + 1$ is an integer. However, they are not able to conclude that fact based on the current assumptions we know about k. This solution only works if it is certain that $3|k$, however that fact was never concluded.

   Proof: Assume that $3a + 1$ is odd such that $3a + 1 = 2\ell + 1$ and $\ell \in \mathbb{Z}$. Consider the expression $5a + 2$.

$$
\begin{aligned}
5a + 2 &= 2a + (3a + 1) + 1 \\
2a + (3a + 1) + 1 &= 2a + (2\ell + 1) + 1 \\
2a + (2\ell + 1) + 1 &= 2(a + \ell + 1) \\
2(a + \ell + 1) &= 2t & t \in \mathbb{Z}
\end{aligned}
$$

   Thus, we can see that the expression $5a + 2$ is of the form $2t$ under the assumption that $3a + 1$ is odd. Hence, $5a + 2$ is even.

7. Solution for question 7.

Proof: Let $a > 0$ and $b > 0$ be funky numbers such that a^ℓ and $b^k \in \mathbb{Z}$, for some $\ell, k \in \mathbb{N}$. Consider the expression $\left(\sqrt{ab}\right)^t$ and let $t = 2k\ell$ such that $t \in \mathbb{N}$.

$$\begin{aligned}
\left(\sqrt{ab}\right)^t &= \left(\sqrt{ab}\right)^{2k\ell} \\
\left(\sqrt{ab}\right)^{2k\ell} &= (ab)^{k\ell} \\
(ab)^{k\ell} &= a^{k\ell}b^{k\ell} \\
a^{k\ell}b^{k\ell} &= \left(a^\ell\right)^k\left(b^k\right)^\ell \\
\left(a^\ell\right)^k\left(b^k\right)^\ell &= iq \qquad\qquad i, q \in \mathbb{Z}
\end{aligned}$$

It follows that $\left(\sqrt{ab}\right)^t = iq$ such that $i, q \in \mathbb{Z}$. This follows since the product of an integer is an integer. Thus the expression $\left(\sqrt{ab}\right)^t$ with $t = 2k\ell$ is of the form iq under the assumption that a^ℓ and b^k are integers. Hence, $\sqrt{ab}$ is a funky number.