

## Solutions to homework 5:

### 1. Solutions

- (a)  $u_1 = \frac{4}{3}, u_2 = \frac{10}{7}, u_3 = \frac{24}{17}$
- (b) *Proof*

#### Proof by Induction:

**Base Case:**  $n = 0, u_0 = 2$  so  $1 \leq u_0 \leq 2$ . Base case holds.

**I.H.** for  $k \geq 0, 1 \leq u_k \leq 2$

#### I.S.

##### ► Lower Bound

$$u_k \geq 1 \quad \text{By I.H.}$$

$$u_k + 2 \geq u_k + 1$$

$$\frac{u_k + 2}{u_k + 1} \geq 1 \quad \text{By I.H. } u_k + 1 > 0$$

$$u_{k+1} \geq 1$$

##### ► Upper Bound

$$u_k \leq 2 \quad \text{By I.H.}$$

$$u_k + 2 \leq 2u_k + 2$$

$$\frac{u_k + 2}{u_k + 1} \leq 2 \quad \text{By I.H. } u_k + 1 > 0$$

$$u_{k+1} \leq 2$$

Hence, by induction  $1 \leq u_k \leq 2$ .

### 2. Solution

#### • Proof by Induction:

► **Base Case:**  $n = 1, 1^{\frac{1+1}{2}} = 1^3, 1 = 1$ . Base case holds.

► **I.H.**

$$\sum_{k=1}^{\ell} k^3 = \left( \frac{\ell(\ell+1)}{2} \right)^2$$

► **I.S.**

$$\begin{aligned}
\sum_{k=1}^{\ell} k^3 &= \left( \frac{\ell(\ell+1)}{2} \right)^2 \\
\sum_{k=1}^{\ell} k^3 + (\ell+1)^3 &= \left( \frac{\ell(\ell+1)}{2} \right)^2 + (\ell+1)^3 \\
\sum_{k=1}^{\ell+1} k^3 &= \frac{\ell^2(\ell+1)^2 + 4(\ell+1)^3}{4} \\
&= \frac{(\ell+1)^2(\ell^2 + 4\ell + 4)}{4} \\
&= \frac{(\ell+1)^2(\ell+2)^2}{4} \\
&= \left( \frac{(\ell+1)(\ell+2)}{2} \right)^2
\end{aligned}$$

Hence by induction  $\forall n \in \mathbb{N}, \sum_{k=1}^{\ell} k^3 = \left( \frac{\ell(\ell+1)}{2} \right)^2$ .

### 3. • **Proof by Induction:**

► **Base Case:**  $n = 1, \frac{1}{1} \leq 2 - 1, 1 \leq 1$ . Base case holds.

► **I.H.**

$$\sum_{i=1}^k \frac{1}{i^2} \leq 2 - \frac{1}{n}$$

► **I.S.**

$$\sum_{k=1}^{\ell} k^3 \leq 2 - \frac{1}{n}$$

Hence by induction  $\forall n \in \mathbb{N}, \sum_{k=1}^{\ell} k^3 = \left( \frac{\ell(\ell+1)}{2} \right)^2$ .