Solutions to homework 3:

1. (a) $\exists a \in \mathbb{Z}, ((6 \mid a \text{ and } 8 \mid a) \text{ and } 48 \nmid a)$. The original statement is false, we prove by showing that the negation is false. Choose $a = 24$, which is divisible by 8 and 6. Thus the original statement is false.

   (b) $\forall x \in \mathbb{Z}$ such that $(x \leq 84)$ or $x \not\equiv 75 \bmod 84$. The original statement is true, we can choose $x = 159$ which satisfies the condition $x > 84$ and by arithmetic we can see $159 = 84 + 75$ meaning $159 \equiv 75 \bmod 84$.

   (c) $\forall x, y \in \mathbb{R}$ that $\left(x^2 < y^2 \vee x > y\right)$. The original statement is true, we can choose $x = -1$ and $y = 1$ such that $-1 \leq 1$ and $(-1)^2 \geq (1)^2$.

2. For $a \in \mathbb{R}$, we define the set $S_a = \{x \in \mathbb{R} : (x \geq 0) \wedge (x < a - 2)\}$. Show that $S_a = \emptyset$ if and only if $a \in (-\infty, 2]$.

   Proof:

   - **If $a \in (-\infty, 2]$ then $S_a = \emptyset$:** Assume that $a \in (-\infty, 2]$. We can say $a \leq 2$ so $a - 2 \leq 0$, $S_a = \{x \in \mathbb{R} : (x \geq 0) \wedge (x < a - 2 \leq 0)\}$. Since $\sim (x \geq 0) = (x < 0)$, we can see that no $x \in \mathbb{R}$ can satisfy the the conditions of S_2 since the conditions are a contradiction. Thus, $S_2 = \emptyset$.
   - **If $S_a = \emptyset$ then $a \in (-\infty, 2]$:** Proof by Contrapositive. Assume that $a \notin (-\infty, 2]$. So we can say that, $a > 2$ and $a - 2 > 0$. Since $a - 2 > 0$ we can say that for S_a we can choose $x = \frac{a-2}{2}$ since it satisfies the conditions $x \geq 0$ and $x < a - 2$. Thus $S_a \neq \emptyset$.

   Thus by both directions we can conclude that the set $S_a = \emptyset$ if and only if $a \in (-\infty, 2]$. ■

3. **Let $A = \{n \in \mathbb{N} : 5 \mid n \text{ or } 6 \mid n\}$**

   (a) $\exists x \in A$ **such that** $\exists y \in A$ **such that** $x + y \in A$. This statement is true.

   - **Proof:** Choose $x = 5$ and $y = 10$ such that $x + y = 15$ and $15 \equiv 0 \bmod 5$, thus $x + y \in A$.

   ■

   (b) $\forall x \in A, \forall y \in A, x + y \in A$. This statement is false. We can show by proving the negation $\exists x \in A, \exists y \in A$ **such that** $x + y \notin A$.

   - **Proof:** Choose $x = 6$ and $y = 5$ such that $x + y = 11$ and $11 \equiv 1 \bmod 5$ and $11 \equiv 5 \bmod 6$ thus $11 \notin A$.

   ■

   (c) $\exists x \in A, \forall y \in A, x + y \in A.$ This statement is true.
   - **Proof:** Assume $x, y \in A$. Choose $x = 30$.
     - **Case 1:** Assume that $y = 5k$, for some $k \in \mathbb{Z}$. Consider $x + y = 5k + 30$ and $x + y = 5(k + 6)$, since $(k + 6) \in \mathbb{Z}$ then

$x + y = 5n$, for some $n \in \mathbb{Z}$, meaning that $x + y \in A$.

- **Case 2:** Assume that $y = 6k$, for some $k \in \mathbb{Z}$. Consider $x + y = 5k + 30$ and $x + y = 6(k + 5)$, since $(k + 5) \in \mathbb{Z}$ then

$x + y = 6n$, for some $n \in \mathbb{Z}$, meaning that $x + y \in A$.

Thus, there exists $x \in A$ such that $\forall y \in A$, $x + y \in A$.

■

4. **Solutions**

(a) **$\forall n \in \mathbb{Z}, \exists y \in \mathbb{R}$ such that $y^n < y$.**

- **Answer:** This statement is false we can show this by proving the negation of the statement is $\exists n \in \mathbb{Z}, \forall y \in \mathbb{R}$ such that $y^n \geq y$.
  - **Proof:** Assume that $n \in \mathbb{Z}$ and $y \in \mathbb{R}$.
    - **Case 1:** Let $-1 < y < 1$ we can choose $n = 0$ which satisfies $y^0 \geq y$ which evaluates to $1 \geq y$. Hence, $y^n \geq y$.
    - **Case 2:** Let $y \leq -1$ we can choose $n = 2$, which satisfies $y^2 \geq y$, hence $y^n \geq y$.
    - **Case 3:** Let $y \geq 1$ we can choose $n = 2$, which satisfies $y^2 \geq y$, hence $y^n \geq y$.

    Thus in all cases $\exists n \in \mathbb{Z}, \forall y \in \mathbb{R}$ such that $y^n \geq y$.

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(b) **$\exists y \in \mathbb{R}$ such that $\forall n \in \mathbb{Z}$ with $n > 1$, we have $y^n < y$.**

- **Answer:** This statement is true. The negation of this statement is $\forall y \in \mathbb{R}, \exists n \in \mathbb{Z}$ with $n > 1$, we have $y^n \leq y$.
  - **Proof:** Choose $y = \frac{1}{2}$ such that $y > 0$, it remains true that $y^n > 0$. Now consider the fact that $y^n = \left(2^{-1}\right)^n$ so that $y = \frac{1}{2^n}$, using this fact we know that $\frac{1}{2^n} \leq \frac{1}{2}$ for any $n \in \mathbb{N}$. Hence $y^n \leq y$.

■

5. **$\forall a \in \mathbb{Z}, \exists b \in \mathbb{Z}$ such that $a^2 + b^2 \equiv 1 \bmod 3$.**

Proof: Assume $a, b \in \mathbb{Z}$.

- **Case 1:** Assume that $a \equiv 0 \bmod 3$ such that $a = 3k$, for some $k \in \mathbb{Z}$ and let $b = 1$. Consider $a^2 + b^2 = 9k^2 + 1$ and $a^2 + b^2 = 3(3k^2) + 1$ and since we know that $3k^2$ is an integer then $a^2 + b^2 \equiv 1 \bmod 3$.
- **Case 2:** Assume that $a \equiv 1 \bmod 3$ such that $a = 3k + 1$, for some $k \in \mathbb{Z}$ and let $b = 0$. Consider $a^2 + b^2 = 9k^2 + 6k + 1$ and $a^2 + b^2 = 3(3k^2 + 2k) + 1$, since $(3k^2 + 1)$ is an integer we know that $a^2 + b^2 \equiv 1 \bmod 3$.

- **Case 3:** Assume that $a \equiv 2 \bmod 3$ such that $a = 3k + 2$, for some $k \in \mathbb{Z}$ and let $b = 0$. Consider $a^2 + b^2 = 9k^2 + 12k + 4$ and $a^2 + b^2 = 3(3k^2 + 1) + 1$, since $(3k^2 + 1)$ is an integer we know that $a^2 + b^2 \equiv 1 \bmod 3$.

Hence, by every case $\forall a \in \mathbb{Z}, \exists b \in \mathbb{Z}$ such that $a^2 + b^2 \equiv 1 \bmod 3$.

■

6. **For every positive number ε there is a positive number M for which**

$$\left|1 - \frac{x^2}{x^2+1}\right| < \varepsilon,$$

whenever $x \geq M$.

Negation: There exists a positive number ε and for all positive numbers M there exists x for which

$$\left|1 - \frac{x^2}{x^2+1}\right| \geq \varepsilon,$$

and $x \geq M$.

7. (a) **Every bounded function $f : \mathbb{R} \to \mathbb{R}$ is also linearly bounded. This statement is false.**

Proof: Consider the constant function $f(x) = 1$ for all $x \in \mathbb{R}$.

First, we verify that f is bounded. We have $|f(x)| = |1| = 1 \leq 1$ for all $x \in \mathbb{R}$. Taking $n = 1$, we see that f satisfies Definition 1, so f is bounded.

Now we show that f is not linearly bounded. Suppose, for the sake of contradiction, that f is linearly bounded. Then by Definition 2, there exists $j \in \mathbb{N}$ such that $|f(x)| \leq |jx|$ for all $x \in \mathbb{R}$.

Consider $x = 0$. Then we have:

$$|f(0)| \leq |j \cdot 0|$$

$$1 \leq 0$$

This is a contradiction. Hence not every bounded function $f : \mathbb{R} \to \mathbb{R}$ is also linearly bounded.

(b) **Every linearly bounded function $f : \mathbb{R} \to \mathbb{R}$ is also bounded. This statement is false.**

Proof: Consider the function $f(x) = x$ for all $x \in \mathbb{R}$.

First, we verify that f is linearly bounded. Taking $j = 1$, we have:

$$|f(x)| = |x| = |1 \cdot x| = |jx|$$

So $|f(x)| \leq |jx|$ for all $x \in \mathbb{R}$, which means f satisfies Definition 2. Therefore, f is linearly bounded.

Now we show that f is not bounded. Suppose, for the sake of contradiction, that f is bounded. Then by Definition 1, there exists $n \in \mathbb{N}$ such that $|f(x)| \leq n$ for all $x \in \mathbb{R}$.

Consider $x = n + 1$. Then:

$$|f(n+1)| = |n+1| = n+1 > n$$

This contradicts the assumption that $|f(x)| \leq n$ for all $x \in \mathbb{R}$.

Therefore, $f(x) = x$ is linearly bounded but not bounded, so the statement is false. ■

(c) **A function $f : \mathbb{R} \to \mathbb{R}$ is bounded if and only if f is strictly bounded. This statement is false.**

Proof: We will show that the “only if” direction is false (i.e., bounded does not imply strictly bounded).

Consider the constant function $f(x) = k$ for some fixed $k \in \mathbb{N}$.

First, we verify that f is bounded. We have $|f(x)| = k \leq k$ for all $x \in \mathbb{R}$. Taking $n = k$ in Definition 1, we see that f is bounded.

Now we show that f is not strictly bounded. By Definition 3, for f to be strictly bounded, there must exist $k \in \mathbb{N}$ such that $|f(x)| < k$ for all $x \in \mathbb{R}$.

However, we have $|f(x)| = k$, and $k < k$ is false.

Therefore, $f(x) = k$ is bounded but not strictly bounded, so the “if and only if” statement is false. ■