Solutions to homework 5:

1. **Solutions**
   - (a) $u_1 = \frac{4}{3}, u_2 = \frac{10}{7}, u_3 = \frac{24}{17}$
   - (b) *Proof*

   Proof by Induction:

   Base Case: $n = 0$, $u_0 = 2$ so $1 \leq u_0 \leq 2$. Base case holds.

   I.H. for $k \geq 0$, $1 \leq u_k \leq 2$

   I.S.

   - Lower Bound

   $$\begin{aligned} u_k &\geq 1 && \text{By I.H.} \\ u_k + 2 &\geq u_k + 1 \\ \frac{u_k + 2}{u_k + 1} &\geq 1 && \text{By I.H. } u_k + 1 > 0 \\ u_{k+1} &\geq 1 \end{aligned}$$

   - Upper Bound

   $$\begin{aligned} u_k &\leq 2 && \text{By I.H.} \\ u_k + 2 &\leq 2u_k + 2 \\ \frac{u_k + 2}{u_k + 1} &\leq 2 && \text{By I.H. } u_k + 1 > 0 \\ u_{k+1} &\leq 2 \end{aligned}$$

   Hence, by induction $1 \leq u_k \leq 2$.

2. **Solution**
   - **Proof by Induction:**
     - **Base Case:** $n = 1$, $1\frac{1+1}{2} = 1^3$, $1 = 1$. Base case holds.
     - **I.H.**

     $$\sum_{k=1}^{\ell} k^3 = \left(\frac{\ell(\ell+1)}{2}\right)^2$$

     - **I.S.**

$$\begin{aligned}
\sum_{k=1}^{\ell} k^3 &= \left(\frac{\ell(\ell+1)}{2}\right)^2 \\
\sum_{k=1}^{\ell} k^3 + (\ell+1)^3 &= \left(\frac{\ell(\ell+1)}{2}\right)^2 + (\ell+1)^3 \\
\sum_{k=1}^{\ell+1} k^3 &= \frac{\ell^2(\ell+1)^2 + 4(\ell+1)^3}{4} \\
&= \frac{(\ell+1)^2(\ell^2+4\ell+4)}{4} \\
&= \frac{(\ell+1)^2(\ell+2)^2}{4} \\
&= \left(\frac{(\ell+1)(\ell+2)}{2}\right)^2
\end{aligned}$$

Hence by induction $\forall n \in \mathbb{N}, \sum_{k=1}^{\ell} k^3 = \left(\frac{\ell(\ell+1)}{2}\right)^2$.

3. - **Proof by Induction:**
    - **Base Case:** $n = 1$, $\frac{1}{1} \leq 2 - 1$, $1 \leq 1$. Base case holds.
    - **I.H.**

$$\sum_{i=1}^{k} \frac{1}{i^2} < 2 - \frac{1}{k}$$

    - **I.S.**

$$\sum_{i=1}^{k} \frac{1}{i^2} < 2 - \frac{1}{k}$$

$$\sum_{i=1}^{k} \frac{1}{i^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$$

$$\sum_{i=1}^{k+1} \frac{1}{i^2} < 2 - \frac{1}{k} + \frac{1}{k+1}$$

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i^2} &< 2 - \frac{1}{k} + \frac{2}{k+1} \\
&< 2 + \frac{-(k+1) + 2k}{k(k+1)} \\
&< 2 + \frac{k}{k(k+1)} - \frac{1}{k(k+1)} \\
&< 2 + \frac{k}{k(k+1)} \\
S_{k+1} &< 2 + \frac{1}{k+1}
\end{aligned}$$

Hence by induction $\forall n \in \mathbb{N}, \sum_{i=1}^{n} \frac{1}{i^2} \leq 2 - \frac{1}{n}$

4. • Solutions

(a) $u_{n+1} - 5^2 \cdot u_n = -17 \cdot 2^{3n+1}$

(b) *Proof*

Proof by Induction

Base Case $n = 0, 3 \cdot 5^{0+1} + 2^{0+1} = 17$ and $17 (\operatorname{mod} 17) = 0$. Base case holds.

I.H. $\forall k \in \mathbb{Z}, k \geq 0, u_k = 3 \cdot 5^{2k+1} + 2^{3k+1}$ s.t. $17 \mid u_k$.

Lemma: $\forall n \in \mathbb{N}, u_{n+1} = -17 \cdot 2^{3n+1} + 5^2 \cdot u_n$

I.S.

$$\begin{aligned}
u_{k+1} &= -17 \cdot 2^{3n+1} + 5^2 \cdot u_k && \text{By Lemma} \\
&= -17 \cdot 2^{3n+1} + 5^2 \cdot 17i && \text{By I.H. and } i \in \mathbb{Z} \\
&= 17(-2^{3n+1}) + 17(5^2 \cdot i) \\
&= 17(-2^{3n+1} + 5^2 \cdot i) \\
&= 17\ell && \ell \in \mathbb{Z}
\end{aligned}$$

Hence by induction, $17 \mid u_n$.

5. **Proof by Induction**

Base Case $n = 1, 2^{1^2} > 1!$ so $2 > 1$. Base case holds.

I.H. $\forall k \in \mathbb{N}, 2^{k^2} > k!$.

- **Sub Induction:** $\forall t \in \mathbb{N}, 2^{2t+1} > (t+1)$

  Base Case: $t = 1, 2^{2(1)+1} > 2$. Base case holds.

  I.H. $\forall s \in \mathbb{N}, 2^{2s+1} > (s+1)$

  I.S.

$$\begin{aligned} 2^{2s+1} &> (s+1) \\ 2 \cdot 2^{2s+1} &> 2(s+1) \\ 2^{2s+2} &> 2s+2 \\ 2^{2s+2} &> s+2 \end{aligned}$$

  Thus if $P(s)$ then $P(s+1)$.

  Hence by induction, $\forall t \in \mathbb{N}, 2^{2t+1} > t+1$

Lemma: $\forall k \in \mathbb{N}, 2^{2k+1} > k+1$

I.S.

$$\begin{aligned} 2^{k^2} &> k! \\ 2^{k^2} \cdot (k+1) &> k! \cdot (k+1) && k \geq 1 \\ 2^{k^2} \cdot 2^{2k+1} &> 2^{k^2} \cdot (k+1) > (k+1)! && \text{By Lemma} \\ 2^{k^2+2k+1} &> (k+1)! \\ 2^{(k+1)^2} &> (k+1)! \end{aligned}$$

Thus if $P(k)$ then $P(k+1)$

Hence by induction, $\forall n \in \mathbb{N}, 2^{n^2} > n!$

6. - **Proof by Induction:**
     - **Base Case:** $n = 1, n = 2$ and $a_1 = 3, a_2 = 12$ so $2^1 < 3 < 4^1$ and $2^2 < 12 < 4^2$. Base case holds.
     - **I.H.**

$$\forall k \in \mathbb{N}, k \geq 1, 2^k < a_k < 4^k \text{ s.t. } a_k = 2a_{k-1} + a_{k-2} + a_{k-3}$$

Lemma $: \forall n \in \mathbb{N}, n \geq 3, a_{n+1} = 5a_{n-1} + 3a_{n-2} + 2a_{n-3}$

$$a_{n+1} = 2a_n + a_{n-1} + a_{n-2}$$
$$a_{n+1} = 4a_{n-1} + 2a_{n-2} + 2a_{n-3} + a_{n-1} + a_{n-2}$$
$$a_{n+1} = 5a_{n-1} + 3a_{n-2} + 2a_{n-3}$$

- **I.S.**

  Consider

$$2^k < a_k$$
$$2 \cdot 2^k < 2 \cdot a_k$$
$$2^{k+1} < 2 \cdot (2a_{n-1} + a_{n-2} + a_{n-3})$$
$$2^{k+1} < 4a_{n-1} + 2a_{n-2} + 2a_{n-3}$$
$$2^{k+1} < 4a_{n-1} + 2a_{n-2} + 2a_{n-3} < 5a_{n-1} + 3a_{n-2} + 2a_{n-3}$$
$$2^{k+1} < 5a_{n-1} + 3a_{n-2} + 2a_{n(n-3)} = a_{k+1} \qquad \text{By Lemma}$$
$$2^{k+1} < a_{k+1}$$

  And

$$a_k < 4^k$$
$$4 \cdot a_k < 4 \cdot 4^k$$
$$4 \cdot a_k < 4^{k+1}$$
$$4 \cdot (2a_{k-1} + a_{k-2} + a_{k-3}) < 4^{k+1}$$
$$8a_{k-1} + 4a_{k-2} + 4a_{k-3} < 4^{k+1}$$
$$5a_{k-1} + 3a_{k-2} + 2a_{k-2} < 8a_{k-1} + 4a_{k-2} + 4a_{k-3} < 4^{k+1}$$
$$5a_{k-1} + 3a_{k-2} + 2a_{k-2} = a_{k+1} < 4^{k+1} \qquad \text{By Lemma}$$
$$a_{k+1} < 4^{k+1}$$

Hence by induction $\forall n \in \mathbb{N}, n \geq 1, 2^n < a_n < 4^n$