Math 307

Lecture 14: Orthogonal Projections

2025-10-13

0.0.1 Orthgonal Projections

- Definition. The projection of a vector x onto vector u

$$\text{proj}_u(x) = \frac{\langle x, u\rangle}{\langle u, u\rangle} u$$

- Projections onto u is given by the matrix multiplication

$$\text{proj}_u(x) = P(x) \text{ where } P = \frac{1}{\|u\|^2} \cdot uu^T$$

And $P^2 = P, P^T = P$ and $\text{rank}(P) = 1$

0.0.2 Orthogonal Bases

- **Definition**. Let $U \subseteq \mathbb{R}^n$ be a subspace. A set of vectors is an orthgonal basis for U if it is a basis for U and the vectors are orthogonal such that they are different vectors. Orthonormality also applies to this.

- **Theorem**. If $\{u_1, ..., u_m\}$ is a basis for the subspace $U \subseteq \mathbb{R}^n$. The Gram-Schmidt algorithm constructs an orthogonal basis of U:

$$\begin{aligned}
v_1 &= u_1 \\
v_2 &= u_2 - \text{proj}_{v_1}(u_2) \\
v_3 &= u_3 - \text{proj}_{v_1}(u_2) - \text{proj}_{v_2}(u_3) \\
v_m &= u_m - \text{proj}_{v_1}(u_m) - \text{proj}_{v_2}(u_m) - ... - \text{proj}_{v_{m-1}}(u_m)
\end{aligned}$$

0.0.3 Projection onto a Subspace

- **Definition** Let $U \subseteq \mathbb{R}^n$ be a subspace and let $\{u_1, ..., u_m\}$ is an orthogonal basis of U. We say that the projection of a vector x onto U is

$$\text{proj}_U(x) = \frac{\langle x, u_1\rangle}{\langle u_1, u_1\rangle} \cdot u_1 + \cdots + \frac{\langle x, u_m\rangle}{\langle u_m, u_m\rangle} \cdot u_m$$

  We also recall the previous identity of a projection:

$$\text{proj}_U(x) = P(x) \text{ where } P = \frac{1}{\|u_1\|^2} \cdot u_1 \cdot u_1^T + \cdots + \frac{1}{\|u_m\|^2} \cdot u_m \cdot u_m^T$$

Also we can note the properties such as, $P^2 = P, P^T = P$ and $\text{rank}(P) = m$

These are key properties of an orthogonal projection matrix this must be true: $P^2 = P$ and $P^T = P$.

Theorem. If P is an orthogonal projection onto U, then $I - P$ is the orthogonal projection matrix onto $U^\perp$.

0.0.4 Projection Theorem

- **Theorem.** Let $U \subseteq \mathbb{R}^n$ be a subspace and let $x \in \mathbb{R}^n$. Then

$$x - \text{proj}_U(x) \in U^\perp$$

and $\text{proj}_U(x)$ is the closest vector in U to x.

0.0.5 Examples:

1. **Example** Find the orthogonal matrix P which projects onto the subspace spannes

by the vectors

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} u_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

This is super easy, we just follow the formula above:

$$P = \frac{1}{\|u_1\|^2} \cdot u_1 u_1^T + \frac{1}{\|u_2\|^2} \cdot u_2 u_2^T$$

$$P = \frac{1}{6} \begin{pmatrix} 3 & 0 & -3 \\ 0 & 0 & 0 \\ -3 & 0 & 3 \end{pmatrix} + \frac{1}{6} \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

$$P = \frac{1}{6} \cdot \begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{pmatrix}$$

$$P = \frac{1}{\|u\|\|v\|} \cdot vu^T$$

- **Example:**