MATH 307

Lecture 18: QR Decomposition

2025-10-17

0.1 Orthogonal Matrices

- *Definition*: A matrix A is **orthgonal** if $A^T A = AA^T = I$.

  $A^T A = I$ implies that the columns of A are orthonormal, and AA^T implies that the rows of A are orthonormal.

  - Note $Q^{-1} \Rightarrow Q^{\perp}$

    Similar computation for rows. To show that $Q \cdot Q^{\perp}$
  - *Theorem*: If A is an orthgonal matrix, then $\|Ax\| = \|x\|, \forall x \in \mathbb{R}^n$
    - For the proof we can just compute $\|Ax\|^2 = (Ax)^T \cdot Ax = x^T A^T Ax = x^T x = \|x\|^2$
  - A good example of this type of matrix are the reflection and rotation matrices. What we can get from this is that projections do not change the length / magnitude.
  - We know that the matrix norm of any orthogonal matrix is 1. $\|Q\| = 1$. Aside, if $Q_1 \wedge Q_2$ are orthogonal than so is $Q_1 \cdot Q_2$.
  - For projections (NOT Orthogonal), unless the trivial $P = I$. So this is true 99.9% time of the time, orthogonal matrices are invertible while it is not true that projections are.
  - Reflections: Let $U \subseteq R^n$ be a subspace. The Reflection of $x \in \mathbb{R}^n$ through U is $Rx = 2Px - x$ where P is the projection onto U.

    $$Rx = 2PX - X \Rightarrow R = 2P - I$$
  - Check

    $$R \cdot R^T = (2P - I)(2P - I)^T = (2P - 1)(2P^T - I^T)$$
    $$R \cdot R^T = (2P - I)(2P - I)^T = (2P - 1)(2P^T - I^T)$$
  - To show that a matrix is orthogonal you must show $R \cdot R^T = I$ and $R^T \cdot R = I$. Which also implies $Q^{-1} = Q^{\perp}$.
  - Formula $P = Q \cdot Q^{\perp}$ but is $P = I$?

0.2 QR by Gram-Schmidt

- *Definition*: Let A be an $n \cdot m$ matrix with $\text{rank}(A) = m$ and let $a_1, ..., a_n$ be the columns of A. There exists an orthogonal matrix Q and upper triangle matrix R such that $A = QR$.

  In particular $Q = [Q_1, Q_2]$ and $R = \binom{R_1}{0}$

  Where Q_1 is n x m, Q2 is n x (n-m), R1 is mxm such that Q1 = [w1, wn] is an orthogonal basis of R(A) constructed by Gram-Schmidt applied to the columns of A and Q2 = [w(m+1), wn] is an orthonormal of R(A)^(tack.t). R is mxn

  So if we do the Gram-Schmidt we just get QR automatically, just as if we applied Gaussian elimination on a matrix we just get LU.

  - Example: Take $A = QR$ and $A = Q_1 \cdot R$ since $QR = [Q_1, Q_2] \cdot \binom{R_1}{0}$

$$\boldsymbol{A = Q_1 \cdot R_1}$$

  We run Gram-Schmidt across the basis columns of R(A) and normalize.

  We expand these columns in terms of the orthonormal basis, by projecting the columns onto the orthonormal basis.

$$\begin{aligned} a_1 &= \langle w_1, a_1 \rangle \cdot w_1 \\ a_2 &= \langle w_1, a_2 \rangle \cdot w_1 + \langle w_2, a_2 \rangle \cdot w_2 \\ &\vdots \\ a_n &= \langle w_1, a_n \rangle \cdot w_1 + \langle w_2, a_n \rangle \cdot w_2 + \cdots + \langle w_n, a_n \rangle \cdot w_n \end{aligned}$$

  where $a_k \in \text{span}\{w_1, ..., w_n\}$, and we can write as matrix multiplication

$$A = \{w_1, ..., w_m\} \cdot \begin{bmatrix} \langle w_1, a_1 \rangle & \langle w_1, a_2 \rangle & ... & \langle w_1, a_m \rangle \\ 0 & \langle w_2, a_2 \rangle & ... & \langle w_2, a_m \rangle \end{bmatrix}$$

$$Q_1 = \begin{bmatrix} \\ w_1 ... w_n \\ \\ \end{bmatrix} R_1 =$$

0.3 Lecture