We represent the feature correspondences between the images by a partial permutation matrix $A \in \{0,1\}^{m\times n}$, $P_0 \in \mathbb{R}^{3\times n_0}$, $P_1 \in \mathbb{R}^{3\times n_1}$.

$$
\begin{aligned}
&\|RP_0 + t1^T - P_1A\|_F^2 \\
&= tr\big((t1^T + RP_0 - P_1A)(1t^T + P_0^TR^T - A^TP_1^T)\big) \\
&= tr(n_0t^Tt) + tr(t1^TP_0^TR^T) - tr(t1^TA^TP_1^T) + tr(RP_01t^T) + tr(RP_0P_0^TR^T) - tr(RP_0A^TP_1^T) \\
&\qquad - tr(P_1A1t^T) - tr(P_1AP_0^TR^T) + tr(P_1AA^TP_1^T) \\
&= tr(n_0t^Tt) + 2tr(1^TP_0^TR^Tt) - 2tr(1^TA^TP_1^Tt) - 2tr(RP_0A^TP_1^T) + tr(P_0P_0^TR^TR) \\
&\qquad + tr(P_1^TP_1AA^T) \\
&= n_0t^Tt + 2\cdot(RP_01)^Tt - 2\cdot(P_1A1)^Tt + C
\end{aligned}
$$

$$
\partial = 2n_0t + 2RP_01 - 2P_1A1 = 0 \Longrightarrow t^* = \frac{1}{n_0}(P_1A - RP_0)1
$$

$$
\|RP_0 + t^*1^T - P_1A\|_F^2 = \left\|(RP_0 - P_1A)\left(I_{n_0} - \frac{11^T}{n_0}\right)\right\|_F^2
$$

$$
= tr\left((RP_0 - P_1A)\left(I_{n_0} - \frac{11^T}{n_0}\right)(P_0^TR^T - A^TP_1^T)\right)
$$

$$
= tr\left(\left(I_{n_0} - \frac{11^T}{n_0}\right)(P_0^TR^T - A^TP_1^T)(RP_0 - P_1A)\right)
$$

$$
= tr\left(P_0\left(I_{n_0} - \frac{11^T}{n_0}\right)P_0^TR^TR\right) - 2tr\left(P_0\left(I_{n_0} - \frac{11^T}{n_0}\right)A^TP_1^TR\right) + tr\left(\left(I_{n_0} - \frac{11^T}{n_0}\right)A^TP_1^TP_1A\right)
$$

$$
= \left\|Diag_3\left(I_{n_0} - \frac{11^T}{n_0}\right)\left((R\otimes I_{n_0})vec(P_0) - (P_1\otimes I_{n_0})vec(A)\right)\right\|_2^2
$$

$$
= \|Mx - b\|_2^2
$$

$$
M = Diag_3\left(I_{n_0} - \frac{11^T}{n_0}\right)(P_1\otimes I_{n_0})
$$

$$
M^TM = (P_1^T\otimes I_{n_0})Diag_3\left(I_{n_0} - \frac{11^T}{n_0}\right)(P_1\otimes I_{n_0}) \succcurlyeq 0
$$

A shit but linear version of the A subproblem:

$$
\min_A \|A - \hat{A}\|_F^2 + \left\|(RP_0 - P_1A)\left(I_{n_0} - \frac{11^T}{n_0}\right)\right\|_F^2 \to \max_A \langle A, \hat{A}\rangle + \langle RP_0 - P_1A, 1\rangle
$$

$$\left\langle \begin{bmatrix} 3 & 3 & 3 & 3 \\ -5 & -5 & -5 & -5 \\ 7 & 7 & 7 & 7 \end{bmatrix}, \color{red}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}} \right\rangle \quad \begin{bmatrix} 3 \\ -5 \\ 7 \end{bmatrix} \text{ must not be closed to } \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ for every component}$$ 🙁

$$RP_0 - P_1 A$$

$$= \left[\tilde{p}_0^0, \tilde{p}_0^1, \dots, \tilde{p}_0^{n_0}\right] - \begin{bmatrix} A_{00}p_1^0 + A_{10}p_1^1 + \cdots + A_{n_1 0}p_1^{n_1}, \\ A_{01}p_1^0 + A_{11}p_1^1 + \cdots + A_{n_1 1}p_1^{n_1}, \\ \dots \dots, \\ A_{0n_0}p_1^0 + A_{1n_0}p_1^1 + \cdots + A_{n_1 n_0}p_1^{n_1} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{p}_0^0 - \left(A_{00}p_1^0 + A_{10}p_1^1 + \cdots + A_{n_1 0}p_1^{n_1}\right), \\ \tilde{p}_0^1 - \left(A_{01}p_1^0 + A_{11}p_1^1 + \cdots + A_{n_1 1}p_1^{n_1}\right), \\ \dots \dots, \\ \tilde{p}_0^{n_0} - \left(A_{0n_0}p_1^0 + A_{1n_0}p_1^1 + \cdots + A_{n_1 n_0}p_1^{n_1}\right) \end{bmatrix}$$

$$\langle RP_0 - P_1 A, \color{red}{1} \rangle$$

$$= \color{red}{1^T} \left(\left(\tilde{p}_0^0 + \tilde{p}_0^1 + \cdots + \tilde{p}_0^{n_0}\right) - p_1^0\left(A_{00} + A_{01} + \cdots + A_{0n_0}\right) - p_1^1\left(A_{10} + A_{11} + \cdots + A_{1n_0}\right) - \cdots \right.$$

$$\left. - p_1^{n_1}\left(A_{n_1 0} + A_{n_1 1} + \cdots + A_{n_1 n_0}\right) \right)$$