Problem Of The Day 2022

1. (**21 Mar**) Simplify the algebraic fraction $\dfrac{a^4 - a^2 b^2}{(a-b)^2} \div \dfrac{a(a+b)}{b^2} \times \dfrac{b}{a}$.

Solution:

$$\frac{a^4 - a^2 b^2}{(a-b)^2} \div \frac{a(a+b)}{b^2} \times \frac{b^2}{a}$$

$$= \frac{a^2 \cancel{(a+b)}\cancel{(a-b)}}{(a-b)^{\cancel{2}}} \times \frac{b^2}{\cancel{a(a+b)}} \times \frac{b^2}{\cancel{a}}$$

$$= \frac{b^4}{a-b}$$

2. (**22 Mar**) Factorise $a^4 + a^2 b^2 + b^2$.

Solution:

$$a^4 + a^2 b^2 + b^4 = a^4 + 2a^2 b^2 + b^4 - a^2 b^2$$

$$= (a^2 + b^2)^2 - (ab)^2$$

$$= (a^2 - ab + b^2)(a^2 + ab + b^2)$$

3. (**23 Mar**) Simplify $\dfrac{1}{a-x} - \dfrac{1}{a+x} - \dfrac{2x}{a^2 + x^2} - \dfrac{4x^3}{a^4 + x^4} + \dfrac{8x^7}{a^8 - x^8}$.

Solution:

$$\frac{1}{a-x} - \frac{1}{a+x} - \frac{2x}{a^2 + x^2} - \frac{4x^3}{a^4 + x^4} + \frac{8x^7}{a^8 - x^8}$$

$$= \frac{2x}{a^2 - x^2} - \frac{2x}{a^2 + x^2} - \frac{4x^3}{a^4 + x^4} + \frac{8x^7}{a^8 - x^8}$$

$$= \frac{4x^3}{a^4 - x^4} - \frac{4x^3}{a^4 + x^4} + \frac{8x^7}{a^8 - x^8}$$

$$= \frac{8x^7}{a^8 - x^8} + \frac{8x^7}{a^8 - x^8}$$

$$= \frac{16x^7}{a^8 - x^8}$$

4. **(24 Mar)** Factorise completely $64x^6 - y^{12}$.

Solution:

$$64x^6 - y^{12} = (8x^3 + y^6)(8x^3 - y^6)$$
$$= (2x + y^2)(2x - y^2)(4x^2 + 2xy^2 + y^4)(4x^2 - 2xy^2 + y^4)$$

5. **(25 Mar)** Factorise completely

$x^2(x - 1)^2 + 32(x - x^2) + 60$.

Solution:

$$x^2(x - 1)^2 + 32(x - x^2) + 60$$
$$= x^2(x - 1)^2 - 32x(x - 1) + 60$$
$$= [x(x - 1)]^2 - 32[x(x - 1)] + 16^2 - 14^2$$
$$= [x(x - 1) - 16]^2 - 14^2$$
$$= (x^2 - x - 2)(x^2 - x - 30)$$
$$= (x - 6)(x - 2)(x + 1)(x + 5)$$

6. **(28 Mar)** Simplify $\dfrac{x^2 - 4}{x^2 - 4x + 4} + \dfrac{2 - x}{x + 2}$.

Solution:

$$\frac{x^2 - 4}{x^2 - 4x + 4} + \frac{2 - x}{x + 2}$$
$$= \frac{(x + 2)(x - 2)}{(x - 2)^2} + \frac{-(x - 2)}{x + 2}$$
$$= \frac{(x + 2)^2 - (x - 2)^2}{(x + 2)(x - 2)}$$
$$= \frac{8x}{x^2 - 4}$$

7. **(29 Mar)** An equation in x, $\dfrac{m}{x - 1} + \dfrac{3}{1 - x} = 1$, has a positive solution. Find the possible range of values for m.

Solution:

$$\frac{m}{x-1} + \frac{3}{1-x} = 1$$
$$m - 3 = x - 1$$
$$x = m - 2$$
$$\because x > 0$$
$$\therefore m > 2$$

8. **(30 Mar)** Given that $\frac{1}{x} + \frac{1}{y} = 3$, find the value of $\frac{3x + 4xy + 3y}{x + 2xy + y}$.

Solution:

$$\because \frac{1}{x} + \frac{1}{y} = 3$$
$$\therefore x + y = 3xy$$

$$\frac{3x + 4xy + 3y}{x + 2xy + y} = \frac{\frac{13}{3}\cancel{(x+y)}}{\frac{5}{3}\cancel{(x+y)}}$$
$$= \frac{13}{5}$$

9. **(31 Mar)** A factory scheduled to manufacture 480 toys within a number of days. The factory increased its daily production by 50% since the beginning of the production and finished the whole batch of 480 toys 10 days earlier than the original schedule. How many toys per day did the factory plan to manufacture originally?

Solution: Let x be the number of toys per day the factory planned to manufacture

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originally.

$$\frac{480}{x} - \frac{480}{\frac{3}{2}x} = 10$$

$$\frac{720x - 480x}{\frac{3}{2}x^2} = 10$$

$$240x = 15x^2$$

$$x = 16$$

The company originally planned to manufacture 16 toys per day.

10. (**1 Apr**) Make y the subject of the formula:

$$\sqrt{\frac{x^3 - x + y}{xy}} = x$$

Solution:

$$\sqrt{\frac{x^3 - x + y}{xy}} = x$$

$$x^2 = \frac{x^3 - x + y}{xy}$$

$$x^3 y = x^3 - x + y$$

$$y(x^3 - 1) = x^3 - x$$

$$y = \frac{x^3 - x}{x^3 - 1}$$

$$= \frac{x(x+1)(x-1)}{(x-1)(x^2 + x + 1)}$$

$$= \frac{x(x+1)}{x^2 + x + 1}$$

11. (**4 Apr**) Solve the SLEs:

$$\begin{cases} \dfrac{1}{x} + \dfrac{1}{y} = 5 & \text{(1)} \\ xy = x - y & \text{(2)} \end{cases}$$

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Solution:

From (1):
$$xy = \frac{x+y}{5} \quad (3)$$

Substitute (3) into (2):
$$x + y = 5x - 5y$$
$$4x = 6y$$
$$x = \frac{3}{2}y \quad (4)$$

Substitute (4) into (2):
$$\frac{3}{2}y^2 = \frac{1}{2}y$$
$$y\left(\frac{3}{2}y - \frac{1}{2}\right) = 0$$
$$\therefore y = 0 \left(\text{rej.}, \frac{1}{y} \text{ cannot be undefined}\right) \text{ or } \frac{1}{3}$$

Substitute $y = \frac{1}{3}$ into (2):
$$\frac{1}{3}x = x - \frac{1}{3}$$
$$\frac{2}{3}x = \frac{1}{3}$$
$$x = \frac{1}{2}$$
$$\therefore x = \frac{1}{2} \text{ and } y = \frac{1}{3}$$

12. **(5 Apr)**

    (a) Find the equation of the line l_1 that makes a 45° angle with the positive x-axis and its y-intercept is −3.

    > **Solution:** Since the angle between the positive x-axis and l_1 has to be 45°, the gradient (m) can only be 1. Since the y-intercept is −3, the value of c in the gradient-intercept form has to be −3 as well. Therefore, the equation of line l_1 is $y = x - 3$.

    (b) Hence, find the equation of a vertical line l_2 which intersects with l_1 at a point with a y-coordinate of 10.

> **Solution:**
>
> $$y = x - 3$$
> $$10 = x - 3$$
> $$\therefore x = 13$$
>
> The equation of line l_2 is $x = 13$.

(c) Are the following three points collinear: the intersection point between l_1 and l_2, the origin, and $(-6.5, -5)$?

> **Solution:** There can be a line, l_3, with both the origin and the intersection on it. The equation of this line would thus be:
>
> $$y = \frac{10 - 0}{13 - 0}x = \frac{10}{13}x$$
>
> Substituting $x = -6.5$ and $y = -5$ into that equation, we see that:
>
> $$-5 = \frac{10}{13} \times -\frac{13}{2}$$
>
> Therefore, all three points are collinear.

13. **(7 Apr)** Expand and simplify the expression $\left[x\left(x^4 - y^4\right) - 3xy\,(-y)^3 \right] \cdot x^2 y$.

> **Solution:**
>
> $$\left[x\left(x^4 - y^4\right) - 3xy\,(-y)^3 \right] \cdot x^2 y$$
> $$= x^3 y \left[x^4 - y^4 - 3y\,(-y)^3 \right]$$
> $$= x^3 y \left(x^4 + 2y^4 \right)$$

14. **(8 Apr)** There is a rectangular piece of paper with length $(5a + 4b)$ and breadth $(4a + 3b)$. A square of side length $(a + b)$ is cut away from the four corners of the paper. Using the remaining paper, a box without its top cover can be folded.

(a) Find the total surface area of the folded paper box.

Solution: Draw the diagram.
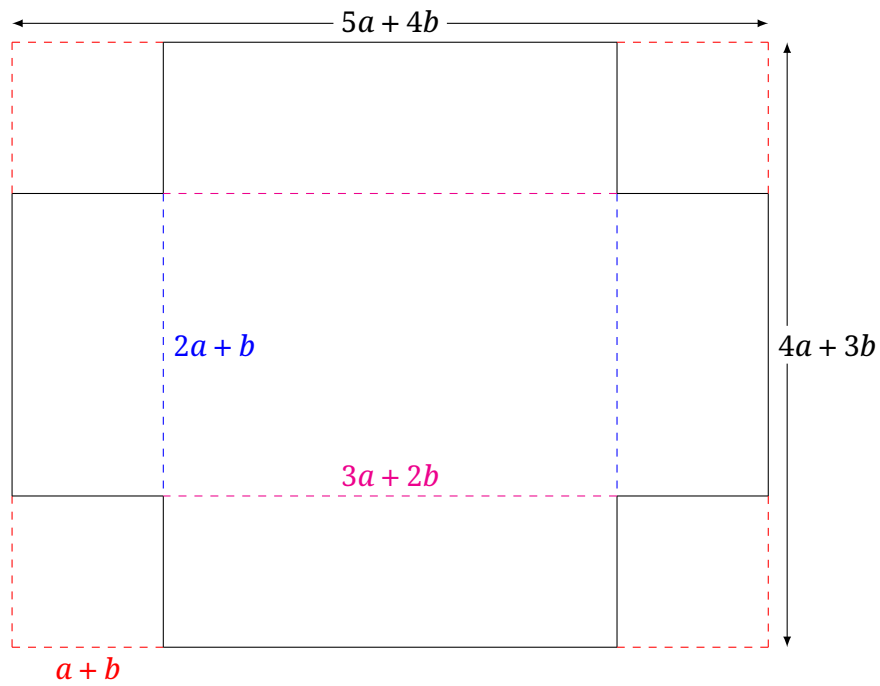
$$\text{total surface area} = 2 \cdot \left[(5a + 4b)\,(4a + 3b) - 4(a + b) \right]$$
$$= 2\left(20a^2 + 15ab + 16ab + 12b^2 - 4\,(a + b) \right)$$
$$= 40a^2 + 62ab + 24b^2 - 8a - 8b$$

(b) Find the volume of the paper box.

Solution:

$$\text{volume of paper box} = (3a + 2b)(2a + b)(a + b)$$
$$= 6a^3 + 13a^2 b + 9ab^2 + 2b^3$$

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