

## Problem Of The Day 2022

1. (27 Jun) If  $y$  varies inversely as  $x$  and can be represented by the equation  $y = (m - 1)x^{m^2 - 2}$ , find the value of constant  $m$ .

**Solution:**

$$y = (m - 1)x^{m^2 - 2} = \frac{k}{x}$$

$$k = (m - 1)x^{m^2 - 1}$$

$$= (m - 1)x^{(m+1)(m-1)} \quad (x \neq 0)$$

By definition,  $y \neq 0$  as well, hence

$$(m - 1)x^{(m+1)(m-1)} \neq 0$$

$$\therefore m \neq 1$$

2. (28 Jun) Which of the following is a possible plot of  $y = x + m$  and  $y = \frac{m}{x}$  on the same axes?  
(The graphs are not drawn to scale.)

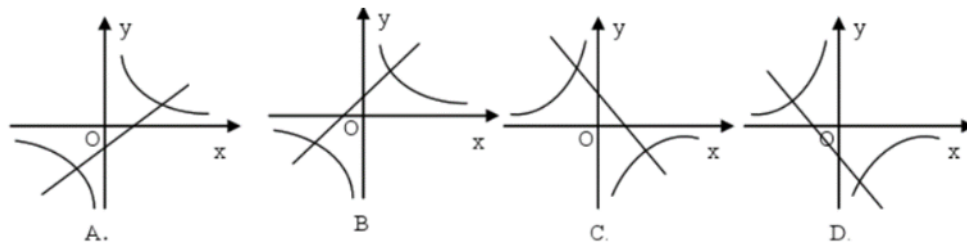


Figure 1:  $y = x + m$  and  $y = \frac{m}{x}$ .

**Solution: B.**

- The straight line should be increasing, since the coefficient of  $x$  is positive. **C** and **D** are eliminated.
- If  $m > 0$ , the  $y$ -intercept of the straight line could not be negative. **A** is eliminated, since the hyperbola in the same graph shows that  $m > 0$ .

3. (29 Jun) Given that points  $A(-2, y_1)$ ,  $B(-1, y_2)$ ,  $C(1, y_3)$  are all on the graph of  $y = -\frac{1}{x}$ , arrange  $y_1$ ,  $y_2$  and  $y_3$  in ascending order.

**Solution:**  $y_3 < y_1 < y_2$ .

Subst.  $x = -2$  into  $y = -\frac{1}{x}$ :

$$\begin{aligned}y_1 &= -\frac{1}{-2} \\&= \frac{1}{2}\end{aligned}$$

Subst.  $x = -1$  into  $y = -\frac{1}{x}$ :

$$\begin{aligned}y_2 &= -\frac{1}{-1} \\&= 1\end{aligned}$$

Subst  $x = 1$  into  $y = -\frac{1}{x}$ :

$$\begin{aligned}y_3 &= -\frac{1}{1} \\&= -1\end{aligned}$$

4. (30 Jun) Given that points  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$  are all on the graph of  $y = \frac{3}{x}$ , also

$x_1 < x_2 < 0 < x_3$ , arrange  $y_1, y_2$  and  $y_3$  in ascending order.

**Solution:**  $y_2 < y_1 < y_3$ .

- $y_3 > 0$  since  $x_3 > 0$ . Hence,  $y_3$  is the greatest.
- $0 > x_2 > x_1$ , hence  $y_2 < y_1 < 0$ .

5. **(1 Jul)** Given that  $y$  varies inversely as  $x$  such that  $y = (a - 2)x^{a^2 - 5}$ , also when  $x > 0$ , as  $x$  increases,  $y$  increases. Find the equation of the hyperbola.

**Solution:**

$$\frac{k}{x} = (a - 2)x^{a^2 - 5}$$

$$k = (a - 2)x^{a^2 - 4}$$

$$= (a - 2)x^{(a+2)(a-2)}$$

$$\therefore y = \frac{(a - 2)x^{(a+2)(a-2)}}{x}$$

$$\therefore (a - 2)x^{(a+2)(a-2)} < 0$$

$$\therefore a - 2 < 0 \text{ and } (a + 2)(a - 2) \geq 0$$

$$\therefore a + 2 = 0$$

$$\therefore a = -2$$

$$\begin{aligned} \therefore y &= -\frac{(-2 - 2) \times x^{(-2+2) \times (-2-2)}}{x} \\ &= -\frac{4}{x} \end{aligned}$$

6. **(5 Jul)** If a straight line  $y = (2m - 1)x$  and a hyperbola  $y = \frac{3 - m}{x}$  has an intersection point each in Quadrant 1 and 3, what is the range of values of the constant  $m$ ?

**Solution:**

$$\therefore y = \frac{3-m}{x}$$

$$\therefore m < 3$$

$$\therefore y = x(2m-1)$$

$$\therefore 2m-1 > 0$$

$$\therefore 0.5 < m < 3$$

7. (6 Jul) Points A and B are on the hyperbola  $y = \frac{k}{x}$ . Right  $\triangle AOC$  and  $\triangle BOD$  are drawn by perpendicular lines drawn from the two points and connecting the points with the origin. Compare the sizes of the areas of the two triangles.

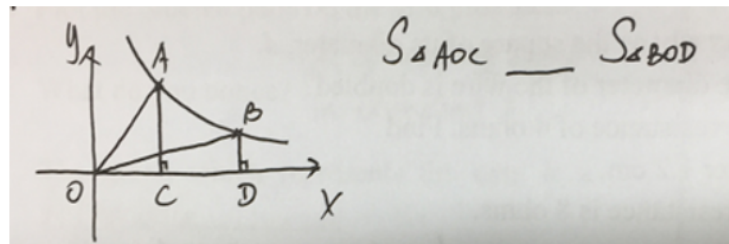


Figure 2:  $\triangle AOC$  and  $\triangle BOD$ .

**Solution:**  $S_{\triangle AOC} < S_{\triangle BOD}$ . As Point B's  $y$ -coordinate approaches 0, its  $x$ -coordinate approaches infinity, resulting in a triangle which approaches an infinite area. Point A is further from  $(\infty, 0)$  than Point B, so  $S_{\triangle AOC} < S_{\triangle BOD}$ .

8. (7 Jul) Points A and B are on the hyperbola  $y = \frac{3}{x}$ . Rectangles are drawn from the two points as shown. The areas bounded by the two rectangles are labelled  $S_1$  and  $S_2$ . If the shaded area is 1 sq. unit, find the sum  $S_1 + S_2$ .

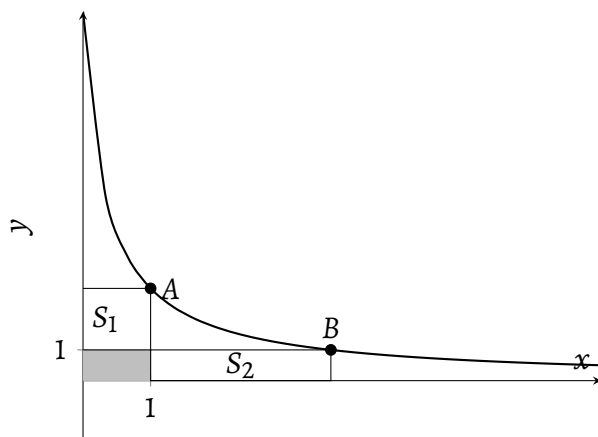


Figure 3: The shaded areas,  $S_1$  and  $S_2$ .

**Solution:**

$$S_1 = (1 - 0) \times \left( \frac{3}{1} - 1 \right)$$

$$= 2 \text{ sq. units}$$

$$S_2 = \left( \frac{3}{3} - 0 \right) \times (3 - 1)$$

$$= 2 \text{ sq. units}$$

$$S_1 + S_2 = 2 + 2$$

$$= 4 \text{ sq. units}$$

9. (8 Jul) If a hyperbola  $y = -\frac{3m}{x}$  and a straight line  $y = kx - 1$  both pass through the point  $P(m, -3m)$ ,

(a) find the coordinates of  $P$  and the equations of the hyperbola and the straight line.

**Solution:**

$$y = -\frac{3m}{x} \tag{1}$$

$$y = kx - 1 \tag{2}$$

Subst.  $x = m$ ,  $y = -3m$  into (1):

$$-3m = -\frac{3m}{m}$$

$$\therefore m = 1$$

$$\therefore P(1, -3) \quad (3)$$

We can substitute the values obtained in (3) into (2):

$$-3 = k - 1$$

$$\therefore k = -2$$

The equations of the hyperbola and the straight line, are, thus:

$$y = -\frac{3}{x}$$

$$y = -2x - 1$$

- (b) If the points  $M(a, y_1)$  and  $N(a + 1, y_2)$  are both on the straight line, explain clearly why  $y_1 > y_2$ .

**Solution:** Substitute the  $x$ - and  $y$ -coordinates of both points into the equation of the line.

$$y_1 = -2a - 1$$

$$y_2 = -2(a + 1) - 1$$

$$= -2a - 3$$

$$\therefore -2a - 1 > -2a - 3, \text{ where } a \in \mathbb{R}$$

$$\therefore y_1 > y_2$$

10. **(12 Jul)** The line  $y = x$  meets the hyperbola  $y = \frac{1}{x}$  at points  $A$  and  $C$ . Vertical lines from  $A$  and  $C$  meet the  $x$ -axis at points  $B$  and  $D$  respectively. Find the area of the quadrilateral  $ABCD$ . (The diagram is not drawn to scale.)

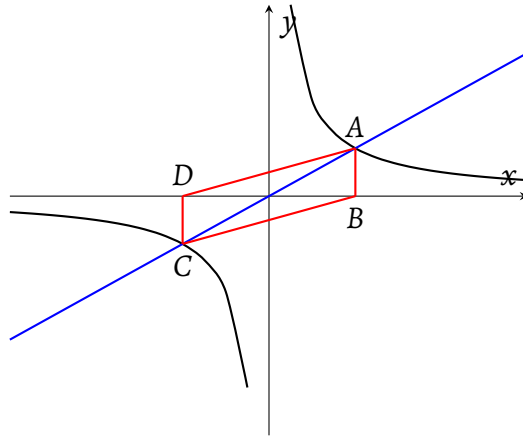


Figure 4: Quadrilateral  $ABCD$ .

**Solution:**

$$x = \frac{1}{x}$$

$$\therefore x = \pm 1$$

$$\therefore A(1, 1) \text{ and } C(-1, -1)$$

$$S_{ABCD} = 1 \times [1 - (-1)]$$

$$= 2 \text{ sq. units}$$

11. **(13 Jul)** A ladder  $AB$  of length 2.5 m has its foot  $B$  1.5 m away from a wall. The ladder is then moved to a new position  $ED$ . The foot of the ladder is moved 0.5 m from the original position  $B$ . Find the distance the top of the ladder drops, the length of  $AE$ .

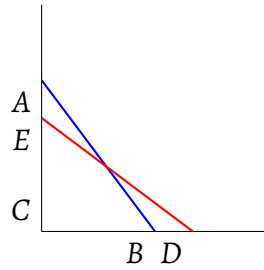


Figure 5: The ladder, before and after.

**Solution:**

$$AC = \sqrt{2.5^2 - 1.5^2}$$

$$= 2 \text{ m}$$

$$EC = \sqrt{2.5^2 - (1.5 + 0.5)^2}$$

$$= 1.5 \text{ m}$$

$$\text{height dropped} = 2 - 1.5$$

$$= 0.5 \text{ m}$$

12. **(14 Jul)** In  $\triangle ABC$ ,  $\angle B = 22.5^\circ$ . The perpendicular bisector of  $AB$  intersects  $BC$  at point  $D$  and  $BD^2 = 72$ .  $AE \perp BC$ . Find  $AE$ .

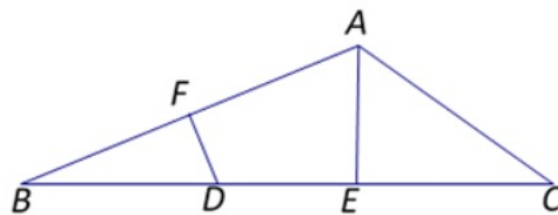


Figure 6: Triangle  $\triangle ABC$ .



**Solution:**

$$\angle FAD = \angle B$$

$$= 22.5^\circ$$

$$BD = AD$$

$$= \sqrt{72}$$

$$\angle FDA = 90 - 22.5$$

$$= 67.5^\circ$$

$$\angle FDB = \angle FDA = 67.5^\circ$$

$$\therefore \angle ADE = 180 - 67.5 \times 2$$

$$= 45^\circ$$

$$\sin \angle ADE = \frac{AE}{\sqrt{72}}$$

$$AE = \sqrt{72} \times \sin 45^\circ$$

$$= \frac{\sqrt{72}}{\sqrt{2}}$$

$$= 6$$

13. **(15 Jul)** In  $\triangle ABC$ ,  $\angle A = 90^\circ$ . The point  $P$  is the midpoint of  $AC$ .  $PD \perp BC$ ,  $BC = 9$  and  $DC = 3$ . Find  $AB$ .

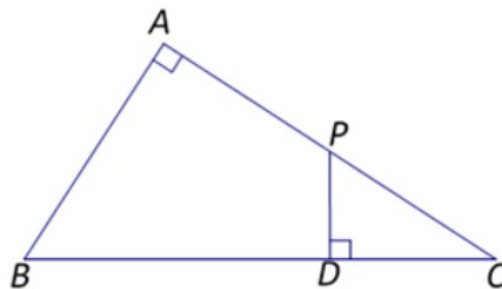


Figure 7: Triangle  $\triangle ABC$ .

**Solution:**

$$\sqrt{3^2 + PD^2} = \sqrt{6^2 + PD^2 - AB^2}$$

$$PD^2 + 9 = PD^2 + 36 - AB^2$$

$$27 - AB^2 = 0$$

$$\therefore AB = \sqrt{27}$$