Simulation of the 7 Qubit Steane Code*

Grant Eberle
Duke University Pratt School Of Engineering

Effective quantum error correction methods are a key area of research in the development of fault tolerant quantum computation. As quantum devices focus on scalability with the goal of solving practical problems, quantum error correction techniques are crucial to mitigating errors in these increasingly large systems. This paper outlines the 7-qubit version of the quantum error correction code proposed by Andrew Steane [3]. The details of the code are explained accompanied by multiple simulations to demonstrate its effectiveness. The simulations focus on the codes ability to correct different levels of single qubit bit and phase flip errors from the code capacity and phenomenological perspectives.

I. Introduction

The 7-qubit Steane code is the smallest example of the Calderbank-Shor-Steane (CSS) stabilizer code family. CSS codes are comprised of specially constructed linear classical codes $C_1 : [n, k_1]$ and $C_2 : [n, k_2]$ such that $C_2 \subset C_1$, C_1 and $C_2^{\perp}$ can correct errors on up to t bits. Then CSS(C_1, C_2) is an $[n, k_1 - k_2]$ quantum code that can correct arbitrary errors on up to t qubits. Implementation of this code only requires the use of Hadamard and CNOT gates which increase linearly with the size of the code [4].

The 7-qubit Steane code is constructed using the [7, 4, 3] classical Hamming code and its dual [7, 3, 4] code. Both codes are concatenated into a single parity check matrix which allows for independent correction of single Pauli X and Pauli Z errors. This parity check matrix is encoded by the generating stabilizers,

$$
\begin{aligned}
S =< &IIIXXXX \\
&IXXIIXX \\
&XIXIXIX \\
&IIIZZZZ \\
&IZZIIZZ \\
&ZIZIZIZ >
\end{aligned}
$$

These stabilizers are then encoded onto 7 qubits that act as a single logical qubit which is safe from arbitrary single qubit errors. The new logical qubit must be encoded into a 7-qubit logical state for computation. The computational basis states are defined on the logical qubit as,

*Quantum Error Correction Final Project
(ECE 621, Instructor: Ken Brown, Ph.D.)

$$|0_L\rangle = \frac{1}{\sqrt{8}}\Big[\,|0000000\rangle + |1010101\rangle + |0110011\rangle + |1100110\rangle$$
$$+ |0001111\rangle + |1011010\rangle + |0111100\rangle + |1101001\rangle\,\Big]$$

$$|1_L\rangle = \frac{1}{\sqrt{8}}\Big[\,|1111111\rangle + |0101010\rangle + |1001100\rangle + |0011001\rangle$$
$$+ |1110000\rangle + |0100101\rangle + |1000011\rangle + |0010110\rangle\,\Big]$$

Computations are performed on these states using transversal gates across the 7 qubits that encode the logical qubit [4].

II. Results

The simulations in this paper were aimed at demonstrating how the effectiveness of the 7-qubit steane code progresses as the level of errors increases. Error level is defined as the number of error gates of each type in the circuit, with 1 Bit flip (Pauli X) and 1 Phase flip (Pauli Z) error for each indicated level. For example, the level 4 simulations include 4 bit flip and 4 phase flip errors randomly placed on the logical qubit register. The errors occur with a physical error rate p that ranges from 0 to 0.5 in each simulation. An additional option is included to change from the code capacity model to the phenomenological error model. This option allows for the inclusion of syndrome measurement errors that occur with the same probability p. The simulations were run on the statevector simulator provided by Qiskit-Aer [5] [1].

First, the logical state $|0_L\rangle$ was prepared using a basic encoding circuit [2].

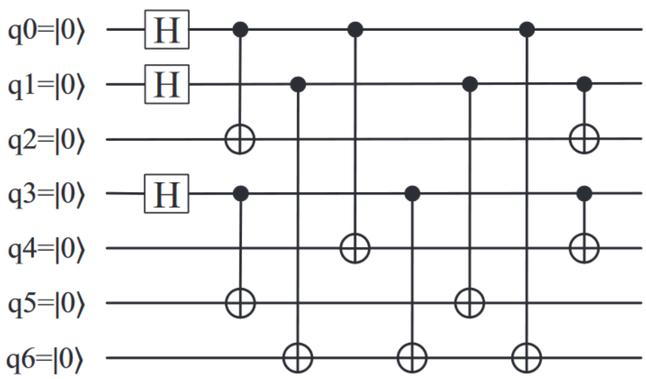

Figure 1: Quantum circuit used to encode the logical $|0_L\rangle$ state, referenced from [2].

The logical $|0_L\rangle$ state can also be prepared by the error correction circuit, but this method allows simulation without resetting the ancillary syndrome bits used to decode the stabilizers in the error correction process. This circuit was tested using a statvector simulation to ensure the correct logical state was encoded.

Next, multiple error filled simulations were preformed using this encoded initial state. The circuits used in the simulations were first encoded into the logical $|0_L\rangle$ state, then errors were introduced and the error correction procedure was applied to the circuit according to the model being tested. Syndrome encoding was accomplished using 6 cat state syndrome qubits, with each corresponding to one stabilizer in the code. The encoding function utilized each syndrome bit in the X-basis to receive the respective errors detected. Decoding was accomplished by measuring each syndrome bit in the X-basis and preforming classically controlled operations according to the binary number measured on the register. This binary number corresponds to the calculated syndrome and thus indicated the location of the error in the logical register.

Both the code capacity and phenomenological error models were tested using a series of 5 simulations where the error level was progressively increased. It should be noted that errors were applied on random qubits in a way that could place two errors such that they cancel out, but this effect seemed to have minimal impact on the simulations. The calculation of each logical error rate in the simulation used a total of 1000 simulated statevector samples. The results for each error model were collected and plotted to display the logical error rate of each error level in comparison to the physical error rates used in the simulations.

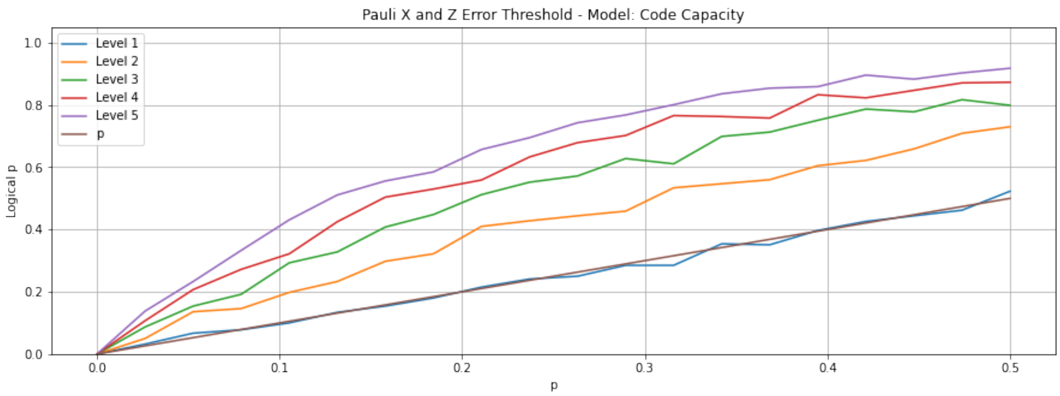

Figure 2: Logical Error rate threshold comparison for code capacity model simulations.

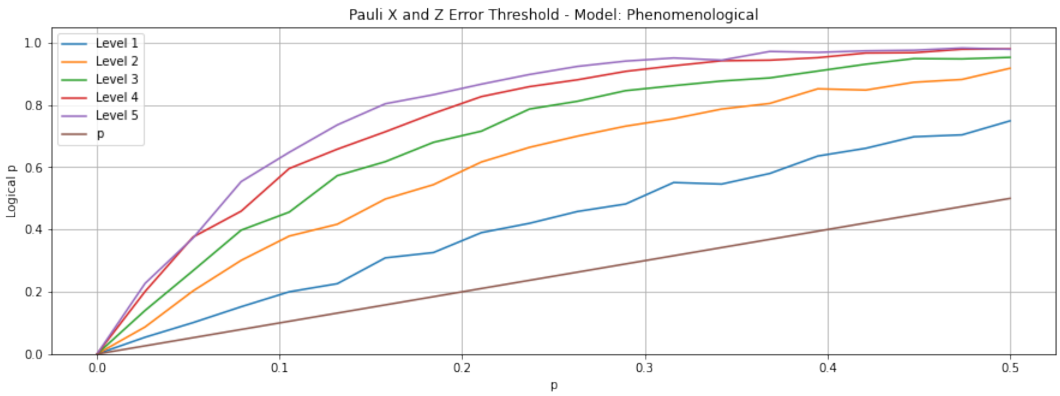

Figure 3: Logical Error rate threshold comparison for phenomenological model simulations.

III. Discussion

The overall results of the simulations are consistent with the theoretical limitations of the 7-qubit Steane code. The implemented error correction procedure was able to successfully detect and correct up to 2 single qubit errors (Single Pauli X and Pauli Z). It is clear from the trend of both graphs (Fig.2 and Fig.3) the code becomes impractical very quickly in terms of increasing error level.

From the code capacity perspective, where encoding and decoding during correction are assumed to be perfectly accurate, the code preforms as expected. Errors that are introduced to the encoded logical $|0_l\rangle$ state are identified and fixed during the error correction process. The effectiveness of the correction process is determined by whether the resulting statvector returns to the logical $|0_l\rangle$ state after the simulation. Moving to the phenomenological perspective, the same level simulations were preformed with the addition of syndrome measurement errors to the circuit.

The simulations preformed under the phenomenological error model produced interesting results. Fig. 3 shows a much steeper increase in the logical error rate when compared to the previous model. This increase indicates the codes ineffectiveness when measurement errors are introduced as the logical error rate is consistently above the physical error rate. By introducing measurement errors during syndrome extraction, the errors that appear from the syndrome register are unable to be corrected. Of course, this is due to the fact that each simulation only preforms one round of error correction on the circuit. In future work, multiple rounds of error correction could be preformed to determine the trade off between preforming another round and introducing more measurement errors.

Executing the circuit under these two error models serves as an example of the basic capabilities of the circuit. For practical applications, both models do not provide a strong estimate of the effectiveness of a code. This is due to the fact that real quantum devices are better understood under the circuit level error model. Under this model, errors can occur anywhere in the circuit, making fault tolerance much more difficult. Fault tolerant state preparation is an effective method used to combat these types of errors. This paper stays within the scope of the code capacity and phenomenological error models, but implementing fault tolerant state preparation in additional simulations would be a great addition from a circuit level perspective. Fault tolerant state preparation of both the logical and syndrome registers was not implemented in these simulations as it has no effect on the errors studied in this paper.

IV. Conclusion

In general, the 7-qubit Steane code is proven effective for correcting common errors which plague quantum circuits. The simulations in this paper provide a basic assessment of the capability of the code in the code capacity and phenomenological models. As previously stated, this work would benefit from the addition of simulations in the circuit level model which implement fault tolerant state preparation. The results of this paper serve there purpose to demonstrate the theoretical capabilities and limits of the 7-qubit Steane code under specific error models. The simulations also provide a general introduction to the 7-qubit code which is intended to be helpful for those new to the field of quantum error correction.

The Steane code has other interesting applications which prove useful for practical error correction such as its interpretation as a 2-D color code [6]. There are much larger CSS codes being studied in hopes of practical applications to scalable quantum devices in the near future. With the basic concepts of the 7-qubit Steane code outlined, and its performance demonstrated, I hope the results of this paper prepare the reader to contribute to the exciting field of quantum error correction.

References

[1] Grant Eberle. *Simulation of the 7 Qubit Steane Code*. `https://github.com/g-ebbs/ece621-steane-code`.

[2] H. Goudarzi et al. "Design of a universal logic block for fault-tolerant realization of any logic operation in trapped-ion quantum circuits". In: *Quantum Information Processing* 13.5 (Jan. 2014), pp. 1267–1299. DOI: `10.1007/s11128-013-0725-3`. URL: `https://doi.org/10.1007%2Fs11128-013-0725-3`.

[3] "Multiple-particle interference and quantum error correction". In: *Proceedings of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences* 452.1954 (Nov. 1996), pp. 2551–2577. DOI: `10.1098/rspa.1996.0136`. URL: `https://doi.org/10.1098%2Frspa.1996.0136`.

[4] Michael A. Nielsen and Isaac L. Chuang. "The stabilizer formalism". In: *Quantum Computation and Quantum Information*. Cambridge University Press, 2021, p. 452.

[5] *Qiskit Aer*. `https://qiskit.org/documentation/apidoc/aer.html`.

[6] Ben W Reichardt. "Fault-tolerant quantum error correction for Steane's seven-qubit color code with few or no extra qubits". In: *Quantum Science and Technology* 6.1 (Nov. 2020), p. 015007. DOI: `10.1088/2058-9565/abc6f4`. URL: `https://doi.org/10.1088%2F2058-9565%2Fabc6f4`.