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CMPT331 - Theory of Programming Languages
Lambda Calculus

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1. **Beta-reduce the following expressions to their normal form:**

   (a) $(\lambda a \lambda y.ya)(zz)$
   $(\lambda y.ya)[(zz)/a]$
   $\lambda y.yzz$

   (b) $(\lambda x \lambda y.(xy))(\lambda z.y)$
   $(\lambda x \lambda a.(xa))(\lambda z.y)$ *Renaming bound variable y to a*
   $(\lambda a.(xa))[(\lambda z.y)/x]$
   $(\lambda a.(\lambda z.y)a)$
   $(\lambda a.(y))[a/z]$ *There is no z to substitute here*
   $\lambda a.y$

   (c) $(\lambda x.(xx))(\lambda y.(yy))$
   $(xx)[(\lambda y.(yy))/x]$
   $(\lambda y.(yy))(\lambda y.(yy))$
   $(yy)[(\lambda y.(yy))/y]$
   $(\lambda y.(yy))(\lambda y.(yy))$
   $(yy)[(\lambda y.(yy))/y]$
   ...
   self-application - infinite recursion

   (d) $K\ x\ y$ *$K \equiv \lambda ab.a$*
   $(\lambda ab.a)xy$
   $(\lambda b.a)[x/a]$
   $(\lambda bx)y$
   $(x)[y/b]$ *There is no b to substitute here*
   x

   (e) $S\ K$ *$S \equiv \lambda xyz.xz(yz)$*
   $(\lambda xyz.xz(yz))K$
   $(\lambda yz.xz(yz))[K/x]$
   $\lambda yz.Kz(yz)$ *K beta-reduces to only the first expression, z*
   $\lambda yz.z \equiv K'$
   K'

   (f) $(S\ K)\ y\ y\ z$
   $(K')yyz$ *As found in part e, $SK \equiv K'$*
   $(\lambda xy.y)yyz$ *$K' \equiv \lambda xy.y$*
   $(\lambda y.y)[y/x]$ *There is no x to substitute, K' beta-reduces to only second expression, y*
   $(\lambda y.y)yz$

$(y)[y/y]$

yz *No more functions to evaluate*

(g) $K'\ y\ y\ z$ *As shown in part f, K' beta-reduces to only the second expression, y*

yz *No more functions to evaluate*

2. **What is the normal form of $(K\ S)(K\ I)$?**

Evaluating (KS):

$\lambda xy.x)S$ *K beta-reduces to only the first expression, S*

$(\lambda y.x)[S/x]$

$\lambda y.S$ *Given any input, this expression outputs S*

$\lambda y.S$

Evaluating $(\lambda y.S)(KI)$: *Given any input, this expression outputs S*

$(S)[(KI)/y]$ *There is no y to substitute here*

S

3. **Prove the following equivalencies by reducing each side to its normal form.**

(a) $I = S\ K\ K$

$I = (\lambda epz.ez(pz))KK$

$I = (\lambda pz.ez(pz))[K/e]$

$I = (\lambda pz.Kz(pz))K$

$I = (\lambda z.Kz(pz))[K/p]$

$I = (\lambda z.Kz(Kz))$

$I = \lambda z.K(z)(Kz)$ *K beta-reduces to only the first expression, z*

$I = \lambda z.z \equiv I$

$I = I$

(b) $S\ K\ K = K\ I\ I$

$I = KII$ *As shown in part a, $SKK = I$*

$I = (\lambda xy.x)II$ *K beta-reduces to only the first expression, I*

$I = (\lambda y.x)[I/x]$

$I = (\lambda y.I)I$

$I = (I)[I/y]$ *There is no y to substitute here*

$I = I$

4. **Given the definition of Church numerals below, what does (m n) do when m and n are Church numerals? For example** $(\bar{2}\ \bar{3})$**. It may be easier to work out as** $\lambda m \lambda n.(mn)$**. Show your work (or at least an example).**

Church Numerals

Let $\bar{0} = \lambda f\ x\ .\ x$
Let $\bar{1} = \lambda f\ x\ .\ (f\ x)$
Let $\bar{2} = \lambda f\ x\ .\ (f\ (f\ x))$
Let $\bar{3} = \lambda f\ x\ .\ (f\ (f\ (f\ x)))$
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Let $\bar{n} = \lambda f\ x\ .\ (f^n.\ x)$

Let successor = $\overline{succ} = \lambda n\ f\ x\ .\ n\ f\ (f\ x)$

$(\ \overline{succ}\ \bar{0}\ ) \equiv (\lambda n\ f\ x\ .\ n\ f\ (f\ x))\ (\bar{0})$
$[\bar{0}/n]$ in $\lambda\ f\ x\ .\ n\ f\ (f\ x)$
$\rightarrow_\beta \lambda\ f\ x\ .\ \bar{0}\ f\ (f\ x)$

The result is going to be λ f x . *something*.
something is $\bar{0}\ f\ (f\ x)$
$\equiv\ (\lambda f\ x\ .\ x)\ f\ (f\ x)$
[f/f in λ x . x]
$\rightarrow_\beta \lambda\ x\ .\ x\ (f\ x)$
[(f x) / x in x]
$\rightarrow_\beta (f\ x)$

plugging *something* back into the above...
$\lambda f\ x\ .\ (f\ x)$
$\equiv \bar{1}$

m = 2, n = 3

Evaluating $(\bar{2}\ \bar{3})$**:**
$m \equiv \lambda f x.f^m(x)$

$(\bar{m}\ \bar{n}) = (\lambda f x.f^m(x))n$
$(\bar{2}\ \bar{3}) = (\lambda f x.f^2(x))3$
$= (\lambda f x.f(f(x)))3$
$= (\lambda x.f(f(x)))[3/f]$
$= (\lambda x.3(3(x)))$
$=$ 3 applied twice to x
$= 3^2$

Evaluating $(\bar{3}\ \bar{2})$**:**
$n \equiv \lambda f x.f^n(x)$

$(\bar{n}\ \bar{m}) = (\lambda f x.f^n(x))m$
$(\bar{3}\ \bar{2}) = (\lambda f x.f^3(x))2$
$= (\lambda f x.f(f(f(x))))2$
$= (\lambda x.f(f(f(x))))[2/f]$
$= (\lambda x.2(2(2(x))))$
$=$ 2 applied three times to x
$= 2^3$

Applying Church numerals to each other results in exponentiation.