Crossing Equation for 6d $\mathcal{N} = (2,0)$

See [1, 2].

We want to focus on the 4-point function of stress-tensor multiplets. The superconformal primary is a spacetime scalar Φ^{IJ}, $I, J = 1, \ldots, 5$, of conformal dimension $\Delta = 4$ which transforms in the symmetric traceless dimension **14** representation of $so(5)_R$.

Schematically a general 4-point function takes the form

$$\langle O_1^A O_2^B O_3^C O_4^D \rangle \;\sim\; \sum_\chi \lambda_{12\chi}\lambda_{34\chi}\left(\sum_R P_R^{ABCD} G_{\chi,R}^{1234}(z,\bar{z})\right) . \tag{0.1}$$

Here χ runs over the irreducible representations appearing in both the $O_1 \times O_2$ and $O_3 \times O_4$ OPE selection rules, R runs over the different $so(5)_R$ representations appearing in the supermultiplet χ, and P_R^{ABCD} are some R-symmetry projection tensors.

The stress tensor multiplet is denoted $\mathcal{D}[2,0] \equiv \mathcal{D}[0,0,0,2,0,4]$ (in general the quantum numbers of a state are given by $[h_1, h_2, h_3, d_1, d_2, \Delta]$ corresponding to eigenvalues of the generators of rotations in three orthogonal planes in $\mathbb{R}^6$, generators of a Cartan subalgebra of $so(5)_R$, and the dilatation generator). The selection rule for the OPE of two stress-tensor multiplets is

$$\begin{aligned}\mathcal{D}[2,0]\times\mathcal{D}[2,0] \;&= \mathbf{1} + \mathcal{D}[4,0] + \mathcal{D}[2,0] + \mathcal{D}[0,4] \\ &\quad + \sum_{l=0,2,\ldots}\left(\mathcal{B}[2,0]_l + \mathcal{B}[0,2]_{l+1}\right) + \sum_{\substack{l=0,2,\ldots\\ \Delta>6+l}} \mathcal{L}[0,0]_{\Delta,l} \, .\end{aligned} \tag{0.2}$$

Note that in the above higher spin conserved currents (corresponding to $\mathcal{B}[0,0]_l$ multiplets) have been dropped due to the assumption that there should be no free decoupled subsector.

There are five families of unitary representations. These families are characterised by linear relations obeyed by the quantum numbers of the superconformal primary state:

$$\begin{array}{ll} \mathcal{L}: \Delta > h_1 + h_2 - h_3 + 2\,(d_1+d_2) + 6, & h_1 \geqslant h_2 \geqslant h_3 \\ \mathcal{A}: \Delta = h_1 + h_2 - h_3 + 2\,(d_1+d_2) + 6, & h_1 \geqslant h_2 \geqslant h_3 \\ \mathcal{B}: \Delta = h_1 + 2\,(d_1+d_2) + 4, & h_1 \geqslant h_2 = h_3 \\ \mathcal{C}: \Delta = h_1 + 2\,(d_1+d_2) + 2, & h_1 = h_2 = h_3 \\ \mathcal{D}: \Delta = 2\,(d_1+d_2)\,, & h_1 = h_2 = h_3 = 0\,. \end{array} \tag{0.3}$$

Back to the 4-point function of primaries. Introduce [1] complex polarisation vectors Y^I and write $\Phi(x,Y) := \Phi^{IJ}(x)Y_IY_J$. It turns out that Y^I can be written as

$$Y^I \;=\; \left(y^i, \frac{1}{2}(1-y^iy_i), \frac{\mathrm{i}}{2}(1+y^iy_i)\right) , \tag{0.4}$$

with y^i an arbitrary 3-vector. We also introduce the analogue of cross-ratios for the polarisation vectors;

$$\frac{1}{\alpha\bar{\alpha}} := \frac{y_{12}^2 y_{34}^2}{y_{13}^2 y_{24}^2}\,, \qquad \frac{(\alpha-1)(\bar{\alpha}-1)}{\alpha\bar{\alpha}} := \frac{y_{14}^2 y_{23}^2}{y_{13}^2 y_{24}^2}\,. \tag{0.5}$$

Then (recalling $\Delta_\Phi = 4$)

$$\langle \Phi(x_1,y_1)\Phi(x_2,y_2)\Phi(x_3,y_3)\Phi(x_4,y_4)\rangle = \frac{y_{12}^4 y_{34}^4}{x_{12}^8 x_{34}^8} G(z,\bar{z},\alpha;\bar{\alpha})\,. \tag{0.6}$$

Since the **14** corresponds to $[2,0]$ in Dynkin labels, the tensor product to two such representations is

$$[2,0]\otimes[2,0] = [0,0]\oplus[2,0]\oplus[4,0]\oplus[0,4]\oplus[0,2]\oplus[2,2]\,, \tag{0.7}$$

where the first 4 are symmetric and the last 2 are anti-symmetric representations.

The function $G(z,\bar{z},\alpha;\bar{\alpha})$ can be written in several ways. First there is the conformal block decomposition:

$$G(z,\bar{z};\alpha,\bar{\alpha}) = \sum_{r\in R}\left(Y^r(\alpha,\bar{\alpha}) \sum_{k_r} \lambda_{k_r}^2 \mathcal{G}_{\Delta_{k_r}}^{(\ell_{k_r})}(z,\bar{z})\right)\,. \tag{0.8}$$

Here $r \in R = \{[0,0],[2,0],[4,0],[0,4],[0,2],[2,2]\}$ and

$$\begin{aligned}
Y^{[4,0]}(\alpha,\bar{\alpha}) &= \sigma^2+\tau^2+4\sigma\tau-\frac{8(\sigma+\tau)}{9}+\frac{8}{63},\\
Y^{[2,2]}(\alpha,\bar{\alpha}) &= \sigma^2-\tau^2-\frac{4(\sigma-\tau)}{7}\\
Y^{[0,4]}(\alpha,\bar{\alpha}) &= \sigma^2+\tau^2-2\sigma\tau-\frac{2(\sigma+\tau)}{3}+\frac{1}{6}\\
Y^{[0,2]}(\alpha,\bar{\alpha}) &= \sigma-\tau\\
Y^{[2,0]}(\alpha,\bar{\alpha}) &= \sigma+\tau-\frac{2}{5}\\
Y^{[0,0]}(\alpha,\bar{\alpha}) &= 1
\end{aligned} \tag{0.9}$$

with $\sigma = \alpha\bar{\alpha}$ and $\tau = (\alpha-1)(\bar{\alpha}-1)$. In addition

$$\begin{aligned}
\mathcal{G}_\Delta^{(\ell)}(\Delta_{12},\Delta_{34};z,\bar{z}) =& \mathcal{F}_{00} - \frac{\ell+3}{\ell+1}\mathcal{F}_{-11} + \frac{(\Delta-4)(\ell+3)}{16(\Delta-2)(\ell+1)}\\
& \frac{(\Delta-\ell-\Delta_{12}-4)(\Delta-\ell+\Delta_{12}-4)(\Delta-\ell+\Delta_{34}-4)(\Delta-\ell-\Delta_{34}-4)}{(\Delta-\ell-5)(\Delta-\ell-4)^2(\Delta-\ell-3)}\mathcal{F}_{02}\\
& -\frac{\Delta-4}{\Delta-2}\frac{(\Delta+\ell-\Delta_{12})(\Delta+\ell+\Delta_{12})(\Delta+\ell+\Delta_{34})(\Delta+\ell-\Delta_{34})}{16(\Delta+\ell-1)(\Delta+\ell)^2(\Delta+\ell+1)}\mathcal{F}_{11}\\
& +\frac{2(\Delta-4)(\ell+3)\Delta_{12}\Delta_{34}}{(\Delta+\ell)(\Delta+\ell-2)(\Delta+\ell-4)(\Delta+\ell-6)}\mathcal{F}_{01}
\end{aligned} \tag{0.10}$$

where

$$
\begin{aligned}
\mathcal{F}_{nm}(z,\bar z) =& \frac{(z\bar z)^{\frac{\Delta-\ell}{2}}}{(z-\bar z)^3}\left(\left(-\frac{z}{2}\right)^{\ell} z^{n+3}\bar z^{m}{}_2F_1\left(\frac{\Delta+\ell-\Delta_{12}}{2}+n,\frac{\Delta+\ell+\Delta_{34}}{2}+n,\Delta+\ell+2n,z\right)\right.\\
&\left.{}_2F_1\left(\frac{\Delta-\ell-\Delta_{12}}{2}-3+m,\frac{\Delta-\ell+\Delta_{34}}{2}-3+m,\Delta-\ell-6+2m,\bar z\right)-(z\longleftrightarrow\bar z)\right)\,.
\end{aligned}
\tag{0.11}
$$

Yet another way of writing the function $G(z,\bar z,\alpha;\bar\alpha)$ is

$$
G(z,\bar z,\alpha;\bar\alpha) \;=\; (z\bar z)^4\Delta_2[(z\alpha-1)(z\bar\alpha-1)(\bar z\alpha-1)(\bar z\bar\alpha-1)a(z,\bar z)]+z^2\bar z^2\mathcal{H}_1^{(2)}(z,\bar z;\alpha,\bar\alpha)
\tag{0.12}
$$

with

$$
\mathcal{H}_1^{(2)}(z,\bar z;\alpha,\bar\alpha) \;=\; D_2\left(\frac{(z\alpha-1)(z\bar\alpha-1)h(z)-(\bar z\alpha-1)(\bar z\bar\alpha-1)h(\bar z)}{z-\bar z}\right)\,,
\tag{0.13}
$$

and Δ_2, D_2 are some second order differential operators which aren't important for us. The function $h(z)$ is fixed to be (this seems to be as a result of the chiral ring)

$$
h(z) \;=\; -\beta_1\left(\frac{z^3}{3}-\frac{1}{z-1}-\frac{1}{(z-1)^2}-\frac{1}{3(z-1)^3}-\frac{1}{z}\right)-\beta_3\left(z-\frac{1}{z-1}+\log(1-z)\right)+\beta_5\,.
\tag{0.14}
$$

The constants β_1,β_3,β_5 are taken to be

$$
\beta_1 \;=\; 1\,,\qquad \beta_3 \;=\; \frac{8}{c}\,,\qquad \beta_5 \;=\; -\frac{1}{6}+\frac{8}{c}\,,
\tag{0.15}
$$

with c the central charge. The central charge is an input in the $(2,0)$ bootstrap and presumably will be for us too. When the gauge group of the $(2,0)$ theory is $SU(N)$, the central charge is given by

$$
c \;=\; 4N^3-3N-1\,.
\tag{0.16}
$$

The crossing constraints rather magically can be phrased only in terms of the function $a(z,\bar z)$ and are

$$
\begin{aligned}
a(z,\bar z) &= \frac{1}{(1-z)^5(1-\bar z)^5}a\left(\frac{z}{z-1},\frac{\bar z}{\bar z-1}\right)\\
z\bar z a(z,\bar z) &= (1-z)(1-\bar z)a(1-z,1-\bar z)+\mathcal{C}_h(1-z,1-\bar z)-\mathcal{C}_h(z,\bar z)\,,
\end{aligned}
\tag{0.17}
$$

where

$$
\mathcal{C}_h(z,\bar z) \;=\; \frac{1}{(z-\bar z)^3}\frac{h(z)-h(\bar z)}{z\bar z}
\tag{0.18}
$$

According to [1] the first equation in (0.17) is easily solved when reformulated in terms of conformal blocks. It is the second equation which is non-trivial and the one we need to apply machine learning to.

The function $a(z,\bar{z})$ can be split into an "atomic" form with contributions from each supermultiplet appearing in the self-OPE (0.2). Defining

$$a^{\text{at}}_{\Delta,\ell}(z,\bar{z}) = \frac{4}{z^6\bar{z}^6(\Delta-\ell-2)(\Delta+\ell+2)}\mathcal{G}^{(\ell)}_{\Delta+4}(0,-2;z,\bar{z})\,, \tag{0.19}$$

we decompose $a(z,\bar{z})$ as follows:

$$a(z,\bar{z}) \;=\; a^{\chi}(z,\bar{z}) + a^{u}(z,\bar{z})\,. \tag{0.20}$$

Here

$$a^{\chi}(z,\bar{z}) \;=\; \sum_{\substack{l=0,2,\dots \\ l \text{ even}}}^{\infty} 2^l b_l\, a^{\text{at}}_{\ell+4,\ell}(z,\bar{z}) \tag{0.21}$$

represents the known contribution (OPE coefficient and conformal dimensions) from the identity, $\mathcal{D}[2,0]$, $\mathcal{D}[4,0]$ and $\mathcal{B}[2,0]_{l-2}$ supermultiplets. The coefficients b_l are determined via the chiral algebra and read

$$\begin{aligned} b_\ell =& \frac{(\ell+1)(\ell+3)(\ell+2)^2\frac{\ell}{2}!\left(\frac{\ell}{2}+2\right)!!\left(\frac{\ell}{2}+3\right)!!(\ell+5)!!}{18(\ell+2)!!(2\ell+5)!!} \\ &+\frac{8}{c}\frac{\left(2^{-\frac{\ell}{2}-1}(\ell(\ell+7)+11)(\ell+3)!!\Gamma\left(\frac{\ell}{2}+2\right)\right)}{(2\ell+5)!!}\,. \end{aligned} \tag{0.22}$$

The remaining part contains unknown data namely the OPE coefficients for the $\mathcal{L}[0,0]_{\Delta,l}$, $\mathcal{B}[0,2]_{l-1}$ and $\mathcal{D}[0,4]$ multiplets and the conformal dimension of $\mathcal{L}[0,0]_{\Delta,l}$. It is given by

$$a^{u}(z,\bar{z}) \;=\; \sum_{\substack{\Delta\geq l+6 \\ l\geq 0,\, l \text{ even}}} \lambda^2_{\Delta,l} a^{at}_{\Delta,l}(z,\bar{z})\,. \tag{0.23}$$

The known conformal dimensions of the $\mathcal{B}[0,2]_{l-1}$ (with $l>0$) and $\mathcal{D}[0,4]$ multiplets are $l+7$ and 8 respectively (l is always an even integer). For us perhaps its better to write

$$a^{u}(z,\bar{z}) \;=(\lambda^{\mathcal{D}[0,4]}_{8,0})^2 a^{at}_{6,0}(z,\bar{z}) + \sum_{l=2,4,\dots}^{l_{\text{cutoff}}}(\lambda^{\mathcal{B}[0,2]_{l-1}}_{l+7,l})^2 a^{at}_{l+6,0}(z,\bar{z}) + \sum_{\substack{l=0,2,\dots \\ \Delta>l+6}}^{l_{\text{cutoff}}}(\lambda^{\mathcal{L}[0,0]_{\Delta_l,l}}_{\Delta_l,l})^2 a^{at}_{\Delta_l,l}(z,\bar{z}) \tag{0.24}$$

where $a^{\rm at}_{\Delta,\ell}(z,\bar{z})$ is as given in (0.19).

Let us define

$$c(z,\bar{z}) \;= z\bar{z}(a^u(z,\bar{z}) + a^\chi(z,\bar{z})) + \mathcal{C}_h(z,\bar{z})\,. \tag{0.25}$$

Then the non-trivial crossing equation can then be written as

$$0 \;= c(z,\bar{z}) - c(1-z,1-\bar{z})\,, \tag{0.26}$$

References

[1] C. Beem, M. Lemos, L. Rastelli and B. C. van Rees, *The (2, 0) superconformal bootstrap*, *Phys. Rev. D* **93** (2016) 025016, [1507.05637].

[2] M. Lemos, B. C. van Rees and X. Zhao, *Regge trajectories for the (2,0) theories*, 2105.13361.