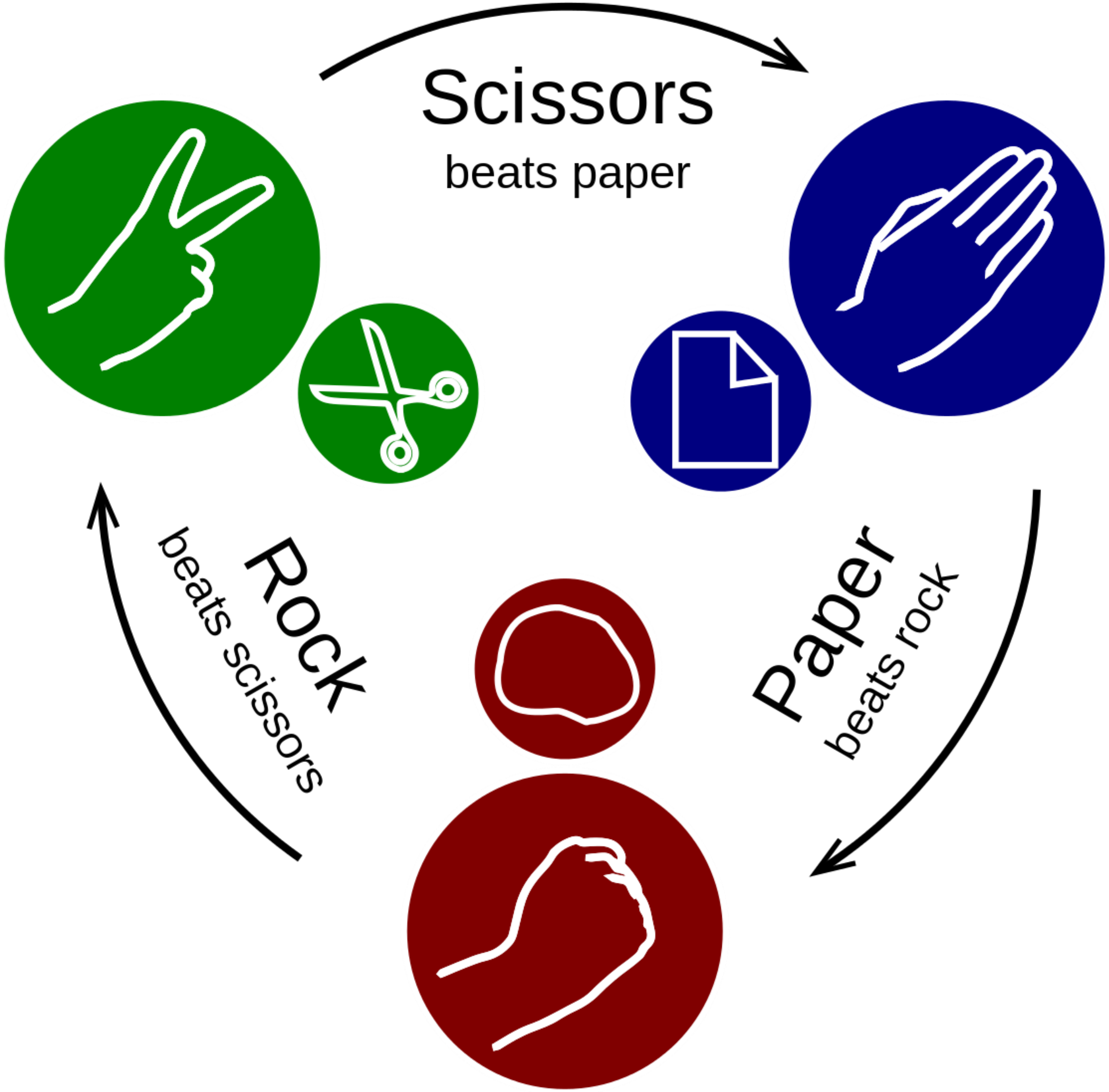
Rock-Paper-Scissor Population Dynamics

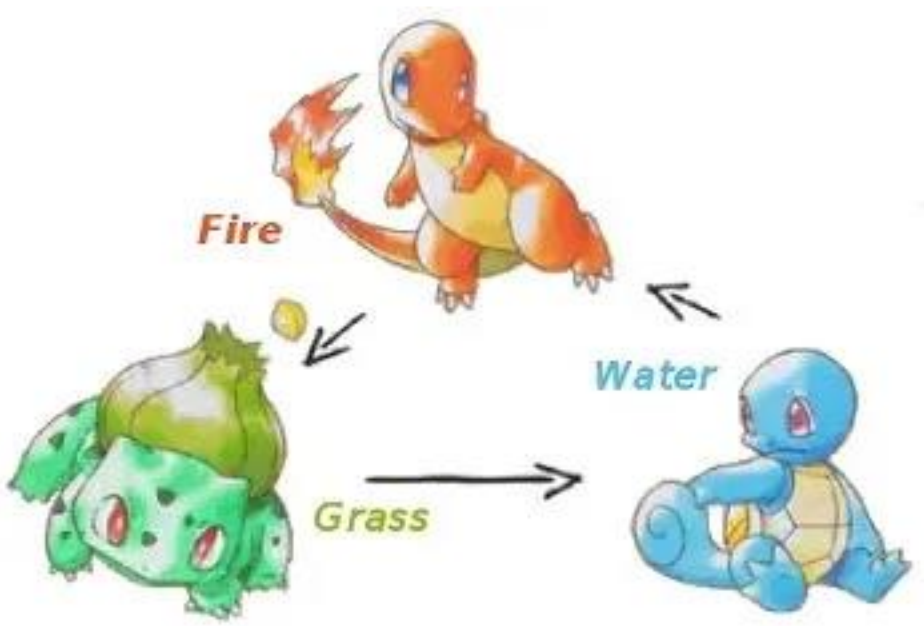


The Problem

- Three populations, r, p, s each one is strong against another, as in the game rock-paper-scissor.

- N sites are available for the three populations.

- Competition for space.



The Model

- No spontaneous death and births
- Each one of the three species has an invasion rate $P_r, P_p, P_s \in [0,1]$.
- When an individual meet another of the weaker species, it predates it with probability P_i.
- The probability of meeting is equal for each pair of individuals.
- We call n_r, n_p, n_s the densities of the populations, and we can normalize them such that $n_r + n_p + n_s = 1$

Mean-field approximation

- As $N \to \infty$, the evolution is well represented by ordinary differential equations (ODEs).

- With the model just described, we can suppose that the rate of encounter of two populations is proportional to their densities

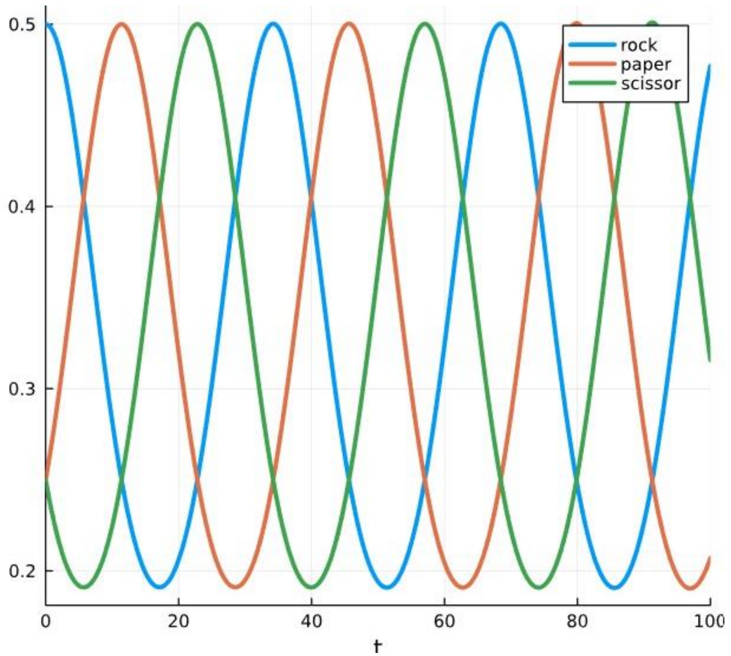
- Rate of change for r: $\frac{dn_r}{d_t} = n_r(P_r n_s - P_p n_p)$

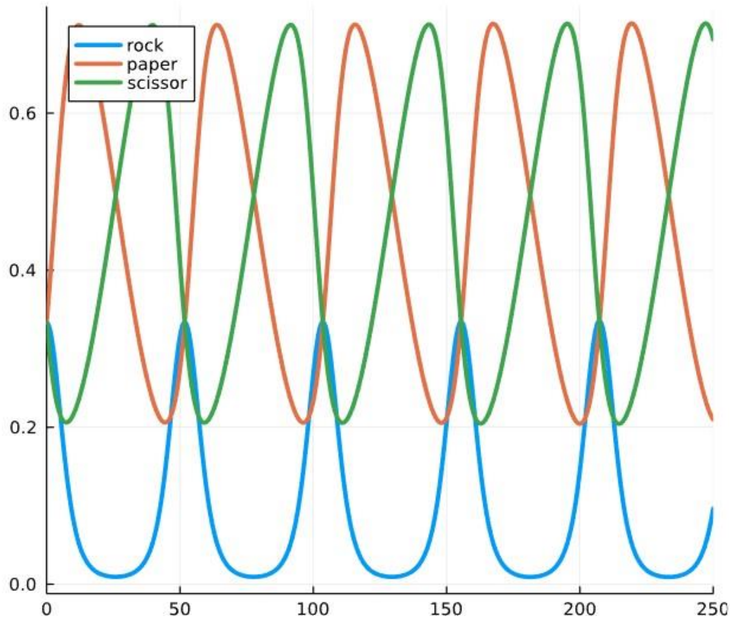
- The other are analogous

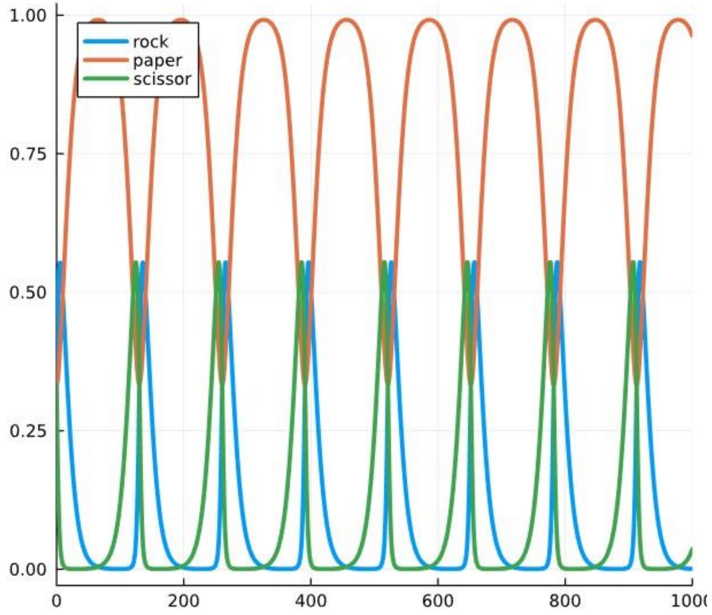
Equilibrium points

- Obviously, there is equilibrium whenever just one species is alive.
- The other equilibrium condition can be obtained by putting $\frac{dn_r}{dt} = \frac{dn_p}{dt} = \frac{dn_s}{dt} = 0$, and we obtain: $n_r = \alpha P_s, n_p = \alpha P_r, n_s = \alpha P_p$, where α is just a normalizing factor.
- The equilibrium point of a species does not depend on its own invasion rate, but rather on the one of the species they invade.
- Therefore, the most aggressive species does not have the highest population at equilibrium.

ODE simulations







$P_r = P_p = P_s$
started from
$(r_0, p_0, s_0) = (0.5, 0.25, 0.25)$

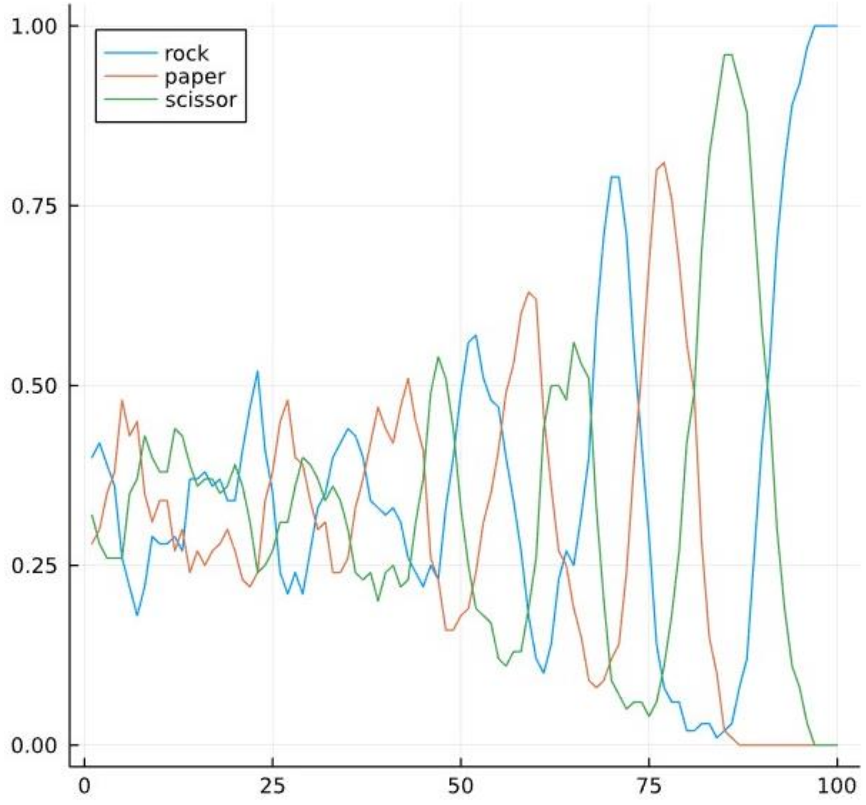
$P_r = P_p = 0.45, P_s = 0.1$

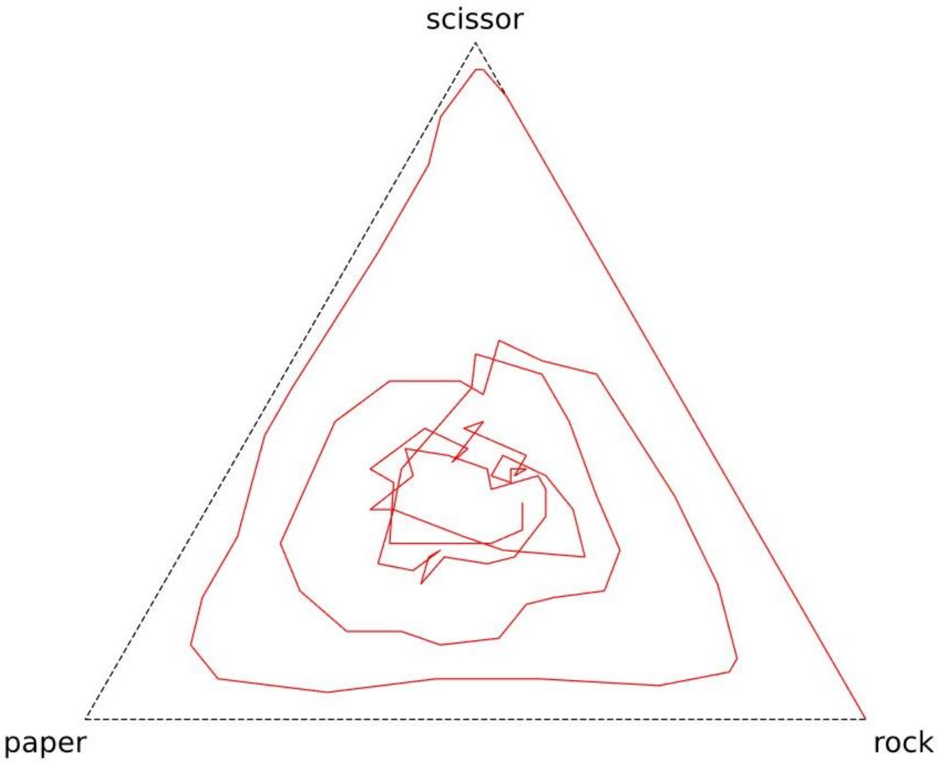
$P_r = 0.8, P_p = P_s = 0.1$

Simulations for a finite N

- The mean-field equation holds for very large N.
- If N is large the simulations seem to follow the ODE,
- For small N, after some epochs (where an epoch is N time steps), only one population will survive.
- It is mathematically proved that extinction will occur in any case.

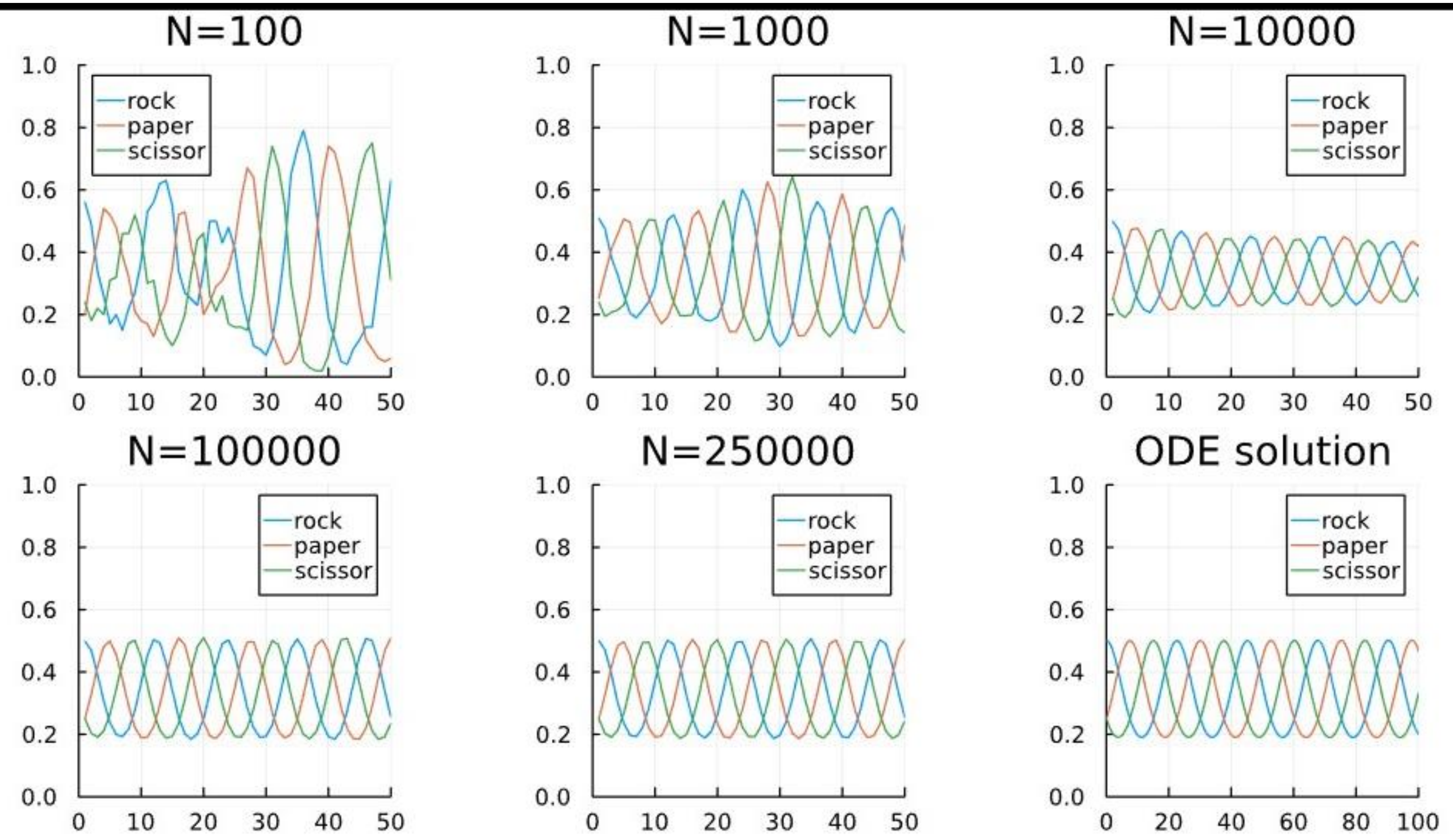
Case $\boldsymbol{P_r = P_p = P_s, N = 100}$





$N = 100$. Even if we started from an initial population near to the equilibrium configuration, eventually only r survived.

Case $P_r = P_p = P_s$, comparison



If N is large enough, the simulation is similar to the ODE. Otherwise, the noise takes over and eventually leads to extinction.

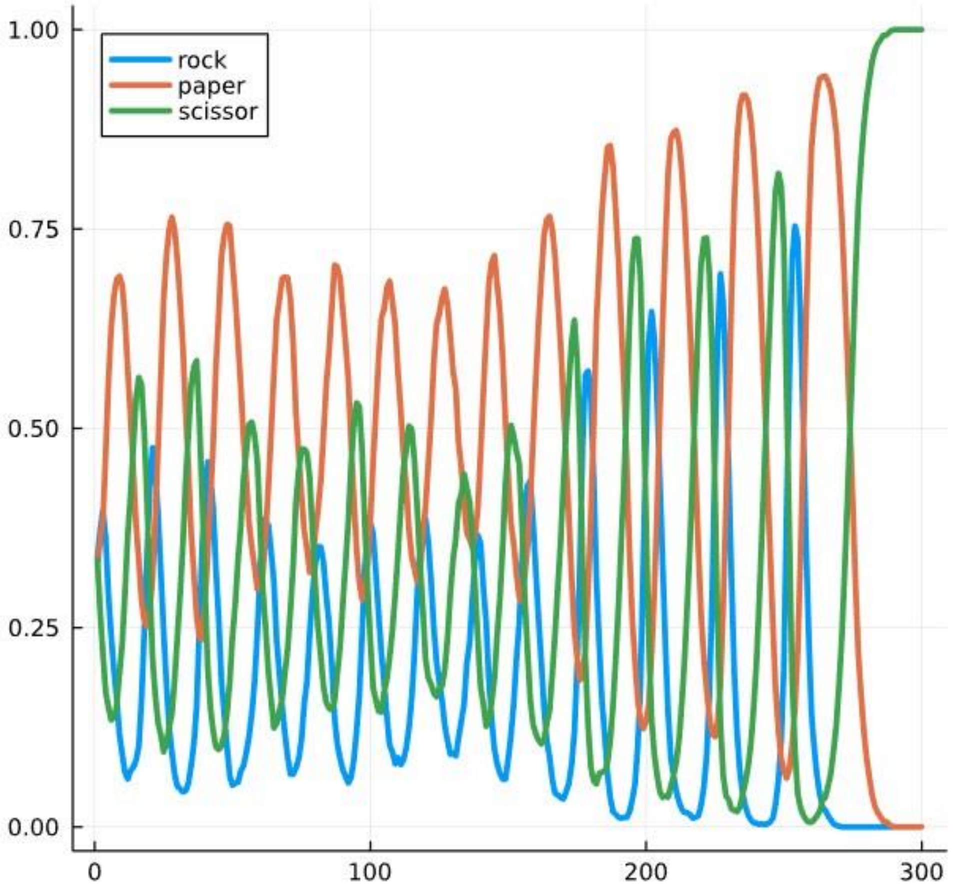
General case: $\boldsymbol{P_r \geq P_p \geq P_s}$

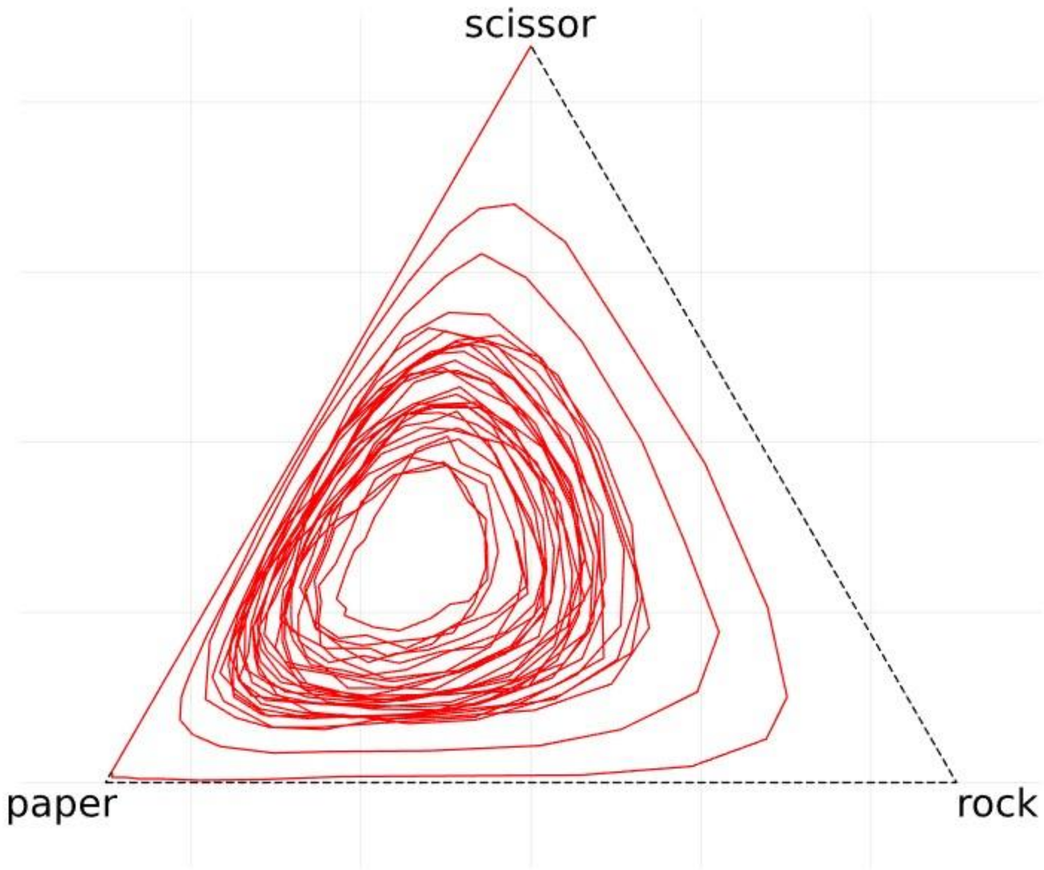
- The equilibrium points are $n_p \geq n_s \geq n_r$.

- The species oscillate around the equilibrium, therefore the species that will go extint more often will be r.

- In this case, the only species that survives is s, that is the weakest species in terms of aggressivity.

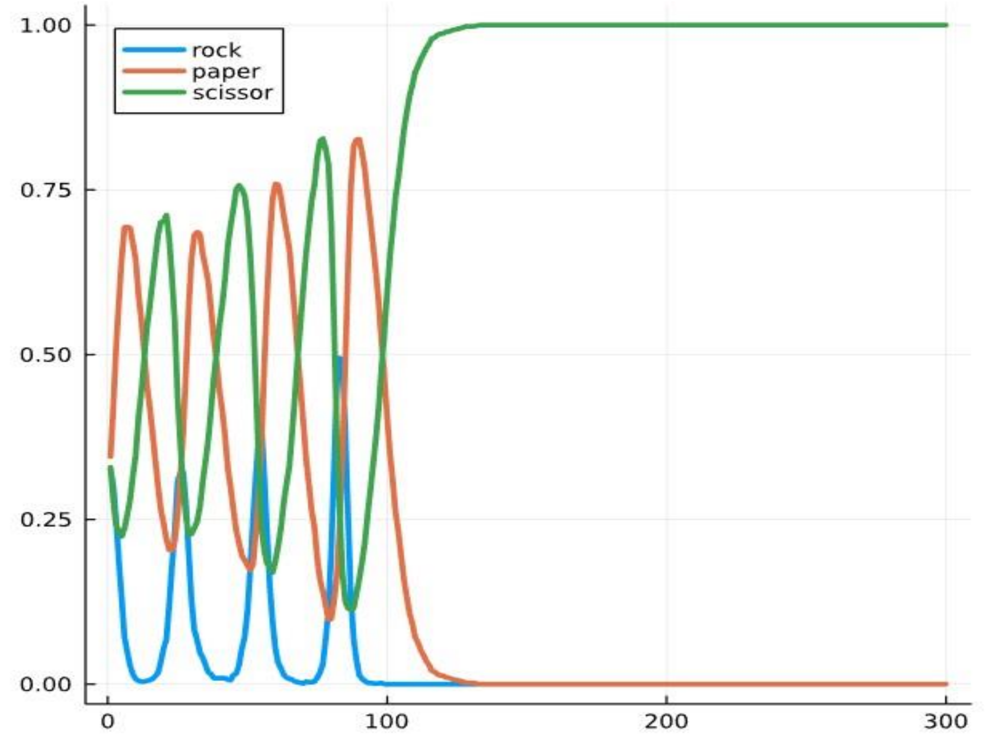
Case $\boldsymbol{P_r > P_p > P_s}$

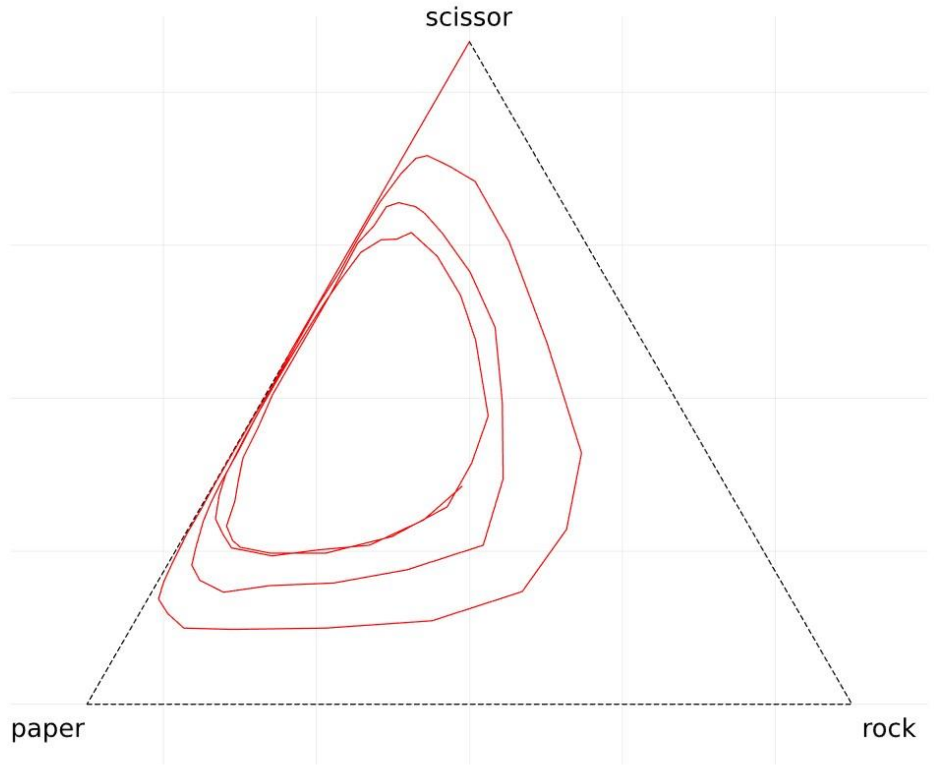




Case $\boldsymbol{P_r = P_p > P_s}$

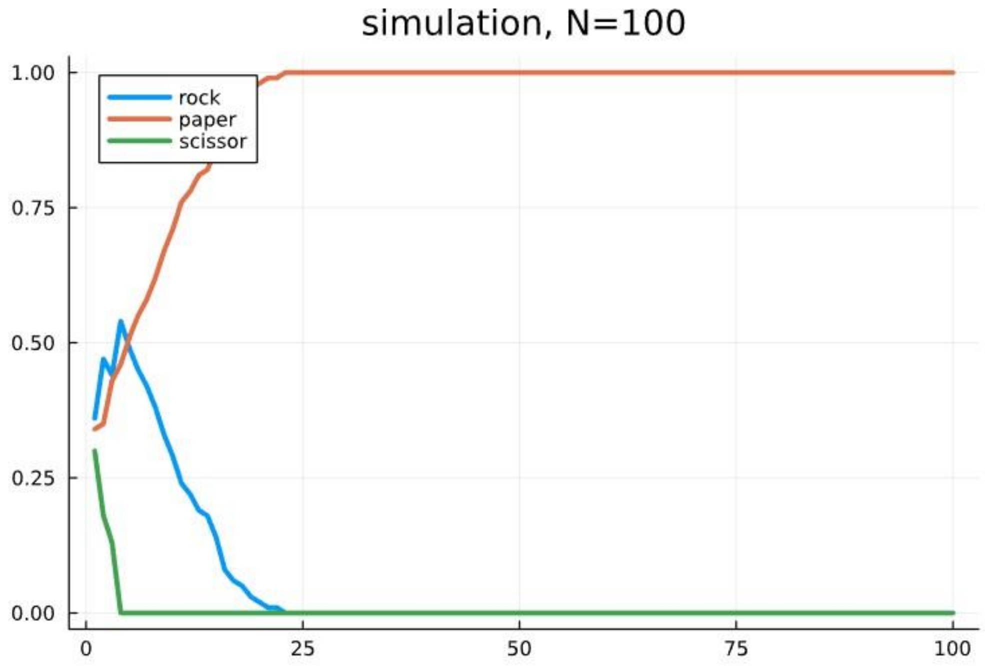
- At the equilibrium point of the ODE system, $n_p = n_s > n_r$
- Again, s is the species that survives more often.

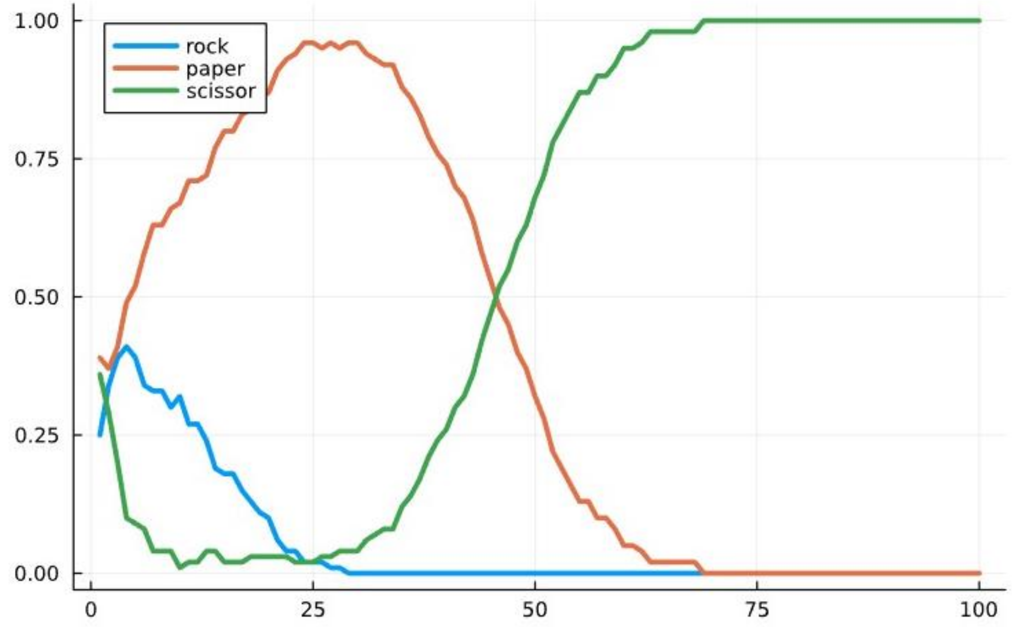




Case $\boldsymbol{P_r > P_p = P_s}$

In this case $n_p > n_s = n_r$. If s goes exctint first, only p survives. If otherwise r goes extinct, only s survives. Either way is one of the weaker species that survives.

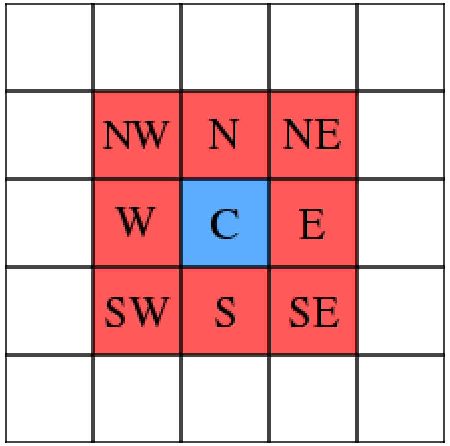




Local interactions

- We can imagine also local interactions rather than global ones.

- We can observe that the dynamics change a lot:
  - Local extinctions
  - Clusterization of the populations



Moore neighborhood

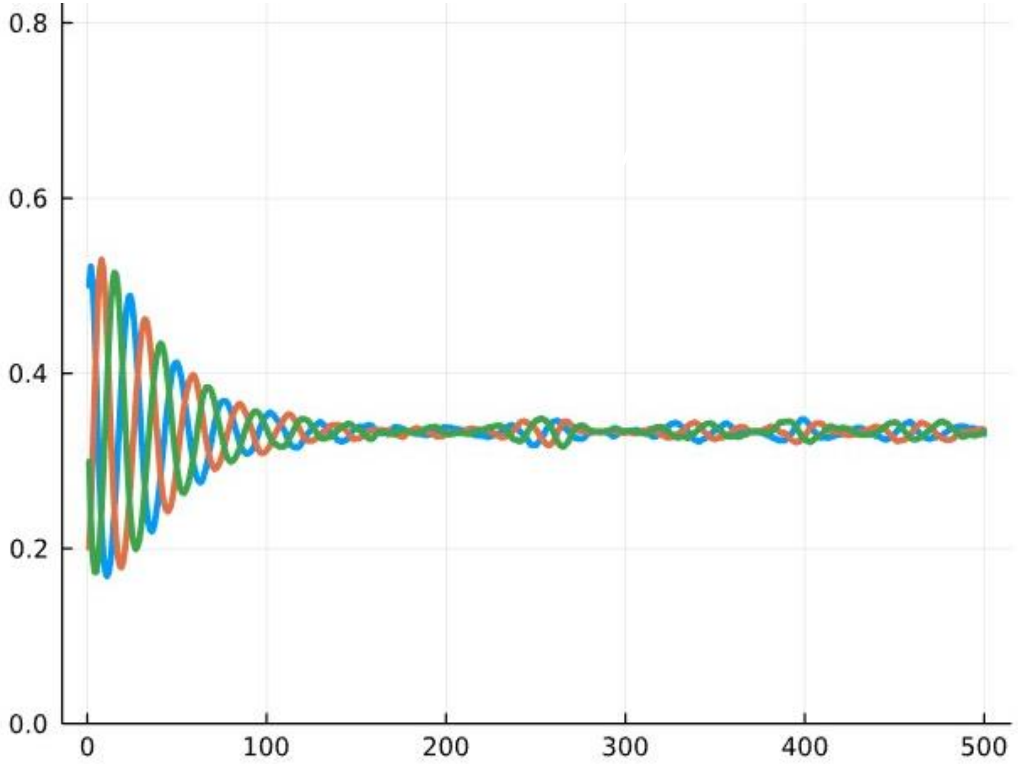
- Equilibrium in terms of population densities is reached, and it is not distant to the theoretical one

Simulations on a $\mathbf{500 \times 500}$ grid

- We used a 500×500 square lattice for the simulations
- Two individuals can interact only if they are neighbor in the environment (we used a Moore Neighborhood)
- Asynchronous updates, one per each time-step

$P_r = P_p = P_s$

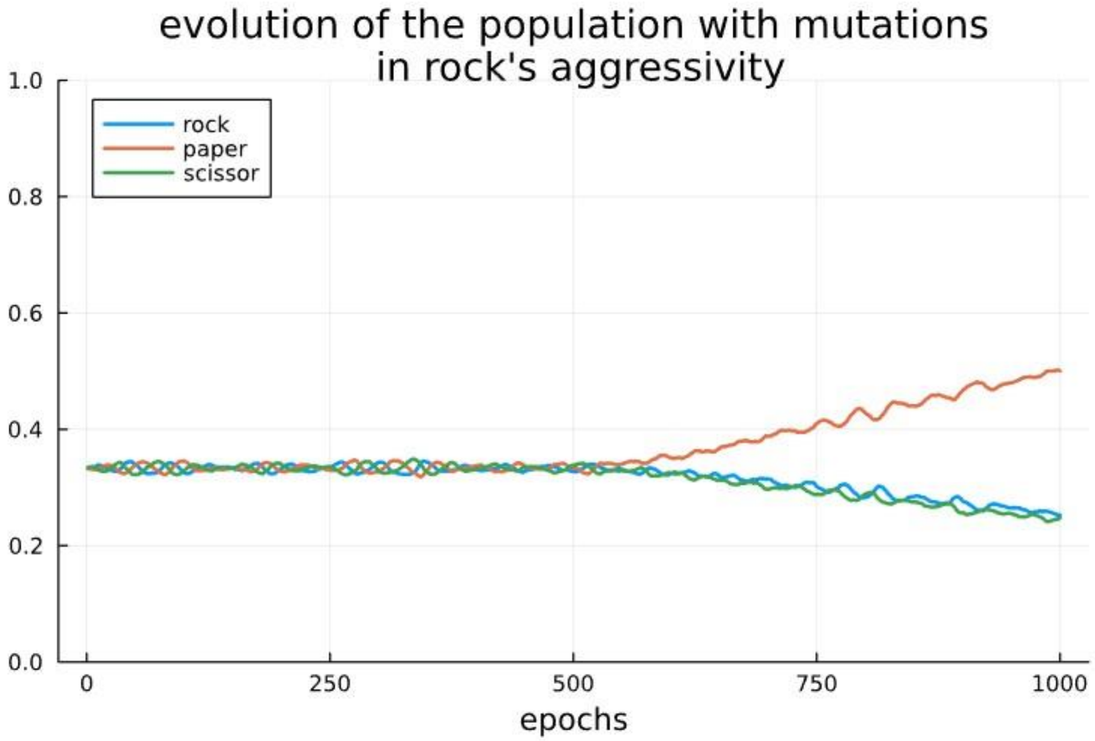
In this case the populations densities seem to converge to the same point, and we see clusters of small size

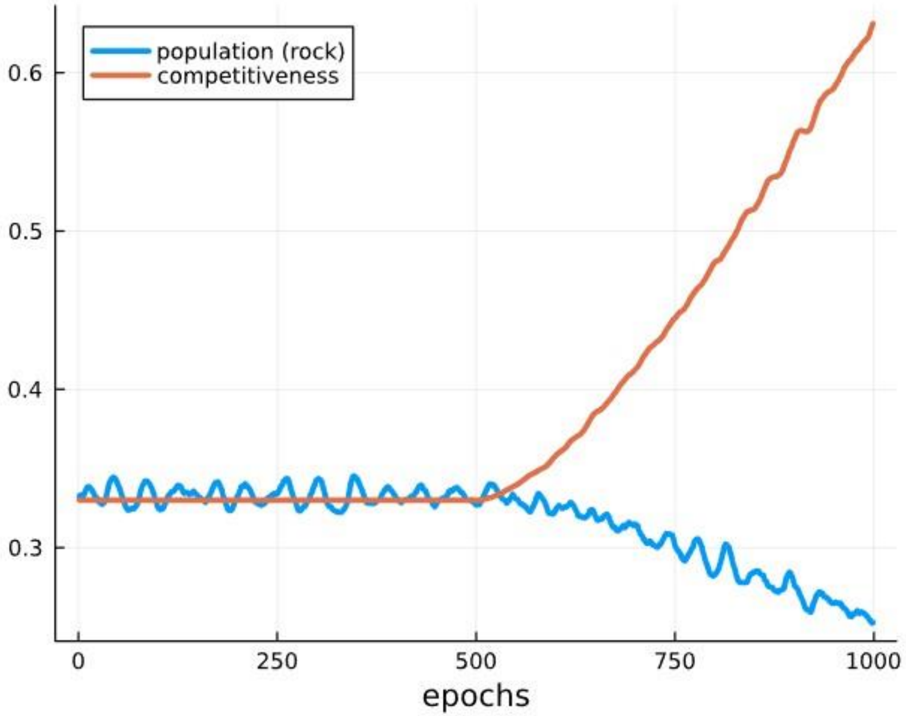


Mutations

- We can add mutations on the new individuals: in particular we changed the invasion rate over time.
- We do that by adding random mutations to newborns starting from the invasion rate of the parent
- This way, the stronger individuals reproduce faster, and the mean invasion rate of the population will surely increase.
- The effect of this is that the size of the population that is subject to mutations decreases over time.

Simulation with mutations



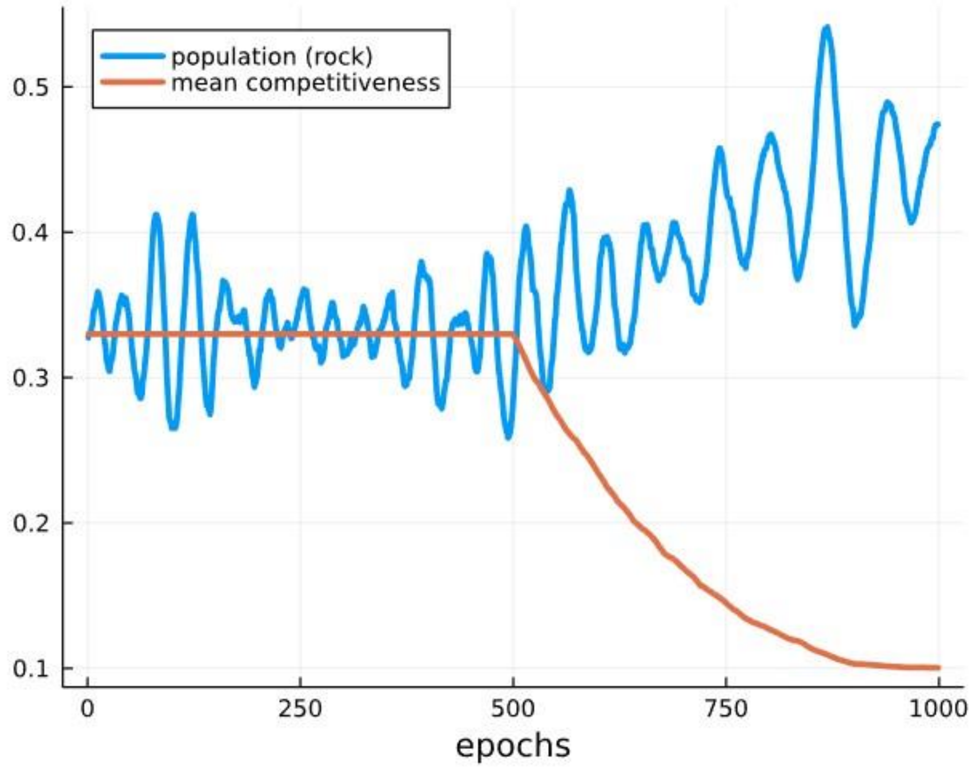


Simulation with mutations: after 500 epochs where the invasion rates were all equat to 1/3, we added to each newborn of rock type's invasion rate a random number sampled from a gaussian with mean 0 and standard deviation 10^{-2}.

Disease effect

- Instead of random mutations, we can think of a disease that weakens one population, lowering its competitiveness.

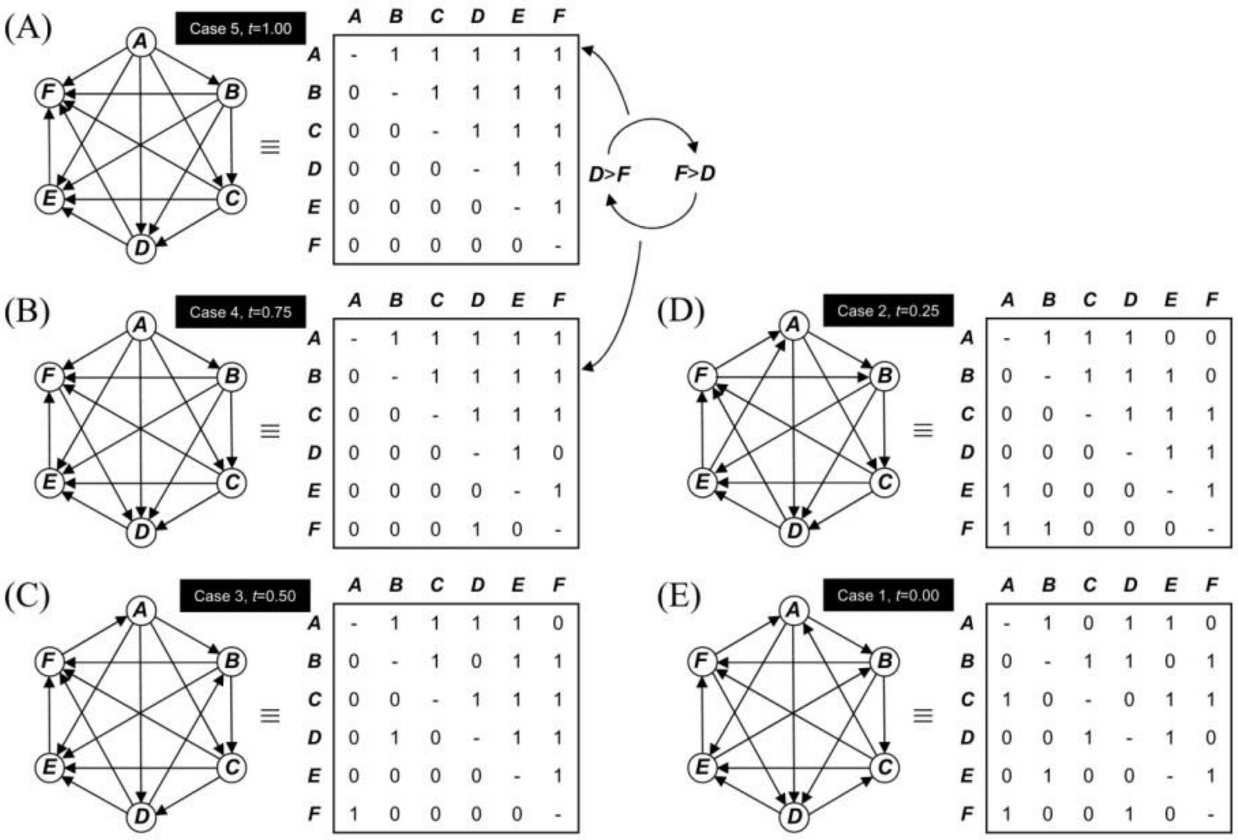
- The paradoxal effect is that, after the disease, the population would increase in size



Generalizations

- The schema could of course be extended for m types of individuals, with various balance of power between them.

- It was find out that the more the relationships are intransitive, the more diversity in the ecosystem.



Conclusions

- Survival of the fittest $\neq$survival of the strongest.
- Competitive intransitiveness may lead to counter-intuitive results.
- These results could explain how biodiversity is mantained: we’ve seen that is often possible to be achieve dynamic equilibrium.
- The complexity of nature could allow such intransitive behaviours.