GRAPHS WITH 2-COLORABILITY DEFECT EQUAL TO THE CHROMATIC NUMBER OF THEIR KNESER GRAPHS

1. Basic terminology in graph theory

A *simple graph* G is an ordered pair (V, E) of finite sets such that $E \subseteq \binom{V}{2}$, i.e., such that the elements of E are pairs of distinct elements of V. The elements of V are *vertices* and those of E are *edges*. An edge is usually written as uv, where $\{u, v\}$ is the corresponding pair of vertices.

A *matching* is a subset M of E such that no two edges in M share a vertex. A *perfect matching* is matching covering all vertices.

A *proper coloring* of G is a map $c : V \to \mathbb{Z}_+$ such that $c(u) \neq c(v)$ whenever $uv \in E$. The integers in $c(V)$ are *colors*. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colors required for the existence of a proper coloring. In other words, $\chi(G) = \min_c |c(V)|$, where the minium is taken over all possible proper colorings c of G. If $\chi(G) \leq k$, then G is *k-colorable*. The 2-colorable graphs are also called *bipartite*.

The complete bipartite graph $K_{m,n}$ is the bipartite graph whose vertices can be partitioned into two sets A and B, with $|A| = m$ and $|B| = n$, so that its edges are precisely all pairs uv with $u \in A$ and $v \in B$. The sets A and B are the *sides* of the bipartite graph.

The graph $K^*_{t,t}$ is the complete bipartite graph $K_{t,t}$ from which a perfect matching has been removed.

2. Context

The 2-*colorability defect* of G, denoted by $\mathrm{cd}_2(G)$, is the minimum number of vertices to be removed so that the remaining graph is 2-colorable. Equivalently,

$$\mathrm{cd}_2(G) = \min\left\{|X| \colon \left(V \setminus X, E \cap \binom{V \setminus X}{2}\right) \text{ is 2-colorable}\right\}.$$

The *Kneser graph associated with* G, denoted by $\mathrm{KG}(G)$, is the simple graph whose vertices are the edges of G and whose edges connect vertices that correspond to disjoint edges of G. Still denoting $G = (V, E)$, the edge set of $\mathrm{KG}(G)$ can thus be written as

$$\{ef \colon e, f \in E \text{ and } e \cap f = \varnothing\}.$$

Notice that a proper coloring of $\mathrm{KG}(G)$ is an assignment of colors to the edges of G such that any pair of edges with the same color shares a vertex.

A special case of a theorem by Dol'nikov [2] is the following inequality

$$\chi(\mathrm{KG}(G)) \geq \mathrm{cd}_2(G). \tag{1}$$

The traditional proof of Dol'nikov's theorem relies on advanced algebraic topology but the inequality (1) can be proved by elementary arguments.

Suppose now that G is such that $\chi(\mathrm{KG}(G)) = \mathrm{cd}_2(G)$. Then, according to a theorem by Alishahi, Hajiabolhassan, and Meunier [1], any proper coloring of $\mathrm{KG}(G)$ with a minimum number of colors contains a $K^*_{\mathrm{cd}_2(G),\mathrm{cd}_2(G)}$ with all colors on each side.

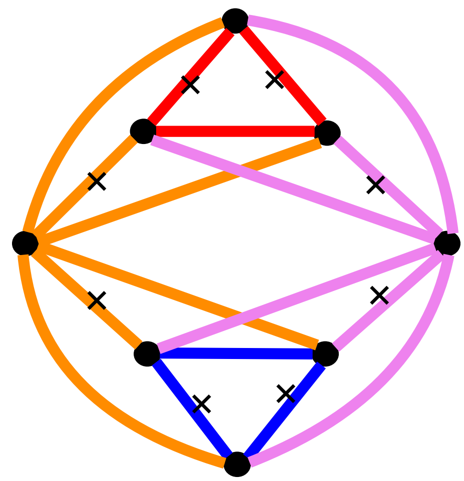

FIGURE 1. Its associated Kneser graph has $K^*_{4,4}$ with all 4 colors on each side.

Figure 1 shows a graph G, with a coloring of its edges that is a proper coloring of the associated Kneser graph. For this graph G, we have $\chi(\mathrm{KG}(G)) = \mathrm{cd}_2(G) = 4$ and the proper coloring presented here uses exactly 4 colors. The edges marked with a small $\times$ are the vertices of a $K^*_{4,4}$ with all the 4 colors on each side.

3. Topic

A characterization of the simple graphs G such that $\chi(\mathrm{KG}(G)) = \mathrm{cd}_2(G)$ is not known. The main objective of the project is to decide whether there is such a characterization. Ultimately, this should clarify the complexity status (existence of a polynomial algorithm vs NP-completeness) of deciding whether G satisfies the above equality.

It would also be interesting to know – and this is probably much easier – whether there is an elementary proof of the existence of the "colorful" $K^*_{\mathrm{cd}_2(G),\mathrm{cd}_2(G)}$ in this case.

References

[1] M. Alishahi, H. Hajiabolhassan, and F. Meunier. Strengthening topological colorful results for graphs. *European Journal of Combinatorics*, 64, 27-44, 2017. `http:https://arxiv.org/pdf/1606.02544v3.pdf`.

[2] V. L. Dolnikov. A certain combinatorial inequality. *Siberian Mathematics Journal*, 29, 375397, 1988