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RAPPORT DE STAGE DE RECHERCHE

Effectué au laboratoire : **CERMICS**
Centre d'Enseignement et de Recherche en MathématIques et Calcul Scientifique

COLORINGS OF KNESER GRAPHS

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Introduction

Présentation du sujet

Le sujet du stage concerne les graphes et hypergraphes, plus précisemment les colorations d'un certain type de graphe et hypergraphe.
En 1950 le mathématicien Kneser introduit une famille particulière d'hypergraphes inspirée des parties à k éléments dans $1, 2, \ldots, n$.
Plus généralement, Kneser introduit les graphes dits "de Kneser" qui se construisent de la façon suivante. A partir d'un premier graphe G, le graphe de Kneser associé à G que l'on note $\mathrm{KG}(G)$ a pour sommmet les arêtes de G et deux sommets sont reliés dans $\mathrm{KG}(G)$ si les arêtes associées dans G sont disjointes, c'est à dire qu'elles n'ont pas de sommet en commun. Dans le cas des hypergraphes du paragraphe précédent, les sommets sont les entiers $1, 2, \ldots, n$ et les arêtes sont les parties à k éléments (qui se représentent par un trait reliant les k entiers dans cette partie) ; ainsi deux arêtes sont disjointes si les parties sont d'intersection vide.
Le mathématicien Dol'nikov a ensuite démontré l'inégalité suivante $\chi(\mathrm{KG}(G)) \geqslant \mathrm{cd}_2(G)$ qui signifie que pour n'importe quel graphe G, le nombre chromatique du graphe de Kneser associé est toujours plus grand que le nombre minimal de sommet qu'il faut retirer à G pour qu'il soit colorable proprement avec deux couleurs. Le premier objectif du stage était de démontrer cette inégalité avec des arguments élémentaires, c'est-à-dire sans utiliser de résultat préalable qui nécessite une profonde connaissance des mathématiques.
Trouver une telle démonstration a comme intérêt de pouvoir donner une idée de la complexité algorithmique de vérifier si un graphe quelconque satisfait le cas d'égalité dans l'inégalité. Ces travaux ainsi que d'autres sont présentés en détail dans la suite.

La motivation de la recherche

Pour monsieur Meunier, cette recherche était motivée par un résultat démontré dans son article 2.5. Ce résultat stipule que les graphes qui vérifient le cas d'égalité dans l'inégalité de Dol'nikov possèdent des propriétés remarquables comme l'existence d'un sous graphe bipartie du graphe de Kneser avec toutes les couleurs de chaque côté.

En revanche, comme ce résultat découle d'une étude plus générale, monsieur Meunier n'avait pas de preuve élémentaire de cette existence ni de méthode pour trouver ce sous graphe. Ma recherche avait donc aussi pour objectif d'éclaircir la situation vis à vis de la complexité et si possible d'apporter une réponse complète (comme un algorithme pour caractériser les graphes par exemple).
D'autre part, j'ai été amené à répondre à une question posée par un autre chercheur, monsieur Gabor Simonyi, qui travaille lui aussi sur la théorie des graphes. Monsieur Simonyi a proposé une méthode pour générer des colorations optimales des graphes de Kneser mais il n'était pas clair si cette méthode permet de générer toutes les colorations ou seulement une partie. Ma recherche a répondu à cette question en montrant l'existence d'autre colorations.
Finalement, l'intérêt de mon travail était aussi de mettre par écrit un certain nombre de résultats dont la plupart des experts avaient l'intuition mais sans pour autant disposer de la preuve écrite. La règle étant que si un résultat n'est pas explicitement démontré avec une preuve revue et approuvée par la communauté scientifique alors on ne peut s'en servir comme tel dans d'autre travaux. Il est donc essentiel de conserver une trace écrite même des petits résultats. D'une certaine façon mon travail permetrait à d'autre chercheurs de ne pas perdre de temps à redémontrer les résultats dont j'ai rédigé les preuves.

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1 Dol'nikov's inequality for graphs

In 1988, Dol'nikov showed, in a theorem, an inequality concerning hypergraphs using advanced algebraic topology; but its application to graphs can be proven by elementary arguments. The aim of this paper is to provide an elementary proof of Dol'nikov's inequality for graphs and some insight on the cases of equality.

1.1 Introduction

Let $G = (V, E)$ be a simple graph (in the rest of this article every graph is considered simple unless mentionned otherwise). The *Kneser graph associated with* G, denoted by $\mathrm{KG}(G)$, is the simple graph whose vertices are the edges of G and whose edges connect vertices that correspond to disjoint edges of G. The set of edges of $\mathrm{KG}(G)$ can be written as

$$\{ef : e, f \in E \text{ and } e \cap f = \varnothing\}.$$

A *proper coloring* of G is a map $c : V \to \mathbb{Z}_+$ such that $c(u) \neq c(v)$ whenever $uv \in E$. The integers in $c(V)$ are *colors*. The *chromatic number* of G, denoted by $\chi(G)$, is the minimum number of colors required for the existence of a proper coloring. When considering Kneser graphs, the chromatic number $\chi(\mathrm{KG}(G))$ will be denoted $\overline{\chi}'(G)$. If $\overline{\chi}'(G) \leqslant k$, then $\mathrm{KG}(G)$ is *k-colorable*. An *optimal coloring* is a proper coloring that uses the minimum number of colors. Graphs that are 2-colorable are also called *bipartite*.

The *2-colorability defect* of G, denoted by $\mathrm{cd}_2(G)$, is the minimum number of vertices to be deleted so that the remaining graph is 2-colorable. Equivalently,

$$\mathrm{cd}_2(G) = \min\{|X| : G[V \setminus X] \text{ is 2-colorable}\}.$$

A special case of a theorem by Dol'nikov can now be presented as the following theorem.

Theorem (Dol'nikov's theorem)**.** *For any simple graph G, the following inequality holds :* $\overline{\chi}'(G) \geqslant \mathrm{cd}_2(G)$.

More generally, Dol'nikov's theorem holds for hypergraphs. The chromatic number and the 2-colorability defect of hypergraphs are defined the same way as they are for graphs. This article provides an elementary proof of Dol'nikov's theorem for graphs in Section 1.2, then it studies more specifically in Section 1.3, the graphs that satisfy the case of equality. Section 1.3.2 presents an improved version of Dol'nikov's theorem for graphs with a girth greater than three. Section 1.4 is a solution to an exercise related to Dol'nikov's inequality given by J. Matoušek. Finally in Appendix A some definitions are given to clarify the notions used in this article.

1.2 Elementary proof of Dol'nikov's theorem in the case of graphs

With elementary arguments it is possible to prove a slightly stronger version of Dol'nikov's theorem in the case of graphs. It can be formulated as follows.

Theorem 1.2.1. *Let G be a graph with edges colored with k colors so that all edges of the same color pairwise intersect. It is possible to delete at most k vertices so that the remaining graph is a forest.*

A forest being bipartite, Dol'nikov's inequality is immediately obtained by applying this theorem to G with an optimal coloring of its Kneser graph.

It is interesting to note the following remark as it is used several time in the article, without further mention.

Remark 1.2.2. If c is an edge-coloring of the graph $G = (V, E)$ such that all edges of the same color pairwise intersect, then every monochromatic subgraph is a star or a triangle.

This remark is shown in Appendix B. Considering this remark, the following notation is introduced. An ST-*coloring* of a graph G is a partition of its edges into stars and triangles. The minimal t such that there exists a partition of $E(G)$ into t parts, each of them being either a star or a triangle is $\overline{\chi}'(G)$.

Proof of Theorem 1.2.1. Let $G = (V, E)$ be a graph and c an edge-coloring such that all edges of the same color pairwise intersect. Let $k = |c(E)|$ be the number of colors of the coloring. The proof works by induction on k.

Consider the case $k = 1$. The graph is either a triangle in which case only one vertex needs to be deleted in order for the graph to become a tree, or a star which is already a tree itself. This shows the theorem for $k = 1$.

Consider then the case $k \geqslant 2$. Suppose there exists a circuit C of length l in G with no two edges of the same color (such a circuit is a *rainbow circuit*). By deleting every vertex of the circuit, the remaining graph contains no edge of a color contained in C. Therefore graph $G' = G[V \setminus \mathrm{V}(C)]$ has $k - l$ colors and by the induction hypothesis at most $k - l$ vertices need to be deleted in order to obtain a forest. This proves that at most k vertices in total need to be deleted in order to obtain a forest.

Suppose G does not contain a rainbow circuit. By deleting the center of each star and one vertex per triangle (k vertices in total) at most one edge of each color remains. The remaining graph is a forest since otherwise it would contain a rainbow circuit which is impossible. This proves by induction that only k vertices need to be deleted in order to obtain a forest. □

1.3 A characterization of the cases of equality

It is known that for most graphs Dol'nikov's inequality is strict and usually by a significant margin. However the graphs that satisfy the case of equality have remarkable properties such as shown by M. Alishahi, H. Hajiabolhassan, and F. Meunier [2]. This section provides several conditions concerning the graphs that satisfy the case of equality.

1.3.1 Necessary conditions

The following lemma is interesting to note as it is use in several proofs later on. It provides a way of reducing the size of a graph while maintaining the property of the case of equality.

Lemma 1.3.1. *Let G be a graph that satisfies* $\overline{\chi}'(G) = \mathrm{cd}_2(G)$. *Supposed that G is optimally edge-colored. By deleting a monochromatic star from G, the remaining graph H also satisfies* $\overline{\chi}'(H) = \mathrm{cd}_2(H)$.

Proof. Let c be an optimal coloring of G. Let H be the remaining graph after a monochromatic star has been deleted from G. The coloring c' induced by c of H uses one color less than c and thus $\overline{\chi}'(H) \leqslant \overline{\chi}'(G) - 1$. As G has only one vertex more, we have $\mathrm{cd}_2(G) \leqslant \mathrm{cd}_2(H) + 1$. Combined with Dol'nikov's indequality it implies that $\overline{\chi}(G) = \mathrm{cd}_2(G)$. □

Necessary conditions for a graph to satisfy the case of equality in Dol'nikov's theorem can be expressed in the following theorem.

Theorem 1.3.2. *Let G be a graph that satisfies* $\overline{\chi}(G) = \mathrm{cd}_2(G)$. *Suppose G is optimally edge-colored.*

- *G contains at least one monochromatic triangle, all monochromatic triangles are vertex disjoint.*
- *No vertex of a monochromatic triangle is the center of a monochromatic star.*
- *Every monochromatic star covers all the vertices of at least one monochromatic triangle.*

Proof. If G does not contain a triangle, repeated applications of Lemma 1.3.1 show that G does not satisfy $\overline{\chi}'(G) = \mathrm{cd}_2(G)$.

Again, the application of Lemma 1.3.1 as many times as there are stars in G shows that it is sufficient to prove the theorem for a union of monochromatic triangles. Suppose then that G is a union of monochromatic triangles. Without loss of generality it is also possible to consider that G is connected, otherwise the same proof can be applied to every connected component. The triangles are monochromatic so an edge cannot be contained by more than one triangle.

Suppose, for a contradiction, that two triangles contain the same vertex. Assume first that G contains a *rainbow circuit* C (every edge has a different color). If G contains more than one rainbow circuit, then by deleting every vertex of C (the number of vertices deleted is the length of C denoted $l(C)$), the $l(C)$ triangles of C are deleted and the remaining graph has $l(C)$ colors less. It is also possible that this process deletes edges from triangles that are not contributing to the circuit, in which case only one edge of each of these triangles remains. This process can be repeated until only one rainbow circuit remains. Let H be the remaining graph created from G by repeating the process described above until only one rainbow circuit C' remains. The triangles in H either have an edge in C' or are edge-disjoint from C'. By deleting every vertex of C' except one, and deleting one vertex of every remaining triangle, H is made bipartite. In total the number of vertices deleted is at most $\overline{\chi}'(G)-1$, which contradicts $\overline{\chi}'(G)=\mathrm{cd}_2(G)$.

Thus G does not contain a rainbow circuit. There exists in G a vertex that is contained by at least two triangles. By deleting this vertex there only remains one edge of each of the two triangles. By deleting arbitrarily one vertex of each remaining triangles at most one edge of each color remains. The remaining graph does not contain any circuit otherwise it would contain a rainbow circuit. Therefore it is a forest. At most $\overline{\chi}'(G)-1$ vertices have been removed which is again in contradiction with $\overline{\chi}'(G)=\mathrm{cd}_2(G)$. It shows the first part of the theorem.

Suppose, for a contradiction, that one vertex in a monochromatic triangle is also the center of a monochromatic star. By deleting every center of star and by deleting one vertex of each remaining triangle (at most $\overline{\chi}'(G)-1$ vertices in total) the graph is made bipartite which is impossible. Therefore no center of monochromatic star is contained by a monochromatic triangle.

Therefore all triangles in G are vertex-disjoint. Suppose for a contradiction, that there exists a monochromatic star S that does not cover all the vertices of a monochromatic triangle. According to Lemma 1.3.1 it is possible to consider that G contains only one star, S. By deleting one vertex of every triangle intersecting the star, such that the vertex is covered by both the star and the triangle, and one arbitrary vertex of every triangle disjoint from the star ($\overline{\chi}(G)-1$ vertices in total), the graph is made bipartite, a contradiction. □

It is interesting to note that the existence of a "colorful" $K^*_{\mathrm{cd}_2(G),\mathrm{cd}_2(G)}$ (defined at the end of Appendix A), as mentionned in the article by M. Alishahi, H. Hajiabolhassan, and F. Meunier [2], is a corollary of this Theorem 1.3.2. Indeed, by choosing, arbitrarily, two edges from each triangle (respectively assigned with a side: left or right), it is also possible to choose two edges from each star (respectively assigned with a side: left or right) such that any edge assigned with the left (respectively right) side only intersects other edges assigned with the left (respectively right) side as well. This process creates a $K^*_{\mathrm{cd}_2(G),\mathrm{cd}_2(G)}$ with all colors on each side.

Figure 1.1 shows a graph G such that $\overline{\chi}'(G)=\mathrm{cd}_2(G)=4$, the thick edges correspond to the vertices of a $K^*_{4,4}$ contained by the Kneser graph associated with G.

Remark 1.3.3. A wrong intuition is to think that if G is a graph satisfying $\overline{\chi}'(G)=\mathrm{cd}_2(G)$ such that for any optimal proper coloring of $\mathrm{KG}(G)$, monochromatic stars cover either three or zero vertices of any monochromatic triangle, then it is possible to find a partition of G in cliques containing exactly one monochromatic triangle. A

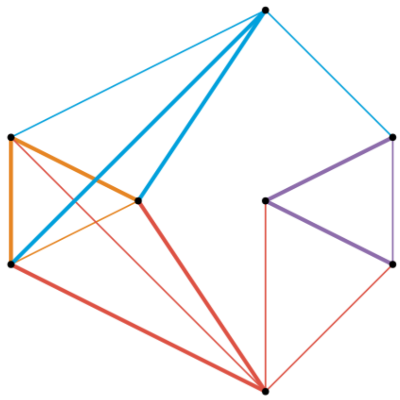

Figure 1.1 – Example of a graph satisfying the equality.

counterexample is provided below with Figure **??**.

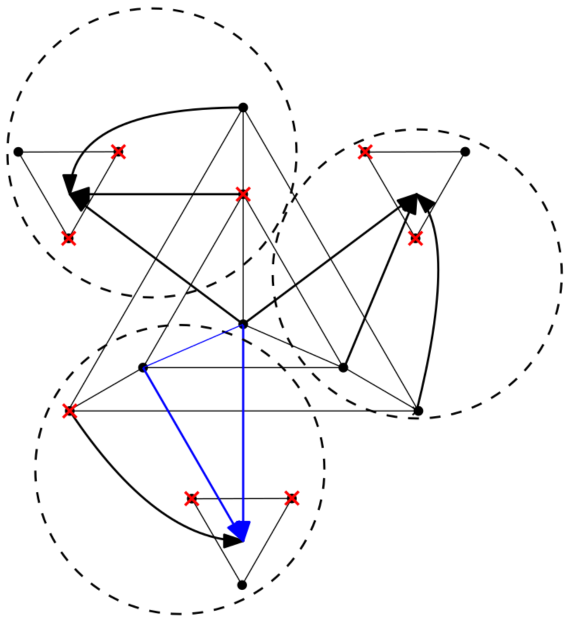

Figure 1.2 – Counterexample to the false intuition

In this graph G a wide arrow represents the fact that the vertex it comes from spans an edge with every vertex form the triangle it points towards. The dashed circles contain the cliques of G.

This graph is such that $\overline{\chi}'(G) \leqslant 10$: a way to color it using 10 colors is to color the 3 external triangles and then all the other vertices are centers of stars.

If the vertex in the center v is deleted the remaining graph H satisfies $\overline{\chi}'(H) = \mathrm{cd}_2(H) = 9$ as it can be decomposed into three K_5 and each K_5 satisfies $\overline{\chi}'(K_5) = \mathrm{cd}_2(K_5) = 3$.

Therefore $\mathrm{cd}_2(G) \geqslant 9$. Suppose for a contradiction that it is possible to find a set B of 9 vertices such that $G \setminus B$ is bipartite. If v is in B, then H only requires 8 vertices to be removed to become bipartite wich is a contradiction. Therefore v is not in B, and it is necessary to delete at least 2 vertices from each outer triangle, and at least 3 vertices in total in each K_5. It is also necessary to delete at least one vertex from each inner triangle. One possible way to do this is shown in red on Figure **??**. In any case there still remains a triangle, shown in blue. This shows that $\mathrm{cd}_2(G) \geqslant 10$. With Dol'nikov's inequality we have $\overline{\chi}'(G) \geqslant \mathrm{cd}_2(G)$ Therefore $\overline{\chi}'(G) = \mathrm{cd}_2(G) = 10$ and yet it is not possible to find a partition of G in cliques containing exactly one monochromatic triangle, thus disproving the intuition.

1.3.2 An improved inequality for graphs with girth greater than 3

A corollary of Theorem 1.3.2 is that, if G does not contain a triangle, $\overline{\chi}'(G) \geqslant \mathrm{cd}_2(G) + 1$. This inequality can be improved by forbiding circuits of greater length.

The *girth* of a graph G, denoted by $\mathrm{g}(G)$, is the length of its smallest circuit. Dol'nikov's inequality can be improved as follows.

Theorem 1.3.4. *Let G be a graph. If $\mathrm{g}(G) \geqslant 4$, then $\overline{\chi}'(G) \geqslant \mathrm{cd}_2(G) + \lfloor \frac{\mathrm{g}(G)}{2} \rfloor$.*

Proof. Let G be a graph such that $\mathrm{g}(G) \geqslant 4$. Let c be an optimal coloring of $\mathrm{KG}(G)$. According to Remark 1.2.2, G is covered by a set of $k = \overline{\chi}'(G)$ stars $S_1 \cup \cdots \cup S_k$ whose centers are $v_1, \ldots, v_k$ respectively.

If $\mathrm{g}(G)$ is an even number, then by deleting $v_1, \ldots, v_{k-\frac{\mathrm{g}(G)}{2}}$, the remaining graph $S_{k-\frac{\mathrm{g}(G)}{2}+1} \cup \cdots \cup S_k$ contains only $\frac{\mathrm{g}(G)}{2}$ stars, does not contain any circuit of length strictly less than $\mathrm{g}(G)$ by definition, and does not contain any circuit of length strictly greater than $\mathrm{g}(G)$ because it could then not be covered with $\frac{\mathrm{g}(G)}{2}$ stars. This shows that the remaining graph only contains circuits of length $\mathrm{g}(G)$ which is even. Therefore G is bipartite.

If $\mathrm{g}(G)$ is an odd number, then by deleting $v_1, \ldots, v_{k-\lfloor \frac{\mathrm{g}(G)}{2} \rfloor}$, the remaining graph $S_{k-\lfloor \frac{\mathrm{g}(G)}{2} \rfloor+1} \cup \cdots \cup S_k$ contains only $\lfloor \frac{\mathrm{g}(G)}{2} \rfloor$ stars and does not contain any circuit. Indeed, suppose the remaining graph contained a circuit, by definition of $\mathrm{g}(G)$ the length of the circuit would be at least $\mathrm{g}(G)$ and could not be covered with $\lfloor \frac{\mathrm{g}(G)}{2} \rfloor$ stars as $\mathrm{g}(G)$ is an odd number. Therefore the remaining graph is a forest and is bipartite.

The conclusion of both cases is that $\mathrm{cd}_2(G) \leqslant \overline{\chi}'(G) - \lfloor \frac{\mathrm{g}(G)}{2} \rfloor$ which can be reformulated as $\overline{\chi}'(G) \geqslant \mathrm{cd}_2(G) + \lfloor \frac{\mathrm{g}(G)}{2} \rfloor$. □

1.3.3 Complexity of verifying the equality.

In this section it is interesting to introduce the notion of *join* of two graphs. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. A graph denoted $G_1 * G_2 = (V, E)$ can be created such that

$$V = V_1 \cup V_2$$

$$E_1 \subseteq E$$

$$E_2 \subseteq E$$

and E contains also all the possible edges between vertices of G_1 and of G_2. This operation satisfies the properties of the following lemma where for any graph G, the number of vertices is denoted $v(G)$.

Lemma 1.3.5. *Let G_1 and G_2 be two graphs.*

$$\mathrm{cd}_2(G_1 * G_2) = \min(v(G_1) + \mathrm{cd}_2(G_2), \mathrm{cd}_2(G_1) + v(G_2), \tau(G_1) + \tau(G_2)) \tag{1.1}$$

$$\min(v(G_1) + \overline{\chi}'(G_2), \overline{\chi}'(G_1) + v(G_2)) \geqslant \overline{\chi}'(G_1 * G_2) \tag{1.2}$$

$$\overline{\chi}'(G_1 * G_2) \geqslant \overline{\chi}'(G_1) + \overline{\chi}'(G_2) \tag{1.3}$$

.

Proof of Equation (1.1). In this proof, $G_1 * G_2$ is denoted G. Let X be a set such that $G - X$ is bipartite and such that $|X|$ is minimal. Suppose $G_1 - X$ contains an edge. Then $G_2 \subseteq X$, otherwise a triangle would remain in $G - X$. In which case the minimal way to have such a situation requires $\mathrm{cd}_2(G_1) + v(G_2)$ vertices to be removed. The symetrical situation where the roles of G_1 and G_2 are inverted requires $\mathrm{cd}_2(G_2) + v(G_1)$ vertices to be removed. Suppose then that $G_1 - X$ and $G_2 - X$ do not contain any edge. The minimal way to have such a situation requires $\tau(G_1) + \tau(G_2)$ vertices to be removed. □

Proof of Equations (1.2) *and* (1.3). Equation (1.3) is shown by the fact that a coloring of $\mathrm{KG}(G)$ induces a coloring of both $\mathrm{KG}(G_1)$ and $\mathrm{KG}(G_2)$.

A possible way to choose a coloring of $\mathrm{KG}(G)$ is to color $\mathrm{KG}(G_2)$ with an optimal coloring and to consider every vertex of G_1 as the center of a star. This shows that $v(G_1) + \overline{\chi}'(G_2) \geqslant \overline{\chi}'(G)$. Symetrically $v(G_2) + \overline{\chi}'(G_1) \geqslant \overline{\chi}'(G)$. Therefore $\min(v(G_1) + \overline{\chi}'(G_2), v(G_2) + \overline{\chi}'(G_1)) \geqslant \overline{\chi}'(G)$, which is Equation (1.2). □

Lemma 1.3.6. *Let G be a graph such that $\overline{\chi}'(G) = \mathrm{cd}_2(G)$. If there exists an optimal proper coloring of G with only one monochromatic triangle, then G is a complete graph.*

Proof. According to Theorem 1.3.2 it is sufficient to show the lemma when G is connected. If G contains less than 5 vertices the result is immediate and G is complete.

Suppose that G has $l \geqslant 5$ vertices, 3 vertices are part of the monochomatic traingle, the remaining $l-3$ are centers of stars. According to lemma 1.3.1 it is possible to delete vertices until only the triangle and 2 stars remain. The remaining graph satisfies the equality, has only one monochormatic triangle and only 5 vertices. Therefore it is a complete graph. By applying this process while keeping any two centers of stars, this shows that any two vertices span an edge. Therefore the graph G is complete. □

A natural question is whether graphs realizing Dol'nikov's inequality as an equality can be characterized. The following theorem is a complexity result, which shows that there is no real hope for a "good" characterization. This settles an open question of Alishahi et al [2].

Theorem 1.3.7. *The problem of deciding whether $\overline{\chi}'(G) = \mathrm{cd}_2(G)$ given a graph G in input is NP-hard.*

The proof proceeds by a reduction from the problem of determining the stability number of a connected triangle-free graph, which is a NP-complete problem [6]. Let H be a connected triangle-free graph. For an integer $g \geqslant 1$, denoted by H_k is the graph obtained by taking the join of H with k vertex-disjoint triangles $T_1, \ldots, T_k$. The key step is to prove that

$$\alpha(H) \leqslant k \qquad \Longleftrightarrow \qquad \overline{\chi}'(H_k) = \mathrm{cd}_2(H_k) \tag{1.4}$$

, where $\alpha(H)$ is the size of the largest independent set of H. This equivalence implies immediatley the desired result. The following lemmas are useful to get the equivalence. The size of the smallest vertex cover of H is denoted by $\tau(H)$ and the number of vertices of H by $v(H)$. The equality $v(H) = \alpha(H) + \tau(H)$ is common knowledge and will be used without further mention.

Lemma 1.3.8. *This equality holds* $\mathrm{cd}_2(H_k) = \min(\mathrm{cd}_2(H) + 3k, \tau(H) + 2k, v(H) + k)$.

Proof. Let X be a subset of $V(H_k)$ of minimal size such that the subgraph of H_k induced by $V(H_k) \setminus X$ is bipartite. It satisfies $|X| = \mathrm{cd}_2(H_k)$.
If X contains all the vertices of the T_i's, then clearly $|X| = \mathrm{cd}_2(H) + 3k$.
If X contains all the vertices of H, then clearly $|X| = v(H) + k$.
If X contains neither all the vertices of the T_i's, nor those of H, then X has to be incident to each edge of H (it is a vertex cover of H) and it has to contain at least two vertices of each T_i's, otherwise it would have at least one triangle. At best, in such

a case, $X \cap V(H)$ is a vertex cover of H of minimal size and X intersects each T_i's in exactly 2 vertices and $|X| = \tau(H) + 2k$. $\square$

Lemma 1.3.9. *If* $\overline{\chi}'(H_k) = \mathrm{cd}_2(H_k)$, *then* $\overline{\chi}'(H_k) = \min(\tau(H) + 3k, \upsilon(H) + k)$.

Proof. Suppose that $\overline{\chi}'(H_k) = \mathrm{cd}_2(H_k)$ and consider an optimal ST-coloring of H_k. It induces an ST-coloring of H that consists only of stars (H does not contain any triangle). Their centers form a cover C of H. If $C = V(H)$, then $\overline{\chi}'(H_k) = \upsilon(H) + k$. Otherwise, according to Theorem 1.3.2, the edges of H_k incident to the vertices in $V \setminus C$ belong to stars of the ST-coloring whose center are not in C, and thus $\overline{\chi}'(H_k) = |C| + 3k$ (all vertices of the T_i's have to be centers of these stars). The conclusion follows from the fact that an ST-coloring can be achieved with C being a cover of H of minimal size. $\square$

Proof of Theorem 1.3.7. The following is the proof of equivalence 1.4.

Consider first the case when $\alpha(H) \leqslant k$. Since $\mathrm{cd}_2(H) + 2\alpha(H) \geqslant \upsilon(H)$, then $\mathrm{cd}_2(H) + 3k \geqslant \upsilon(H) + k$ and $\tau(H) + 2k \geqslant \upsilon(H) + k$ which implies with Lemma 1.3.8 that $\mathrm{cd}_2(H_k) = \upsilon(H) + k$. Consider the ST-coloring whose stars are those whose centers are the vertices of H and whose triangles are the T_i's. Its existence implies that $\overline{\chi}'(H_k) \leqslant \upsilon(H) + k$ and Dol'nikovinequality allows to conclude.

Consider now the case $\overline{\chi}'(H_k) = \mathrm{cd}_2(H_k)$. Suppose for a contradiction that $\alpha(H) > k$. It is possible to start by even supposing that $\alpha(H) \geqslant 2k$. Lemma 1.3.9 implies that $\overline{\chi}'(H_k) = \tau(H) + 3k$. Using the equality $\overline{\chi}'(H_k) = \mathrm{cd}_2(H_k)$ and Lemma 1.3.8 it ensues that $\mathrm{cd}_2(H_k) = \mathrm{cd}_2(H) + 3k$, and thus that $\tau(H) = \mathrm{cd}_2(H)$, which implies that H is an independent set. Since H is supposed to be connected, H has at most one vertex, which contradicts $\alpha(H) > k$.

Suppose now that $k < \alpha(H) < 2k$. Lemma 1.3.9 implies that $\overline{\chi}'(H_k) = \upsilon(H) + k$. Using the equality $\overline{\chi}'(H_k) = \mathrm{cd}_2(H_k)$ and Lemma 1.3.9 it ensues that $\tau(H) + 2k \geqslant \upsilon(H) + k$ which is in contradiction with $\alpha(H) > k$. $\square$

1.4 Proof of an exercise given by Matoušek

Dol'nikov's inequality is true even for hypergraphs and the general proof of this inequality uses advanced arguments of algebraic topology [2.5]. This section provides an elementary proof of Dol'nikov's inequality when the chromatic number is at most 2.

A *hypergraph* $\mathscr{H}$ is an ordered pair $(\mathscr{V}, \mathscr{E})$ such that $\mathscr{E} \subseteq \mathscr{P}(\mathscr{V})$ where $\mathscr{P}(\mathscr{V})$ is the set of subsets of $\mathscr{V}$. As for graphs, the elements of $\mathscr{V}$ are the vertices and the elements of $\mathscr{E}$ are the edges. Most definitions given for graphs in Appendix A can be extended easily to hypergraphs and will be used in this section without being given again. It is important to note that the *Kneser graph associated with* a hypergraph is a graph and that the *2-colorability defect* of a hypergraph is the minimum number of vertices to be deleted so that the hypergraph is *bipartite*, which means that there exists a coloring of the vertices with two colors such that every edge of the hypergraph covers a vertex of each color.

1.4.1 Chromatic number $\overline{\chi}'(\mathscr{H}) = 1$

Let $\mathscr{H}$ be a hypergraph such that $\overline{\chi}'(\mathscr{H}) = 1$, let e be an edge such that for any edge e', if $e' \subseteq e$ then $e' = e$. The vertices of e can be colored arbitrarily in blue and the other vertices of $\mathscr{H}$ in red. As $\overline{\chi}'(\mathscr{H}) = 1$, any other edge of $\mathscr{H}$ intersects e and contains a blue vertex but is not a subset of e and therefore also contains a red vertex. With this coloring, every edge contains both a blue and a red vertex except e. By deleting one vertex of e the hypergraph becomes bipartite. This shows that $\mathrm{cd}_2(\mathscr{H}) \leqslant 1$ and proves Dol'nikov's inequality.

1.4.2 Chromatic number $\overline{\chi}'(\mathscr{H}) = 2$

Let $\mathscr{H}$ be a hypergraph such that $\overline{\chi}'(\mathscr{H}) = 2$. Without loss of generality we can suppose that no two edges e and f can be found such that $e \subseteq f$. This hypothesis can be made without loss of generality for – noting $\mathscr{H}' = (\mathscr{V}, \mathscr{E} \setminus f)$ – the following equalities holds: $\overline{\chi}'(\mathscr{H}') = \overline{\chi}'(\mathscr{H})$ and $\mathrm{cd}_2(\mathscr{H}') = \mathrm{cd}_2(\mathscr{H})$. Indeed let c be a coloring of $\mathrm{KG}(\mathscr{H}')$, then $\mathscr{H}$ can be colored using the same colors as $\mathscr{H}'$ with f of the same color as e. This immediately shows that $\overline{\chi}'(\mathscr{H}') = \overline{\chi}'(\mathscr{H})$. If a vertex of e has to be deleted (among other vertices) for $\mathscr{H}'$ to become bipartite, then by removing the same vertices $\mathscr{H}$ will become bipartite as well. This shows that $\mathrm{cd}_2(\mathscr{H}) \leqslant \mathrm{cd}_2(\mathscr{H}')$ while $\mathrm{cd}_2(\mathscr{H}) \geqslant \mathrm{cd}_2(\mathscr{H}')$ is also true as $\mathscr{H}' \subseteq \mathscr{H}$.

As $\overline{\chi}'(\mathscr{H}) = 2$, there exists a proper coloring c of $\mathrm{KG}(\mathscr{H})$ using two colors, arbitrarily blue and red. Let b be an edge colored in blue and r an edge colored in red such that the number $|b \cup r|$ is minimum. To color the vertices, the first step is to give the color noted $+$ to every vertex in $b \cup r$ and to every vertex not in $b \cup r$ the color $-$. The previous hypothesis guarantees that there exists $x \in b \setminus r$ and $y \in r \setminus b$. The remaining of the proof consists in showing that if x and y are deleted H is made bipartite. Let e be an edge that does not cover x or y (if such an edge does not exist the proof is immediate). Suppose without loss of generality that e is blue. Therefore e intersects b and covers a vertex colored $+$, but $e \not\subseteq b \cup r$ as the number $|b \cup r|$ is a minimum and therefore e covers a vertex colored $-$. This shows that deleting x and y makes $\mathscr{H}$ bipartite and that $\mathrm{cd}_2(\mathscr{H}) \leqslant 2$.

The study of both cases $\overline{\chi}'(\mathscr{H}) = 1$ and $\overline{\chi}'(\mathscr{H}) = 2$ shows Dol'nikov's inequality for $\overline{\chi}'(\mathscr{H}) \leqslant 2$.

2 Colorings of the usual Kneser graph

G. Simonyi has given a method for generating optimal proper colorings of Kneser graphs. In this chapter it is shown that the colorings generated by this method are not the only colorings. A complete characterization of the optimal proper colorings is still missing.

2.1 Introduction

In this chapter $\mathrm{KG}(n,k)$ is the graph whose vertices are the k-subsets of $\{1,\ldots,n\}$ and the edges are the pairs of disjoint k-subsets. To simplify the study it is also assumed that $n \geqslant 2k$ otherwise the graph has no edge. Lovász has shown that $\chi(\mathrm{KG}(n,k)) = n-2k+2$. The set of integers ranging from 1 to n is denoted $[n]$. The graphs are also considered to be vertex labeled graphs, meaning that each vertex is labeled with the k-subset it represents so that all vertices are distinguishable.

2.2 A method for producing optimal proper colorings

A method given by Simonyi [7] allows to produce optimal colorings of $\mathrm{KG}(n,k)$.

Partition $[n]$ into $n-2k+2$ odd cardinality subsets $T_1,\ldots,T_{n-2k+2}$. This step is always possible as n and $n-2k+2$ have the same parity.

For any k-subset A in $\binom{[n]}{k}$ there exists at least one $i \in [n]$ such that $|A\cap T_i| > \frac{1}{2}|T_i|$ otherwise $|A| = \sum_{j=1}^{n-2k+2} \left|A\cap T_j\right| \leqslant \sum_{j=1}^{n-2k+2} \frac{|T_j|-1}{2} = \sum_{j=1}^{n-2k+2} \frac{|T_i|}{2} - \frac{n-2k+2}{2} = k-1$ which is a contradiction. Such an integer i is chosen to be the color of A (in the case of a tie between multiple colors the choice is made arbitrarily). It is interesting to note that a partition does not necessarily induce a unique coloring. Sometimes there are k-subsets A such that $|A \cap T_i| > \frac{1}{2}|T_i|$ for multiple i and each of the different possible choices induces a different coloring.

These colorings are proper and optimal. If two k-subsets A and B have the same color i then $A \cap B \neq \emptyset$ as both A and B contain more than half of T_i.

We call such colorings *Simonyi colorings*.

A *graph automorphism of* G is a permutation α of the vertices such that, for any pair of vertices (u, v), the vertices $\alpha(u)$ and $\alpha(v)$ form an edge if and only if u and v form an edge as well. The set of automorphisms of any graph is a group under composition.

Lemma 2.2.1. *For any pair* (n, k) *with* $n \geqslant 2k+1$*, there is a natural group isomorphism between the set of automorphisms of* $\mathrm{KG}(n,k)$ *denoted* $\mathrm{Aut}(\mathrm{KG}(n,k))$ *and* $\mathscr{S}_n$ *the set of permutations of* $[n]$*. This group isomorphism is defined as follow,*

$$\begin{array}{rccc} \Phi: & \mathrm{Aut}(\mathrm{KG}(n,k)) & \longrightarrow & \mathscr{S}_n \\ & \alpha & \longmapsto & \sigma_\alpha \end{array}$$

where for any vertex $v = \{i_1, \ldots, i_k\}$ *we have* $\alpha(v) = \{\sigma_\alpha(i_1), \ldots, \sigma_\alpha(i_k)$.

Proof. According to the Erdös-Ko-Rado theorem in the case $n \geqslant 2k+1$, the maximal independent sets of $\mathrm{KG}(n,k) = (V, E)$ are all the $I_j = \{v \in V : j \in v\}$ for j in $[n]$ (the set of vertices that all contain the element j). An important preliminary remark is that for any k distinct integers $i_1, \ldots, i_k$, the set $I_{i_1} \cap \cdots \cap I_{i_k}$ is of cardinality 1 and contains only the vertex $\{i_1, \ldots, i_k\}$. This is equivalent to saying that a vertex is completely characterized by the k independent it intersects.

Notice that for any graph automorphism α of $\mathrm{KG}(n,k)$, if I is an independent set, then $\alpha(I)$ is an independent set of same cardinality, which implies that for any j in $[n]$ there exists an l in $[n]$ such that $\alpha(I_j) = I_l$. In addition, for any integers i, j in $[n]$ we have $\alpha(I_i) \neq \alpha(I_j)$. Otherwise it would be possible to choose $k-1$ distinct integers $i_1, \ldots, i_{k-1}$ in $[n] \setminus \{i, j\}$ and would have $\alpha(I_{i_1} \cap \cdots \cap I_{i_{k-1}} \cap I_i) = \alpha(I_{i_1} \cap \cdots \cap I_{i_{k-1}} \cap I_j)$ which would contradict the fact that α is an automorphism. This means that for every graph automorphism α there exists a permutation π of $[n]$ such that $\alpha(I_j) = I_{\pi(j)}$ for any j in $[n]$.

Let Φ be a function defined as follow,

$$\begin{array}{rccc} \Phi: & \mathrm{Aut}(\mathrm{KG}(n,k)) & \longrightarrow & \mathscr{S}_n \\ & \alpha & \longmapsto & \sigma_\alpha \end{array}$$

where for any graph automorphism α, the permutation σ_α is such that $\alpha(I_j) = I_{\sigma_\alpha(j)}$ for all integers j in $[n]$. The above explanations show that Φ is well defined.

The objective is to show that Φ is an isomorphism. Let α and α' be two different graph automorphisms of $\mathrm{KG}(n,k)$. There exists a vertex v such that $\alpha(v) \neq \alpha'(v)$. According to the preliminary remark there exists an integer i in $[n]$ such that $\alpha(I_i) \neq \alpha'(I_i)$ otherwise $\alpha(v) = \alpha(I_{i_1} \cap \cdots \cap I_{i_k}) = \alpha(I_{i_1}) \cap \cdots \cap \alpha(I_{i_k}) = \alpha'(I_{i_1}) \cap \cdots \cap \alpha'(I_{i_k}) = \alpha'(v)$. This shows that $\Phi(\alpha) \neq \Phi(\alpha')$ and that Φ is injective.

For a permutation π of $[n]$, the function $\alpha\colon \mathrm{KG}(n,k) \to \mathrm{KG}(n,k)$ defined for a vertex $v = \{i_1,\dots,i_k\}$ by $\alpha(v) = v'$ where $v' = \{\pi(i_1),\dots,\pi(i_k)\}$ is a graph automorphism. Indeed, as π is a permutation, if the sets $\{i_1,\dots,i_k\}$ and $\{i'_1,\dots,i'_k\}$ are disjoint, then the sets $\{\pi(i_1),\dots,\pi(i_k)\}$ and $\{\pi(i'_1),\dots,\pi(i'_k)\}$ are disjoint as well, for any distinct integers $i_1,\dots,i_k,i'_1,\dots,i'_k$ in $[n]$. Therefore $\Phi(\alpha) = \pi$ and Φ is surjective.

Let α and β be two graph automorphisms of $\mathrm{KG}(n,k)$, let $\sigma_\alpha = \Phi(\alpha)$, $\sigma_\beta = \Phi(\beta)$ and $\sigma_{\alpha\circ\beta} = \Phi(\alpha\circ\beta)$ be three permutations of $[n]$. Let i be an integer in $[n]$. There exists an integer j in $[n]$ such that $()\alpha\circ\beta)(I_i) = I_j$, which means that $\sigma_{\alpha\circ\beta}(i) = j$. Moreover $(\alpha\circ\beta)(I_i) = \alpha(\beta(I_i))$ and there exists an integer l in $[n]$ such that $\beta(I_i) = I_l$ which means that $\sigma_\beta(i) = l$. Therefore $\alpha(I_l) = I_j$ which means that $\sigma_\alpha(l) = j$ and that $(\sigma_\alpha\circ\sigma_\beta)(i) = \sigma_{\alpha\circ\beta}(i) = j$. This equality holds for every integer i in $[n]$. Therefore $\Phi(\alpha\circ\beta) = \Phi(\alpha)\circ\Phi(\beta)$.

The definition of Φ makes it clear that $\Phi(\mathrm{id}) = \mathrm{id}$. This shows that Φ is a group isomorphism between $\mathrm{Aut}(\mathrm{KG}(n,k))$ and $\mathscr{S}_n$. □

When given a coloring c of $\mathrm{KG}(n,k)$ and a graph automorphism α, it is possible to create a coloring c' such that $c'(\alpha(v)) = c(v)$ for any vertex v in $\mathrm{KG}(n,k)$. The previous Lemma 2.2.1 allows to show the following Proposition 2.2.2.

Proposition 2.2.2. *Let c be a Simonyi coloring of $G = \mathrm{KG}(n,k)$. Let α be a graph automorphism of G. Then $c\circ\alpha$ is a Simonyi coloring as well.*

Proof. There exists a partition of $[n]$ into $T_1,\dots,T_{n-2k+2}$ with odd cardinality such that c is a Simonyi coloring associated with this partition. Let $\sigma_\alpha = \Phi(\alpha)$ with Φ defined as in Lemma 2.2.1. For any vertex $v = \{i_1,\dots,i_k\}$ in G, the vertex $\alpha(v) = \{\sigma_\alpha(i_1),\dots,\sigma_\alpha(i_k)\}$ and therefore it appears that $c\circ\alpha$ is a Simonyi coloring associated with the partition $\sigma_\alpha(T_1),\dots,\sigma_\alpha(T_k)$. □

2.3 Colorings in the case $k = 2$

For $k = 2$, coloring $\mathrm{KG}(n,2)$ is equivalent to coloring the edges of the complete graph K_n such that two edges that are disjoint have different colors.

Proposition 2.3.1. *For $n \geqslant 5$, any optimal proper coloring of $\mathrm{KG}(n,2)$ is a Simonyi coloring.*

Proof. As seen in Lemma 1.3.6 the optimal edge-colorings of complete graphs have one monochromatic triangle and the other vertices are centers of stars. Therefore all the colorings of $\mathrm{KG}(n,2)$ are Simonyi colorings with $|T_1| = |T_2| \cdots = |T_{n-3}| = 1$ and $|T_{n-2}| = 3$. □

2.4 Other optimal proper colorings

For $k \geqslant 3$, according to G. Simonyi personal communications, an argument given by S. Thomassé provides with a lower bound for the number of colorings of $\mathrm{KG}(n,k)$.

Consider the case $|T_1| = \cdots = |T_{n-2k+1}| = 1$ and $|T_{n-2k+2}| = 2k-1$. The subgraph induced by the last two color classes T_{n-2k+1} and T_{n-2k+2} is a matching. The number of edges in the matching is $M = \binom{2k-1}{k}$. For each edge of the matching it is possible to swap the colors of the two vertices and thus create a new coloring. Therefore there are at least 2^M different colorings of $\mathrm{KG}(n,k)$.

2.4.1 The example of $\mathrm{KG}(7,3)$

The graph $\mathrm{KG}(7,3)$ has 35 vertices and $\chi(\mathrm{KG}(7,3)) = 3$. There are only two ways to partition [7] into three parts of odd cardinality, in one case two parts have a cardinality of 1 and the third has a cardinality of 5, in the other case one part has a cardinality of 1 and the other two have a cardinality of 3.

Consider now the colorings of $\mathrm{KG}(7,3)$ that have a color class of 15 vertices (this class is called Red, the other are Blue and Green). Such colorings exist according to the Erdös-Ko-Rado theorem, indeed the maximal independent sets have 15 vertices (they are of the form $I_j = \{v \in V : j \in v\}$ for j in $[n]$). The argument given by S. Thomassé shows there are at least $2^{10} = 1024$ colorings with a color class of 15 vertices.

As stated before there are two ways to partition $[n]$ to create Simonyi colorings.

- Case $1,1,5$. The color class associated with the subset of 5 elements can only contain $\binom{5}{3} = 10$ elements so the color of this class can be Blue or Green. There are $\binom{7}{5} = 21$ ways to create this subset of 5 elements. There are also 2 ways to select the elements for the other class Blue or Green, the last element being associated with Red. It is important to note that there is no problem of indetermination here as any vertex that contains the element associated with Red must have the color Red otherwise there can be no color class of 15 vertices as stated by the Erdös-Ko-Rado theorem. Therefore there are $2 \times 21 \times 2 = 84$ colorings of this sort.

- Case $1,3,3$. The color class Red can only be associated with the part of cardinaliy 1 as only 13 vertices can intersect more than two elements of a subset of 3 elements. Therefore Blue and Green are associated with the parts of cardinality 3. There are $\binom{7}{3} = 35$ ways to choose the elements for the part associated with Blue and 4 ways to choose the elements for the part associated with Green. In total there are $35 \times 4 = 140$ colorings of this sort.

In total there are $84 + 140 = 224$ Simonyi colorings which is less than 1024, therefore the Simonyi colorings cannot be the only colorings of $\mathrm{KG}(7,3)$.

2.5 More general Simonyi colorings

The argument given by Thomassé relies on the fact that the last two color classes create a matching and thus these color classes created multiple connected components in the graph. It is then possible to switch the colors in each connected component independently from the other components and thus creating more colorings. This process can be generalized to any coloring, including all the Simonyi colorings. If two color classes create multiple connected components then it is possible to create different colorings by switching the colors independently in each connected component.

The question whether all colorings of $\mathrm{KG}(n,k)$ with $n \geqslant 2k+1 \geqslant 3$ can be obtained from a Simonyi coloring by applying this process has not yet been aswered.

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A Preliminaries on graphs

The definitions in this section are, for the most part, extracted from Combinatorial optimization by Alexander Schrijver [2.5].

Let $H = (V, E)$ be a simple graph. An edge uv *connects* u and v, those are the *ends* of the edge uv. If uv is an edge of H, then H *covers* u and v. Two vertices u and v are *adjacent* if uv is an edge. the graph H is *complete* if it is simple and any two distinct vertices are adjacent. If both ends of an edge e belong to $U \subseteq V$, then U *spans* e.

A *matching* is a subset M of E such that no two edges in M share a vertex. A *perfect matching* is a matching covering all vertices.

Proper colorings have already been defined in the introduction. In addition *edge-coloring* is a map $c : E \to \mathbb{Z}_+$. The elements of $c(E)$ are, as for vertices, the *colors*.

A graph $H' = (V', E')$ is a *subgraph* of H if $V' \subseteq V$ and $E' \subseteq E$. A subgraph whose edges are of the same color for a coloring is *monochromatic*, a *maximum monochromatic subgraph* has the set of its vertices spanning the edges of any monochromatic subgraph of its color ; the context should make clear which coloring is used. If $V' = V$, then H' is a *spanning subgraph* of H and if E' consists of all edges of H spanned by V', then H' is the *subgraph induced by* V'. In notation,

$$H[V'] := \text{subgraph of } H \text{ induced by } V',$$
$$E[V'] := \text{family of edges spanned by } V'.$$

So $H[V'] = (V', E')$ and for any vertex v, subset $U \subseteq V$, edge e, and subset $F \subseteq E$

$$H - v = H[V \setminus v], \qquad H - U = H[V \setminus U],$$
$$H - e = (V, E \setminus e), \qquad H - F = (V, E \setminus F).$$

These operations are, respectively, *deletions* of u, U, e, or F. It is expected that the context will always make it clear whether the deletion $H - e$ is considering e as a set of vertices or as an edge. Two subgraphs are *disjoint* if they share no vertex and *edge-disjoint* if they share no edge. If $V' \subseteq V$ and $E' \subseteq E$, then $H = (V, E)$ *contains* $H' = (V', E')$; in notation $H' \subseteq H$. Two graphs *intersect* if they share a vertex. A *clique* is a complete subgraph of H.

A *walk* in a graph $H = (V, E)$ is a sequence

$$P = (v_0, e_1, v_1, \ldots, e_k, v_k),$$

where $k \geqslant 0$, the elements $v_0, v_1, \ldots, v_k$ are vertices and e_i is an edge connecting v_{i-1} and v_i for $i = 1, \ldots, k$. If $v_0, v_1, \ldots, v_k$ are all distinct, the walk is a *path*. The number k is the *length* of the walk. The walk P is *closed* if $v_0 = v_k$, and a *circuit* if it is closed, $k \geqslant 1$, $v_1, \ldots, v_k$ are all distinct, and $e_1, \ldots, e_k$ are all distinct. If $k = 3$, the circuit is a *triangle*. A graph is *connected* if there is a path connecting any two vertices.

A *cover* is a set of vertices such that any edge in H covers at least one element of the cover. The minimal number for the cardinality of a cover of H is denoted $\tau(H)$.

A graph is a *forest* if it has no circuit, a *tree* is a connected forest. A graph is a *star* if there exists one vertex (the *center*) covered by all the edges.

The complete bipartite bipartite graph $K_{m,n}$ is the bipartite graph whose vertices can be partitionned into two sets A and B, with $|A| = m$ and $|B| = n$, so that its edges are precisely all pairs uv with u in A and v in B. The sets A and B are the *sides* of the bipartite graph.

The graph $K^*_{t,t}$ is the complete bipartite graph $K_{t,t}$ from which a perfect matching has been removed.

B Secondary results

Remark 1.2.2. If c is an edge-coloring of the graph $G = (V, E)$ such that all edges of the same color pairwise intersect, then every monochromatic subgraph is a star or a triangle.

Proof. Let H' be a monochromatic subgraph and k the number of its edges, if $k \leqslant 2$ the proof is immediate. If $k = 3$, as each edge contains a vertex of all the other edges, if the shared vertex is the same for both remaining edges then H' is a star, otherwise it is a triangle.

If $k \geqslant 4$, suppose that every monochromatic subgraph of H' with $k-1 \geqslant 3$ edges is either a triangle or a star. The remaining edge e must share a vertex with all the other edges, if they are in the shape of a star then e covers the center (otherwise e has to cover $k-1 \geqslant 3$ vertices which is impossible) and if they are in the shape of a triangle then e is one of them already (as every pair on vertices in a triangle are already forming and edge) which is not allowed as H is a simple graph. Therefore H' is a star.

By induction this shows the remark. □

Corollary B.0.1. *If c is an optimal proper coloring, by noting p the number of maximum monochromatic subgraphs shaped like a star and q the number of maximum monochromatic subgraphs shaped like a triangle, as every edge has a color $\chi(\mathrm{KG}(H)) = p + q$.*

Proposition B.0.2. *A graph is bipartite if and only if it does not contain a circuit of odd length.*

Proof. Let $H = (V, E)$ be a graph, if it contains a circuit of odd length then at least three colors are needed for a proper coloring of the vertices to exist, by contraposition if H is bipartite then it does not contain any circuit of odd length.

If H does not contain any circuit of odd length, suppose H is not empty and also connected. Let r be a vertex of H, the following coloring is a proper coloring with two colors,

$$\begin{array}{rccl} \Phi: & V & \longrightarrow & \{0,1\} \\ & v & \longmapsto & \begin{cases} 0 \text{ if there exists a walk between } v \text{ and } r \text{ of even length} \\ 1 \text{ otherwise.} \end{cases} \end{array}$$

Indeed, let v be a vertex of H, P and P' two paths between v and r. Let H' bet the subgraph induced by the set of edges $\mathrm{E}(P)\Delta\mathrm{E}(P')$. Every vertex u in H' is covered by an even number of edges, indeed if u belonged to only one path then it is covered by two edges, if it belonged to both paths then it is covered by either four, two or no edges depending on the edges that belonged to both paths. According to Proposition B.0.3, H' is a union of paths that do not share any edge two by two, therefore $\left|\mathrm{E}(P)\Delta\mathrm{E}(P')\right|$ is an even number and

$$\left|\mathrm{E}(P)\Delta\mathrm{E}(P')\right| = |\mathrm{E}(P)| + \left|\mathrm{E}(P')\right| - 2\left|\mathrm{E}(P)\cup\mathrm{E}(P')\right|$$

so the lengths of P and P' have the same parity. This shows that every path between two vertices of H have lengths of the same parity.

Let w be a vertex such that $vw \in E$, if $w \notin P$ then by adding $\{w, uw\}$ at the beginning of P a path is created between w and r which length is not of the same parity as P, if $w \in P$ then by deleting the part of P between v and w a path is created between w and r which length is not of the same parity as P for an odd number of edges has been deleted from P (the deleted part added to $\{v, vw\}$ is a circuit of even length). This shows that if $vw \in E$ then $\Phi(v) \neq \Phi(w)$.

Φ is therefore a proper coloring of H using two colors which means that H is bipartite. If H is not connected then by working separately on each *maximum connected subgraph* (i.e. it contains every connected subgraph it intersects) such a coloring can again be created. This shows the proposition. □

Proposition B.0.3. *If any vertex of a graph is covered by an even number of edges then the graph is a union of circuits that do not share any edge two by two.*

Proof. Let $H = (V, E)$ be a graph such that any vertex of a graph is covered by an even number of edges. If V or E is empty then the result is immediate. Suppose H is not empty, let v_0 be a vertex of H such that v_0 is covered by an edge e_1, then v is covered by at least two edges because it is covered by an even number of edges. There exists then a vertex v_1 such that $v_0 v_1 = e_1$ and an edge $e_2 \neq e_1$ covering v_1. By iterations a path can be created, until the path meets a vertex that it has already covered (this will necessarily happen as H is finite) thus creating a circuit. By deleting the edges of the circuit and then the vertices that are not covered by any edge anymore from the graph, H is the union of the created circuit and the remaining subgraph whose vertices are also covered by an even number of edges, these subgraphs do not share any edge. By induction in regard to the number of edges in H this shows the proposition. $\square$