* let $C \subseteq \mathbb{R}^3$ be a closed bounded convex set.
* let $B = \partial C$ be its boundary.
* let $D = \{ p+q \mid p, q \in B \}$

show that D is also convex.

Proof:

let $x_1, x_2 \in D$ and $\lambda \in [0,1]$

we want to show that $x = \lambda x_1 + (1-\lambda) x_2$ is in D.

step 1: we show that $\frac{1}{2} x$ is in C

we can write $x_1 = a_1 + b_1$

and $x_2 = a_2 + b_2$

where $a_1, b_1, a_2, b_2 \in B$

then
$$\frac{1}{2} x = \frac{1}{2} \lambda x_1 + \frac{1}{2} (1-\lambda) x_2$$
$$= \frac{1}{2} \lambda (a_1 + b_1) + \frac{1}{2} (1-\lambda)(a_2 + b_2)$$
$$= \frac{1}{2} \underbrace{\left[\lambda a_1 + (1-\lambda) a_2 \right]}_{\in C} + \frac{1}{2} \underbrace{\left[\lambda b_1 + (1-\lambda) b_2 \right]}_{\in C}$$

so $\frac{1}{2} x \in C$ by convexity.

Step 2: we show that the intersection of C with the plane orthogonal to x going through the point at $\frac{1}{2}x$ is a non empty closed bounded convex set.

* $\frac{1}{2}x$ is in the intersection so it is not empty.
* the plane is also closed and convex so the intersection is closed and convex.
* the intersection is bounded because C is bounded.

we denote I to be this intersection.
I is either the point $\{\frac{1}{2}x\}$, a segment or a closed bounded convex surface.

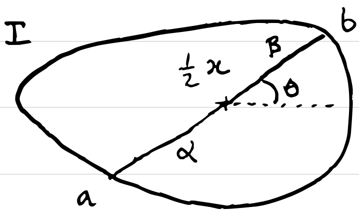
the first two cases are easier because $\frac{1}{2}x$ is in the boundary and $x = \frac{1}{2}x + \frac{1}{2}x$ is in D.

we then consider the third case.

step 3 : we find two points in the boundary of I to construct a parallelogram.

* If $\frac{1}{2}x$ is on the boundary then by the previous argument x is in D.

* If $\frac{1}{2}x$ is in the interior then we are in the following situation:



tracing a line going through $\frac{1}{2}x$ we can look at the quantity $\beta - \alpha$ as a function of θ.

this function is continuous and

$f(\theta + \pi) = - f(\theta)$ so by continuity there must exist a θ_0 such that $f(\theta_0) = 0$

we denote a and b the associated points.

By construction a and b are in B and $(0, a, b, x)$ is a parallelogram and $\vec{Oa} + \vec{Ob} = \vec{x}$

therefore $x \in D$. QED