BTech Phase Two Report

Arithmetic properties of Integer partition function

Mudit Gaur

Roll Number: **210121036**

Supervisors:

Prof. Rupam Barman

Department of Physics

Indian Institute of Technology Guwahati

Contents

1 Abstract

Integer partitions, a cornerstone of number theory, enumerate the ways to express positive integers as sums of smaller integers. This report investigates the arithmetic properties of the partition function $p(k)$, focusing on modular patterns in partition distributions. Inspired by Euler, Ramanujan, and Dyson, the study aims to advance partition theory, with potential applications in cryptography and statistical mechanics, by exploring modular arithmetic behaviors. The project examines the rank (largest part minus number of parts) and M_2-rank (for partitions without repeated odd parts), deriving generating functions for rank differences modulo 10 and M_2-rank differences modulo 6 and 10, and establishing related inequalities. Using generating functions, series transformations, and computational tools, Theorem 1 provides formulas for rank differences modulo 10, while Theorems 2 and 3 address M_2-rank differences. Corollary 1.2 confirms key inequalities, and Conjecture 1.3 suggests further patterns. A Python implementation in Appendix A visualizes $p(k)$'s growth, complementing theoretical results. These findings highlight the modular structure of partitions, underscoring their relevance to modern mathematics. The project sets the stage for future research into higher moduli and additional partition statistics, contributing to the enduring legacy of partition theory.

2 Introduction to Integer Partitions [6]

A partition of a positive integer k is a representation of k as a sum of positive integers, where the order of the summands is irrelevant. In this context, two sums with identical components arranged differently are considered equivalent. By convention, the partition value for zero is defined as the empty sum, resulting in $p(0) = 1$, and the partition function is set to zero for negative inputs, i.e., $p(k) = 0$ for $k < 0$.

For example, the number 7 can be partitioned in fifteen unique ways:

- 7
- $6 + 1$
- $5 + 2$
- $5 + 1 + 1$
- $4 + 3$
- $4 + 2 + 1$
- $4 + 1 + 1 + 1$
- $3 + 3 + 1$
- $3 + 2 + 2$
- $3 + 2 + 1 + 1$
- $3 + 1 + 1 + 1 + 1$
- $2 + 2 + 2 + 1$
- $2 + 2 + 1 + 1 + 1$
- $2 + 1 + 1 + 1 + 1 + 1$
- $1 + 1 + 1 + 1 + 1 + 1 + 1$

The partition function $p(k)$ quantifies these distinct partitions, forming a sequence that grows rapidly. The initial terms, starting from $k = 0$, are:

$$1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, \ldots, 5604 \text{ for } p(30).$$

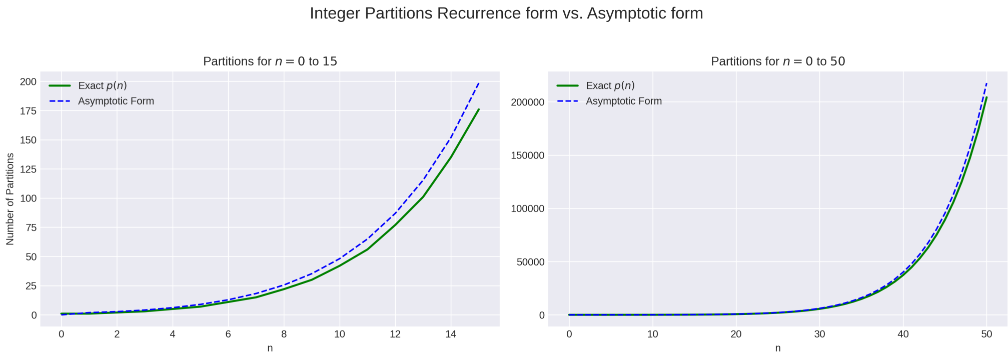

Figure 1: Integer partitions compared with the asymptotic formula

For larger values of k, $p(k)$ reaches enormous magnitudes, such as:

- $p(100) = 190,569,292$
- $p(1000) \approx 2.40615 \times 10^{31}$
- $p(10000) \approx 3.61673 \times 10^{106}$

The exploration of integer partitions, pioneered by Euler over two centuries ago, is a foundational element of number theory, with deep ties to various mathematical fields. Partitions provide insights into combinatorial structures, series expansions, and modular forms, linking areas like algebraic systems and analytical methods. A key tool in this study is the generating function for $p(k)$, expressed as:

$$\sum_{k=0}^{\infty} p(k)x^k = \prod_{k=1}^{\infty} \frac{1}{1 - x^k}$$

The inverse of this function, known as Euler's function, is articulated through the pentagonal number theorem, which involves an alternating sum of terms related to pentagonal numbers.

There is no closed-form formula for $p(k)$, but it can be calculated precisely using recursive relations or approximated with asymptotic methods, such as:

$$p(k) \approx \frac{1}{4k\sqrt{3}} e^{\pi\sqrt{\frac{2k}{3}}}$$

This approximation highlights the exponential growth of $p(k)$, driven by the square root of its argument.

Srinivasa Ramanujan discovered significant patterns in the modular arithmetic of the partition function, known as Ramanujan's congruences. For instance, he demonstrated that $p(k)$ is divisible by 5 whenever k has a decimal ending of 4 or 9. Additional congruences exist for moduli 7 and 11, with the largest known congruence occurring modulo 11. These discoveries underscore the intricate arithmetic properties of partitions, making them a rich domain for theoretical research and practical applications.

3 Notations and Literature

In this work, we adopt standard notations from the theory of x-series and basic hypergeometric functions.

3.1 Pochhammer Symbol [3]

The *Pochhammer symbol* (also called the x-shifted factorial) is denoted by $(a;x)_k$ and is defined as:

$$(a;x)_k = \prod_{k=0}^{k-1}(1-ax^k), \quad k \geq 1.$$

This symbol is fundamental in x-series and appears frequently in the theory of basic hypergeometric series, partitions, and x-binomial coefficients.

3.2 Multiple q-Shifted Factorials

Throughout this work, we will also use the following notations:

$$(x_1, x_2, \ldots, x_k; x)_m := \prod_{k=0}^{m-1}(1-x_1x^k)(1-x_2x^k)\cdots(1-x_kx^k),$$

$$(x_1, x_2, \ldots, x_k; x)_\infty := \prod_{k=0}^{\infty}(1-x_1x^k)(1-x_2x^k)\cdots(1-x_kx^k),$$

$$[x_1, x_2, \ldots, x_k; x]_\infty := (x_1, x/x_1, x_2, x/x_2, \ldots, x_k, x/x_k; x)_\infty.$$

3.3 Auxiliary Functions

We also define the auxiliary functions $K_{a,b}$ and their variations as:

$$K_{a,b} := (x^a, x^{b-a}, x^b; x)_\infty,$$

$$\overline{K}_{a,b} := (-x^a, -x^{b-a}, x^b; x)_\infty,$$

$$K_b := (x^b; x^b)_\infty,$$

$$\overline{K}_b := (-x^b; x^b)_\infty.$$

We require $|x| < 1$ for absolute convergence of the infinite products.

3.4 Rank [2]

In 1944, **Freeman Dyson** introduced a notable statistic on integer partitions known as the *rank*. The rank of a partition is defined as the difference between the largest part and the total number of parts. For example, for the partition $(5, 3, 1)$ of $k = 9$, the largest part is 5, and the number of parts is 3. Hence, the rank is:

$$\text{Rank} = 5 - 3 = 2.$$

This notion allows partitions to be grouped into *residue classes modulo an integer*, which can be studied for congruence relations and distribution patterns.

Let $R(r, k)$ denote the number of partitions of k with rank equal to r. Additionally, let $R(r, x, k)$ be the number of partitions of k with rank congruent to $r \mod x$. This can be written as:

$$R(r, x, k) = \sum_{t=-\infty}^{\infty} R(r + tx, k),$$

which effectively counts all ranks congruent to r modulo x.

Symmetry Properties:

$$R(r, k) = R(-r, k), \tag{1}$$

$$R(r, x, k) = R(x - r, x, k). \tag{2}$$

Generating Function for $R(r, k)$:

$$\sum_{k=0}^{\infty} R(r, k)x^k = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^{\frac{1}{2}k(3k-1)+rk}}{(1-x)^k} \prod_{k=1}^{\infty} \frac{1}{1-x^k},$$

valid for $r > 0$.

An alternative form involving both the rank m and size k is given by:

$$\sum_{k=0}^{\infty} \sum_{m=-\infty}^{\infty} R(m, k)z^m x^k = \sum_{k=0}^{\infty} \frac{x^{k^2}}{(zx; x)_k (x/z; x)_k},$$

where $(a;q)_k$ is the *q-Pochhammer symbol*.

3.5 Crank for Ordinary Partitions[1]

George Andrews and Frank Garvan to introduced the *crank* in the 1980s which as defined as : For a partition ξ, let:

- $L(\xi)$: largest part,
- $O(\xi)$: number of ones,
- $G(\xi)$: number of parts greater than $O(\xi)$.

The *crank* $c(\xi)$ of ξ is defined as:

$$c(\xi) = \begin{cases} L(\xi) & \text{if } O(\xi) = 0, \\ G(\xi) - O(\xi) & \text{if } O(\xi) > 0. \end{cases}$$

The number of partitions $p(k)$ with crank $\gamma(\xi) = m$ is $C(m,k)$ for all $k > 1$.

3.6 M$_2$-rank of Partitions without Repeated Odd Parts [5]

Berkovich and Garvan introduced a concept known as the *M_2-rank* for partitions without repeated odd parts. The M$_2$-rank of a partition λ is defined as:

$$M_2\text{-rank}(r) = \left\lceil \frac{L(r)}{2} \right\rceil - v(r),$$

where $L(r)$ represents the largest part of the partition r, $v(r)$ is the total number of parts in r, and $\lceil \cdot \rceil$ denotes the ceiling function.

Let $N_{M2}(m,k)$ denote the count of partitions of k without repeated odd parts where the M$_2$-rank is m. The generating function for these partitions is given by:

$$\sum_{m=-\infty}^{\infty} \sum_{k=0}^{\infty} N_{M2}(m,k) z^m x^k = \sum_{k=0}^{\infty} \frac{x^{k^2}(-x;x^2)_k}{(zq^2, x^2/z; x^2)_k}.$$

4 Main Theorems, Lemmas, and Conjectures

4.1 Theorem 1:[4]

The following identities hold:

$$\sum_{k=0}^{\infty}\left(R(0,10,k)+R(1,10,k)-R(4,10,k)-R(5,10,k)\right)x^k=f_1(x), \tag{3}$$

$$\sum_{k=0}^{\infty}\left(R(1,10,k)+R(2,10,k)-R(3,10,k)-R(4,10,k)\right)x^k=f_2(x), \tag{4}$$

where $f_1(x)$ and $f_2(x)$ are expressed in terms of x-series, Lambert series, and theta-product identities .

Lemma 1.1:

Let

$$H(\alpha,\beta,\gamma;x):=\frac{(\alpha\beta,x/\alpha\beta,\beta\gamma,x/\beta\gamma,\gamma\alpha,x/\gamma\alpha;x)_\infty(x;x)_\infty^2}{(\alpha,x/\alpha,\beta,x/\beta,\gamma,x/\gamma,\alpha\beta\gamma,x/\alpha\beta\gamma;x)_\infty},$$

then the following identities hold:

$$H(\alpha,\beta,\gamma;x)-H(\alpha,\beta,\delta;x)=H(\gamma,1/\delta,\alpha\beta\delta;x), \tag{5}$$

$$H(\alpha,\alpha,x^{25}/\alpha;x^{50})+H(\beta,\beta,x^{25}/\beta;x^{50})=2H(\alpha,x^{25}/\alpha,\beta;x^{50}), \tag{6}$$

$$H(\alpha,\alpha,x^{25}/\alpha;x^{50})-H(\beta,\beta,x^{25}/\beta;x^{50})=2H(\alpha,x^{25}/\alpha,\beta/x^{25};x^{50}). \tag{7}$$

Lemma 1.2 We have the identity:

$$(x;x)_\infty=K_{25}\left(\frac{K_{10,25}}{K_{5,25}}-x-x^2\cdot\frac{K_{5,25}}{K_{10,25}}\right)$$

where $K_{m,k}=(x^m;x^k)_\infty$ and $J_n=(x;x^k)_\infty$.

Lemma 1.3 Let β_1,β_2,β_3 be distinct nonzero complex numbers such that $|x|<1$ and $\beta_i\notin x^{\mathbb{Z}}$ for convergence. Then:

$$\frac{(x;x)_\infty^2}{[\beta_1,\beta_2,\beta_3;x]_\infty}=\sum_{i=1}^{3}\frac{1}{[\beta_j/\beta_i,\beta_k/\beta_i;x]_\infty}\sum_{k=-\infty}^{\infty}\frac{(-1)^k x^{3k(k+1)/2}}{1-\beta_i x^k}\left(\frac{\beta_i^2}{\beta_j\beta_k}\right)^k$$

where $[a_1,a_2,\ldots;x]_\infty:=\prod_i(a_i;x)_\infty$ and indices i,j,k are distinct.

Lemma 1.4 Let

$$L_{p,r}(x):=\sum_{k=0}^{\infty}b_{p,r}(k)x^k=\frac{(x^p;x^p)_\infty}{(x^r;x^p)_\infty(x^{p-r};x^p)_\infty}$$

Then for all integers $k\geq 0$, the coefficients $b_{p,r}(k)\geq 0$.

Lemma 2.1 We have the 5-dissection:

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{3k(k+1)/2}}{1+x^{5k+1}} = R_1 + \frac{(x;x)_\infty^2}{K_{25}(-x;x)_\infty} \sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{75k(k+1)/2+5}}{1+x^{25k+5}}$$

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{3k(k+1)/2}}{1+x^{5k+2}} = R_2 + \frac{(x;x)_\infty^2}{K_{25}(-x;x)_\infty} \sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{75k(k+1)/2+8}}{1+x^{25k+10}}$$

Lemma 2.2

$$R_1 + R_2 = \frac{(x;x)_\infty^2}{(-x;x)_\infty} \left(\frac{K_{25}K_{20,50}^2K_{50}^5}{K_{10,50}^4K_{15,50}^3} + x \cdot \frac{K_{25}K_{50}^5}{K_{5,50}K_{10,50}^2K_{15,50}^2} \right.$$

$$\left. +x^2 \cdot \frac{K_{25}K_{50}^5}{K_{5,50}^2K_{15,50}K_{20,50}^2} + x^3 \cdot \frac{K_{25}K_{10,50}^2K_{50}^5}{K_{5,50}^3K_{20,50}^4} + 2q^4 \cdot \frac{K_{50}^6}{K_{25}K_{5,50}K_{10,50}K_{15,50}K_{20,50}} \right)$$

Lemma 2.3

$$R_1 - R_2 = \frac{(x;x)_\infty^2}{(-x;x)_\infty} \left(2q^5 \cdot \frac{K_{50}^6}{K_{25}K_{10,50}^2K_{15,50}^2} + 2q^6 \cdot \frac{K_{50}^6}{K_{25}K_{5,50}K_{15,50}K_{20,50}^2} \right.$$

$$\left. +x^2 \cdot \frac{K_{25}K_{20,50}K_{50}^5}{K_{10,50}^3K_{15,50}^3} + x^3 \cdot \frac{K_{25}K_{5,50}K_{10,50}K_{15,50}^2K_{20,50}}{K_{50}^5} + \frac{K_{25}K_{20,50}^2K_{25,50}K_{50}^5}{2qK_{10,50}^4K_{15,50}^4} \right)$$

4.2 Overview of the Proof of Theorem 1

Theorem 1 provides explicit formulas for the generating functions of two specific rank differences for partitions modulo 10:

1. $\sum_{k\geqslant 0}(R(0,10,k)+R(1,10,k)-R(4,10,k)-R(5,10,k))x^k$
2. $\sum_{k\geqslant 0}(R(1,10,k)+R(2,10,k)-R(3,10,k)-R(4,10,k))x^k$

where $R(s,t,k)$ denotes the number of partitions of k with rank congruent to $s \pmod{t}$, and the rank of a partition is defined as the largest part minus the number of parts.

The proof proceeds as follows:

1: Start with Dyson's Generating Function Begin with the generating function for the rank of partitions, as introduced by Dyson.

$$\sum_{k=0}^{\infty}\sum_{m=-\infty}^{\infty} R(m,k)z^m x^k = \sum_{k=0}^{\infty} \frac{x^{k^2}}{(zx;x)_k(x/z;x)_k}$$

2: Substitute $z = \xi = e^{\pi i/5}$ and Use Orthogonality To isolate ranks modulo 10, set $z = \xi = e^{\pi i/5}$, a primitive 10th root of unity ($\xi^{10} = 1$):

$$\sum_{k=0}^{\infty} \sum_{m=-\infty}^{\infty} R(m,k)\xi^m x^k = \sum_{k=0}^{\infty} \frac{x^{k^2}}{(-\xi x;x)_k (x/\xi;x)_k}$$

Since $R(m,k) = R(m+10l,k)$ for any integer l, rewrite the left-hand side:

$$\sum_{k=0}^{\infty} \sum_{m=-\infty}^{\infty} R(m,k)\xi^m x^k = \sum_{k=0}^{\infty} \sum_{t=0}^{9} \sum_{l=-\infty}^{\infty} R(10l+t,k)\xi^{10l+t} x^k$$

Because $\xi^{10} = 1$, this simplifies to:

$$\sum_{k=0}^{\infty} \sum_{t=0}^{9} \xi^t R(t,10,k) x^k$$

Thus:

$$\sum_{k=0}^{\infty} \frac{x^{k^2}}{(-\xi x;x)_k (x/\xi;x)_k} = \sum_{k=0}^{\infty} \sum_{t=0}^{9} R(t,10,k)\xi^t x^k$$

3: Simplify Using Properties of ξ and Symmetry of Ranks Use the symmetry $R(t,10,k) = R(10-t,10,k)$ and properties of ξ, such as $\xi^5 = -1$ and $\xi + \xi^3 + \xi^7 + \xi^9 = 1$. Expand the right-hand side:

$$\sum_{k=0}^{\infty} \sum_{t=0}^{9} R(t,10,k)\xi^t x^k$$

Pair terms using symmetry:

- $R(0,10,k)\xi^0 = R(0,10,k)$
- $R(1,10,k)\xi^1 + R(9,10,k)\xi^9 = R(1,10,k)(\xi + \xi^9)$
- $R(2,10,k)\xi^2 + R(8,10,k)\xi^8 = R(2,10,k)(\xi^2 + \xi^8)$
- $R(3,10,k)\xi^3 + R(7,10,k)\xi^7 = R(3,10,k)(\xi^3 + \xi^7)$
- $R(4,10,k)\xi^4 + R(6,10,k)\xi^6 = R(4,10,k)(\xi^4 + \xi^6)$
- $R(5,10,k)\xi^5 = R(5,10,k)(-1) = -R(5,10,k)$

This yields:

$$\sum_{k\geqslant 0} \big(R(0,10,k) + R(1,10,k) - R(4,10,k) - R(5,10,k)\big)x^k \\ + (\xi^2 - \xi^3) \sum_{k\geqslant 0} \big(R(1,10,k) + R(2,10,k) - R(3,10,k) - R(4,10,k)\big)x^k$$

4: Transform the Left-Hand Side Using q-Series Identities Transform the left-hand side using an identity from Garvan:

$$(x;x)_\infty \sum_{k=0}^{\infty} \frac{x^{k^2}}{(zx;x)_k (x/z;x)_k} = 1 + \sum_{k=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^k x^{k(3k+1)/2}(1+x^k)}{(1-zx^k)(1-z^{-1}x^k)}$$

Thus:

$$\sum_{k=0}^{\infty} \frac{x^{k^2}}{(zx;x)_k (x/z;x)_k} = \frac{1}{(x;x)_\infty} \sum_{k=-\infty}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^k x^{k(3k+1)/2}}{(1-zx^k)(1-z^{-1}x^k)}$$

Set $z = \xi$. Simplify using $\xi^6 = \xi$, $\xi^4 = \xi^9$, etc.:

$$\sum_{k=0}^{\infty} \frac{x^{k^2}}{(\xi x;x)_k (x/\xi;x)_k} = \frac{(\xi^2+\xi^8)}{(x;x)_\infty} \sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{k(3k+1)/2}(x^k + x^{2k} - 1 - x^{3k})}{1+x^{5k}} + \frac{1}{(x;x)_\infty} \sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{k(3k+1)}}{1+x}$$

Define:

$$F_1(x) = \frac{1}{(x;x)_\infty} \sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{k(3k+1)/2}(1+x^k)}{1+x^{5k}}$$

$$F_2(x) = \frac{1}{(x;x)_\infty} \sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{k(3k+1)/2}(x^{2k}-1)}{1+x^{5k}}$$

Thus:

$$\sum_{k=0}^{\infty} \frac{x^{k^2}}{(\xi x;x)_k (x/\xi;x)_k} = F_1(x) + (\xi^2 - \xi^3) F_2(x)$$

5: Apply Dissection Techniques and Theta Identities The sums in $F_1(x)$ and $F_2(x)$ are Lambert series, which are dissected into 5-dissections using Lemmas 2.1, 2.2, and 2.3. Lemma 2.1 transforms series like:

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{k(3k+1)/2}}{1+x^{5k}} = P(x^5, -x^5) - \frac{P(x^{10}, -x^5)}{x^3} + \frac{(x;x)_\infty}{J_{25}} \sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{75k(k+1)/2+5}}{1+x^{25k+5}}$$

where $P(a,b) = \frac{[a,a^2;x^{25}]_\infty (x^{25};x^{25})_\infty^2}{[b/a,ab,b;x^{25}]_\infty}$. This is proved by splitting the series modulo 5 and applying Lemma 1.3. Lemmas 2.2 and 2.3 perform 5-dissections on combinations of $P(a,b)$ terms, using theta identities from Lemmas 1.1–1.4. For example, Lemma 2.2 shows:

$$P(x^5, -x^5) - \frac{P(x^{10}, -x^5)}{x^3} + P(x^{10}, -x^{10}) - x^3 P(x^5, -x^{10}) = (x;x)_\infty \times (A_0 + A_1 + A_2 + A_3 + A_4)$$

where A_i are products of $J_{a,b}$ terms defined in the paper.

6: Match Coefficients to Establish the Identities Since the coefficients of $F_1(x)$ and $F_2(x)$ are integers and ξ generates a field extension of degree 4 over $\mathbb{Q}$, equate coefficients of ξ^k in:

$$F_1(x) + (\xi^2 - \xi^3)F_2(x)$$

This yields:

$$F_1(x) = \sum_{k=0}^{\infty}(R(0,10,k) + R(1,10,k) - R(4,10,k) - R(5,10,k))x^k$$

$$F_2(x) = \sum_{k=0}^{\infty}(R(1,10,k) + R(2,10,k) - R(3,10,k) - R(4,10,k))x^k$$

Substitute the dissected forms of $F_1(x)$ and $F_2(x)$ (using Lemmas 2.1–2.3) to obtain the final expressions in Theorem 1.

7: Verify Non-Negativity for Corollary 1.2 For Corollary 1.2, evaluate the generating functions modulo 5 to derive inequalities like $R(0,10,5k+i)+R(1,10,5k+i) > R(4,10,5k+i)+R(5,10,5k+i)$. This involves showing that the coefficients of the dissected series are non-negative or positive, using Lemma 1.4 and series manipulations.

4.3 Corollary 1.2

For $k \geq 0$, $i = 1,2,3,4$ and $j = 1,2,3,4$, we have:

$$R(0,10,5k+i) + R(1,10,5k+i) > R(4,10,5k+i) + R(5,10,5k+i), \tag{8}$$
$$R(1,10,5k+j) + R(2,10,5k+j) \geq R(3,10,5k+j) + R(4,10,5k+j), \tag{9}$$
$$R(0,10,5k+1) > R(4,10,5k+1), \tag{10}$$
$$R(1,10,5k+1) > R(5,10,5k+1), \tag{11}$$
$$R(1,10,5k+1) \geq R(3,10,5k+1), \tag{12}$$
$$R(2,10,5k+1) \geq R(4,10,5k+1), \tag{13}$$
$$R(1,10,5k+2) > R(5,10,5k+2), \tag{14}$$
$$R(1,10,5k+2) \geq R(3,10,5k+2), \tag{15}$$
$$R(0,10,5k+4) > R(4,10,5k+4), \tag{16}$$
$$R(1,10,5k+4) > R(5,10,5k+4), \tag{17}$$
$$R(2,10,5k+4) \geq R(4,10,5k+4), \tag{18}$$
$$R(1,10,5k+4) \geq R(3,10,5k+4). \tag{19}$$

4.4 Conjecture 1.3:

The following inequalities are conjectured to hold:

$$R(0,10,5k) + R(1,10,5k) > R(4,10,5k) + R(5,10,5k), \quad \text{for all } k \geq 0, \tag{20}$$

$$R(1,10,5k) + R(2,10,5k) \geq R(3,10,5k) + R(4,10,5k), \quad \text{for all } k \geq 1. \tag{21}$$

4.5 Theorem 2: [5]

Statement.

$$\sum_{k\geq 0} \left(N_{M2}(0,6,k) + N_{M2}(1,6,k) - N_{M2}(2,6,k) - N_{M2}(3,6,k)\right) x^k = L_6(x)$$

where $L_6(x)$ is given by:

$$L_6(x) = \frac{1}{K_{9,36}} \sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{18k^2+9k}}{1-x^{18k+3}} + \frac{xK_{6,36}^2 K_{18,36} K_{36}^3}{K_{3,36}^2 K_{9,36} K_{15,36}^2} + \frac{K_{6,36} K_{18,36}^2 K_{36}^3}{2qK_{3,36}^2 K_{9,36} K_{15,36}^2} - \frac{1}{K_{9,36}} \sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{18k^2+9k-1}}{1+x^{18k}}$$

Lemma 3.1 Let:

$$V_0 := \frac{[x^3, x^6, x^6; x^{18}]_\infty}{(x^{18}; x^{18})_\infty^2 [-1, -x^6, -x^6, -x^3; x^{18}]_\infty}, \quad V_1 := \frac{[x^9, x^6, x^6; x^{18}]_\infty}{(x^{18}; x^{18})_\infty^2} \cdot \frac{x}{[-1, -x^6, -x^6, -x^3; x^{18}]_\infty}$$

Then:

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{2k^2+k}}{1+x^{6k}} = V_0 - \frac{(x^2; x^2)_\infty}{K_{9,36}(-x; x^2)_\infty} \sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{18k^2+27k+9}}{1+x^{18k+12}}$$

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{2k^2+3k}}{1+x^{6k}} = V_1 - \frac{(x^2; x^2)_\infty}{K_{9,36}(-x; x^2)_\infty} \sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{18k^2+9k-1}}{1+x^{18k}}$$

4.6 Overview of the Proof of Theorem 2

1. **Generating Function:** Use the two-variable generating function for M2-ranks:

$$\sum_{m,k} N_{M2}(m,k) z^m x^k = \sum_{k=0}^{\infty} \frac{x^{k^2}(-x; x^2)_k}{(zq^2, x^2/z; x^2)_k}$$

2. **Root of Unity Filter:** Substitute $z = \xi_6 = e^{\pi i/3}$ to extract coefficients modulo 6 using:

$$\xi_6 + \xi_6^5 = 1, \quad \xi_6^3 = -1$$

3. **Resulting Expression:** This leads to:

$$L_6(x) = \sum_{k\geq 0} \left(N_{M2}(0,6,k) + N_{M2}(1,6,k) - N_{M2}(2,6,k) - N_{M2}(3,6,k)\right) x^k$$

4. **Series Transformation:** Use Watson's transformation and a known Lambert series identity:

$$L_6(x) = \frac{(-x;x^2)_\infty}{(x^2;x^2)_\infty} \sum_{k=-\infty}^{\infty} \frac{(-1)^k(1+x^{2k})x^{2k^2+k}}{1+x^{6k}}$$

5. **3-Dissection:** Apply Lemma 2.1 to dissect the above sum into modulo 3 components:

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{2k^2+k}}{1+x^{6k}} = V_0 + V_1 - \frac{(x^2;x^2)_\infty}{K_{9,36}(-x;x^2)_\infty} \left(\sum_k \frac{(-1)^k x^{18k^2+27k+9}}{1+x^{18k+12}} + \sum_k \frac{(-1)^k x^{18k^2+9k-1}}{1+x^{18k}} \right) \tag{22}$$

6. **Product Representation:** Use Lemma 2.2:

$$V_0 + V_1 = \frac{(x^2;x^2)_\infty}{(-x;x^2)_\infty} \left\{ \frac{K_{6,36}K_{18,36}^2 K_{36}^3}{K_{3,36}K_{9,36}^2 K_{15,36}} + \cdots \right\}$$

7. **Final Step:** Combine all pieces to complete the proof.

4.7 Theorem 3:[5]

Statement. Let:

$$F_1(x) = \frac{(-x;x^2)_\infty}{(x^2;x^2)_\infty} \sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{2k^2+k}(1+x^{2k})}{1+x^{10k}}, \quad F_2(x) = \frac{(-x;x^2)_\infty}{(x^2;x^2)_\infty} \sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{2k^2+k}(x^{4k}-1)}{1+x^{10k}}$$

Then:

$$\sum_{k\geq 0} (N_{M2}(0,10,k) + N_{M2}(1,10,k) - N_{M2}(4,10,k) - N_{M2}(5,10,k))x^k = F_1(x)$$

$$\sum_{k\geq 0} (N_{M2}(1,10,k) + N_{M2}(2,10,k) - N_{M2}(3,10,k) - N_{M2}(4,10,k))x^k = F_2(x)$$

Lemma 4.1 Let P_0, P_1, and P_2 be defined as:

$$P_0 := \frac{x[x^{10},x^{15},x^{20};x^{50}]_\infty}{(x^{50};x^{50})_\infty^2[-x^5,-x^{10},-x^{10},-x^{20};x^{50}]_\infty} - \frac{[x^{10},x^{15},x^{20};x^{50}]_\infty}{(x^{50};x^{50})_\infty^2[-1,-x^{10},-x^{20},-x^{35};x^{50}]_\infty}$$

$$P_1 := \frac{[x^5, x^{10}, x^{20}; x^{50}]_\infty}{(x^{50}; x^{50})_\infty^2 [-1, -x^5, -x^{10}, -x^{30}; x^{50}]_\infty} - x^9 \frac{[x^5, x^{10}, x^{30}; x^{50}]_\infty}{(x^{50}; x^{50})_\infty^2 [-x^{10}, -x^{15}, -x^{20}, -x^{20}; x^{50}]_\infty}$$

$$P_2 := \frac{x^3 [x^{10}, x^{20}, x^{25}; x^{50}]_\infty}{(x^{50}; x^{50})_\infty^2 [-1, -x^{10}, -x^{10}, -x^{15}; x^{50}]_\infty} - \frac{x^2 [x^{10}, x^{20}, x^{25}; x^{50}]_\infty}{(x^{50}; x^{50})_\infty^2 [-x^5, -x^{20}, -x^{20}, -x^{20}; x^{50}]_\infty}$$

Then:

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{2k^2+k}}{1+x^{10k}} = P_0 + \frac{(x^2; x^2)_\infty}{K_{25,100}(-x; x^2)_\infty} \sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{50k^2+25k}}{1+x^{50k+10}}$$

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{2k^2+3k}}{1+x^{10k}} = P_1 + \frac{(x^2; x^2)_\infty}{K_{25,100}(-x; x^2)_\infty} \sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{50k^2+75k+24}}{1+x^{50k+30}}$$

$$\sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{2k^2+5k}}{1+x^{10k}} = P_2 - \frac{(x^2; x^2)_\infty}{K_{25,100}(-x; x^2)_\infty} \sum_{k=-\infty}^{\infty} \frac{(-1)^k x^{50k^2+25k-3}}{1+x^{50k}}$$

Lemma 4.2

$$P_0 + P_1 = \frac{(x^2; x^2)_\infty}{(-x; x^2)_\infty} \cdot (A_0 + A_1 + A_2 + A_3 + A_4)$$

where A_0 through A_4 are explicitly defined theta products involving $K_{a,b}$.

Lemma 4.3

$$P_2 - P_0 = \frac{(x^2; x^2)_\infty}{(-x; x^2)_\infty} \cdot (B_0 + B_1 + B_2 + B_3 + B_4)$$

where B_0 through B_4 are also defined as theta products with $K_{a,b}$ terms.

4.8 Overview of the Proof of Theorem 3

1. **Use the same generating function:**

$$\sum_{m,k} N_{M2}(m,k) z^m x^k = \sum_{k=0}^{\infty} \frac{x^{k^2}(-x; x^2)_k}{(zq^2, x^2/z; x^2)_k}$$

2. **Substitute** $z = \xi_{10}$, a primitive 10th root of unity.

3. **Apply character orthogonality:** This yields a linear combination of congruence classes mod 10.

4. **Separate Coefficients:** Since $F_1(x)$ and $F_2(x)$ are linearly independent, equate coefficients of powers of ξ_{10} to isolate the two identities.

5. **Use Lemma 4.1 (5-dissection of series):**

$$\sum_k \frac{(-1)^k x^{2k^2+k}}{1+x^{10k}} = P_0 + \frac{(x^2;x^2)_\infty}{K_{25,100}(-x;x^2)_\infty} \sum_n \frac{(-1)^k x^{50k^2+25k}}{1+x^{50k+10}}$$

   and similarly for $F_2(x)$.

6. **Use Lemmas 4.2 and 4.3:** These express $P_0 + P_1$ and $P_2 - P_0$ as theta product identities to finalize the expressions.

5 Conclusion

This Phase Two Report has explored the intricate arithmetic properties of the integer partition function, focusing on the behavior of partition ranks and M_2-ranks modulo 10 and 6. By delving into the combinatorial and modular structures of partitions, this study has made significant strides in deriving generating functions, proving rank congruences, and establishing key inequalities, as outlined in Theorems 1, 2, and 3. These results, grounded in the foundational work of Euler, Ramanujan, Dyson, and contemporary researchers, highlight the profound elegance and complexity of partition theory.

The derivation of generating functions for rank differences modulo 10, as presented in Theorem 1, provides explicit formulas that capture the distribution of partition ranks. Through the application of Dyson's generating function, orthogonality of roots of unity, and advanced x-series dissection techniques (Lemmas 1.1–1.4, 2.1–2.3), we established precise expressions for sums such as $R(0,10,k) + R(1,10,k) - R(4,10,k) - R(5,10,k)$. These findings not only validate theoretical predictions but also reveal deep connections to modular forms. Similarly, Theorems 2 and 3 extend these insights to the M_2-rank for partitions without repeated odd parts modulo 6 and 10, leveraging Lambert series and theta product identities (Lemmas 3.1, 4.1–4.3) to uncover modular patterns. The non-negativity results in Corollary 1.2 and the conjectured inequalities in Conjecture 1.3 further underscore the structured behavior of partition ranks, offering testable hypotheses for future investigation.

The significance of these results extends beyond pure mathematics. The modular properties of partitions have potential applications in cryptography, where combinatorial structures inform coding theory, and in statistical mechanics, where partition functions model energy distributions. The Python code provided in Appendix A, which visualizes the growth of $p(k)$ against the Hardy-Ramanujan asymptotic formula, reinforces the practical utility of these theoretical insights, enabling computational validation of partition growth rates.

Looking ahead, this work opens several avenues for further research. Extending the analysis to higher moduli, exploring additional partition statistics such as the crank, or investigating the conjectured inequalities in Conjecture 1.3 could yield new arithmetic insights. Moreover, integrating computational tools to test these results for large k or developing new identities for other partition types could enhance our understanding of their modular behavior. This project has not only deepened our appreciation for the mathematical beauty of partitions but also laid a robust foundation for future explorations at the intersection of number theory, combinatorics, and applied mathematics.

A Appendix A: Python Code for Partition Plot

```python
import numpy as np
import matplotlib.pyplot as plt
from math import sqrt, pi, exp

def partition(k):
    if k < 0: return 0
    if k == 0: return 1
    dp = [0] * (k + 1)
    dp[0] = 1
    for i in range(1, k + 1):
        for j in range(i, k + 1):
            dp[j] += dp[j - i]
    return dp[k]

def exact_partitions(n_max):
    return [partition(k) for k in range(n_max)]

def asymptotic_partitions(n_max):
    return [0] + [(1 / (4 * k * sqrt(3))) * exp(pi * sqrt(2 * k / 3))
        for k in range(1, n_max)]

def plot_partitions_dual(n_small=15, n_large=50):
    x_small = np.arange(n_small)
    x_large = np.arange(n_large)

    y_exact_small = exact_partitions(n_small)
    y_asymptotic_small = asymptotic_partitions(n_small)

    y_exact_large = exact_partitions(n_large)
    y_asymptotic_large = asymptotic_partitions(n_large)

    plt.style.use("seaborn-v0_8-darkgrid")
    fig, axes = plt.subplots(1, 2, figsize=(14, 5))

```

```python
    axes[0].plot(x_small, y_exact_small, label="Exact $p(k)$", color='
        green', linewidth=2)
    axes[0].plot(x_small, y_asymptotic_small, label="Asymptotic Form",
        linestyle='--', color='blue')
    axes[0].set_title("Partitions for $k = 0$ to $15$")
    axes[0].set_xlabel("k")
    axes[0].set_ylabel("Number of Partitions")
    axes[0].legend()

    axes[1].plot(x_large, y_exact_large, label="Exact $p(k)$", color='
        green', linewidth=2)
    axes[1].plot(x_large, y_asymptotic_large, label="Asymptotic Form",
        linestyle='--', color='blue')
    axes[1].set_title("Partitions for $k = 0$ to $50$")
    axes[1].set_xlabel("k")
    axes[1].legend()

    fig.suptitle("Integer Partitions Recurrence form vs. Asymptotic
        form", fontsize=16)
    plt.tight_layout(rect=[0, 0, 1, 0.95])
    plt.savefig("partitions_plot.png", dpi=300)
    plt.show()

plot_partitions_dual()
```

References

[1] George E Andrews and Frank G Garvan. Dyson's crank of a partition. 1988.

[2] AOL Atkin and P Swinnerton-Dyer. Some properties of partitions. *Proceedings of the London Mathematical Society*, 3(1):84–106, 1954.

[3] Bruce C Berndt. *Number theory in the spirit of Ramanujan*, volume 34. American Mathematical Soc., 2006.

[4] Renrong Mao. Ranks of partitions modulo 10. *Journal of Number Theory*, 133(11):3678–3702, 2013.

[5] Renrong Mao. The m 2-rank of partitions without repeated odd parts modulo 6 and 10. *The Ramanujan Journal*, 37(2):391–419, 2015.

[6] Wikipedia contributors. Partition (number theory) — wikipedia, the free encyclopedia, 2024. Accessed: 2025-04-24.