Study of Rank & Crank for parition function p(n)

BTech Phase One Report

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Contents

1 Abstract

This report explores the arithmetic properties of partition functions, focusing on their historical development and the profound contributions of mathematicians such as Euler, Ramanujan, and Dyson. Partition functions, which count the ways an integer can be expressed as a sum of positive integers, have deep connections to number theory, combinatorics, and modular forms.

A central theme of this work is Ramanujan's congruences, which revealed surprising divisibility properties of partition numbers modulo 5, 7, and 11. While Dyson's rank provided a combinatorial explanation for the congruences modulo 5 and 7, it failed for modulo 11. This limitation led to the introduction of the *crank* by Andrews and Garvan, which successfully resolved the divisibility of $p(11n + 6)$ by 11 through a more refined classification.

The report presents detailed proofs, both classical and modern, of these congruences, highlighting their combinatorial elegance and structural symmetries. It also discusses advanced tools like generating functions, ranks, and cranks, which have significantly advanced the understanding of partition functions. This study not only revisits key results but also emphasizes their broader implications, laying the groundwork for further exploration in number theory and related fields.

2 Introduction

The concept of integer partitions is an important and intriguing area within number theory, with profound connections to combinatorics, algebra, and mathematical analysis. A partition of a positive integer n is defined as a way of expressing n as a sum of positive integers, where the order of the addends is not considered. For instance, the integer 7 can be expressed in fifteen distinct partitions:

- 7
- $6 + 1$
- $5 + 2$
- $5 + 1 + 1$
- $4 + 3$
- $4 + 2 + 1$
- $4 + 1 + 1 + 1$
- $3 + 3 + 1$
- $3 + 2 + 2$
- $3 + 2 + 1 + 1$
- $3 + 1 + 1 + 1 + 1$
- $2 + 2 + 2 + 1$
- $2 + 2 + 1 + 1 + 1$
- $2 + 1 + 1 + 1 + 1 + 1$
- $1 + 1 + 1 + 1 + 1 + 1 + 1$

The number of distinct partitions of n is represented by the partition function $p(n)$, which counts the number of ways in which n can be expressed as a sum of positive integers, without regard to the order of the terms. In this case, $p(7) = 15$.

Integer partitions not only serve as a fundamental concept in number theory but also exhibit deep connections to other mathematical fields. Their study provides insights into combinatorial identities, q-series, and modular forms. Over centuries, mathematicians have uncovered various properties of partition functions, linking them to diverse problems in mathematical analysis and algebraic structures. These connections make partitions a rich area for exploration, with both theoretical and practical implications.

Beyond their mathematical significance, partition functions play a crucial role in understanding physical systems, particularly in statistical mechanics and quantum physics. For instance, the partition function in thermodynamics, though distinct in definition, shares conceptual parallels with integer partitions. It encodes the distribution of energy states in a system, demonstrating how abstract number-theoretic concepts can inform real-world phenomena.

This report focuses on the arithmetic properties of partition functions, emphasizing Ramanujan's groundbreaking contributions and their subsequent development. By examining the generating functions and congruences associated with partitions, we aim to uncover the structural elegance underlying these functions. Additionally, the study revisits classical results, such as Euler's and Jacobi's contributions, while integrating modern combinatorial advancements like Dyson's rank and Andrews and Garvan's crank. This multifaceted exploration highlights the enduring relevance of partitions and their profound impact across mathematics and beyond.

3 A Brief History of Partition Functions

The study of partition functions has a long and captivating history, beginning with the pioneering work of Leonhard Euler in 1748. Euler was among the first to systematically investigate partition functions and their properties. He demonstrated that the generating function for the sequence $\{p(n)\}_{n=0}^{\infty}$, which represents the number of partitions of n, can be expressed as an elegant infinite product:

$$\sum_{n=0}^{\infty} p(n)x^n = \frac{1}{(1-x)(1-x^2)(1-x^3)\cdots}.$$

A major breakthrough in the study of partition functions occurred in the early 20th century. In 1918, Srinivasa Ramanujan and Godfrey Harold Hardy, through their pathbreaking work on asymptotic analysis, and independently James Victor Uspensky in 1920, derived an asymptotic formula for $p(n)$, the partition function. They demonstrated that for large n, $p(n)$ grows rapidly and can be approximated as:

$$p(n) \sim \frac{1}{4\sqrt{3}n} e^{\pi\sqrt{\frac{2n}{3}}}.$$

This result revealed not only the exponential growth of $p(n)$ but also the power of analytic methods in studying combinatorial functions, thus paving the way for further research in analytic number theory.

In addition to asymptotics, Ramanujan's contributions to partition theory included groundbreaking discoveries about its arithmetic properties. In 1919, he identified remarkable congruences satisfied by $p(n)$. For $n \in \{5, 7, 11\}$, he showed:

$$p(nk + \delta_n) \equiv 0 \pmod{n},$$

where δ_n is inverse of 24 modulo n. These congruences are considered among the most remarkable results in the theory of partitions, revealing an unexpected interplay between combinatorics and modular arithmetic.

The search for a combinatorial explanation of Ramanujan's congruences led to the introduction of new ideas. In 1944, F. Dyson, while still an undergraduate at Cambridge, introduced the concept of the "rank" of a partition. The rank, defined as the difference between the largest part and the number of parts in a partition, was designed to shed light on these congruences. Dyson's innovative approach opened new avenues of research and linked the study of partitions more closely with combinatorial statistics.

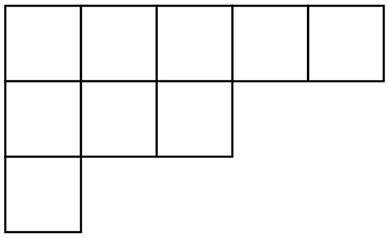
Decades later, in 1988, George Andrews and Frank Garvan extended Dyson's work by introducing the "crank," a more versatile statistic that successfully explained all of Ramanujan's congruences combinatorially. The crank provided a deeper understanding of the partition function's arithmetic properties and resolved longstanding questions in the field. This discovery marked a significant milestone, solidifying the connection between combinatorics, number theory, and modular forms, and inspiring a wealth of further research in partition theory.

4 Notations and Definitions

This section introduces and explains key notations and concepts essential to the study of partition functions and related combinatorial structures. These ideas are commonly used in number theory, combinatorics, and the analysis of integer partitions.

- **Generalized Partition Function** $p_k(n)$: This function represents the number of ways n can be partitioned into exactly k parts. The study of $p_k(n)$ is central to understanding partition congruences and their connections to modular forms.

- **Ferrers Diagram**: A Ferrers diagram visually depicts a partition by arranging rows of boxes, where the number of boxes in each row corresponds to a part of the partition. For example, the partition $(5, 3, 1)$ is represented as:



- **The q-Binomial Coefficient**: Also called the Gaussian binomial coefficient, this coefficient is expressed as:

$$\binom{n}{k}_x = \frac{(1-x^n)(1-x^{n-1})\cdots(1-x^{n-k+1})}{(1-x^k)(1-x^{k-1})\cdots(1-x)}.$$

  It is a significant tool in the study of q-series and generating functions for partitions.

- **Pochhammer Symbol** $(a;x)_n$: Known as the shifted or rising factorial, this symbol is defined as:

$$(a;x)_n = \prod_{k=0}^{n-1}(1-ax^k), \quad n \geq 1.$$

  It is widely used in q-series analysis and is integral to the generalized binomial coefficients.

5 Generating Function for the Partition Function

We can express a number n as a sum of positive integers in various ways, i.e., a partition of n can be written as:

$$n = 1 \cdot k_1 + 2 \cdot k_2 + 3 \cdot k_3 + \cdots + i \cdot k_i + \ldots$$

where k_i denotes the number of occurrences of the part i in the partition. In other words, k_1 counts the number of 1's in the partition, k_2 counts the number of 2's, and so on.

Now, consider the infinite product:

$$
\begin{aligned}
\left(1+x+x^2+x^3+\dots\right)\cdot\left(1+x^2+x^4+x^6+\dots\right)& \\
\cdot\left(1+x^3+x^6+x^9+\dots\right)&\cdot\dots \\
=\prod_{i=1}^{\infty}\sum_{j=0}^{\infty}x^{i\cdot j}&
\end{aligned}
$$

This product can be written more succinctly as:

$$\prod_{i=1}^{\infty}\left(1+x^{i\cdot 1}+x^{i\cdot 2}+x^{i\cdot 3}+\dots\right)$$

Each factor corresponds to a part i in the partition, and the exponent of x reflects how many times part i appears. More specifically, $x^{i\cdot j}$ represents the part i appearing j times in the partition.

Thus, choosing each term in the product represents selecting how many times each part i appears in the partition of n. The product accounts for all possible ways parts can appear multiple times.

Now, let's simplify the product. Each sum over j is a geometric series, which can be written as:

$$\sum_{j=0}^{\infty}x^{i\cdot j}=\frac{1}{1-x^i}\quad\text{for}\quad |x|<1$$

Therefore, the infinite product becomes:

$$\prod_{i=1}^{\infty}\frac{1}{1-x^i}$$

Now, we can expand the product as a power series in x. The coefficient of x^n in this series corresponds to the number of distinct partitions of n. That is, it gives the number $p(n)$, the partition function.

Thus, we have the following generating function for the partition function:

$$\prod_{i=1}^{\infty}\frac{1}{1-x^i}=\sum_{n=0}^{\infty}p(n)x^n$$

This result shows that $p(n)$, can be extracted from the coefficient of x^n in the expansion of the infinite product $\prod_{i=1}^{\infty}\frac{1}{1-x^i}$. Each term in the expanded series corresponds to a partition of n, where the exponent of x represents the total sum of the parts in that partition.

Therefore, we can summarize this relation as:

$$\prod_{i=1}^{\infty}\sum_{j=0}^{\infty} x^{i\cdot j} = \prod_{i=1}^{\infty}\frac{1}{1-x^i} = \sum_{n=0}^{\infty} p(n)x^n$$

$$\sum_{n=0}^{\infty} p(n)x^n = (x;x)_\infty$$

This elegant expression ties together the generating function for the partition function and its combinatorial interpretation, providing a compact way to understand and compute partition numbers for any integer n.

6 Essential Theorems

All theorems in this section are referenced from [3]

Theorem 1 (Theorem of Partitions into Distinct and Odd Parts). *The number of partitions of a positive integer n into distinct parts is equal to the number of partitions of n into odd parts. This relationship can be represented using generating functions as follows:*

$$\sum_{n=0}^{\infty} d(n)x^n = (-x;x)_\infty = \frac{(x^2;x^2)_\infty}{(x;x)_\infty} = \sum_{n=0}^{\infty} o(n)x^n.$$

Here, the L.H.S corresponds to the generating function for partitions into distinct parts, while the R.H.S corresponds to the generating function for partitions into odd parts.

Theorem 2 (Jacobi's triple product). *For $|x| < 1$ and $z \neq 0$, the following elegant identity is true:*

$$\sum_{n=-\infty}^{\infty} z^n x^{n^2} = \prod_{k=0}^{\infty}(1+zx^{2k+1})\prod_{k=0}^{\infty}\left(1+\frac{x^{2k+1}}{z}\right)\prod_{k=1}^{\infty}(1-x^{2k}).$$

Theorem 3 (Euler's pentagonal number theorem). *The generating function for integer partitions in terms of an infinite product is given by:*

$$\sum_{n=-\infty}^{\infty}(-1)^n x^{n(3n-1)/2} = \sum_{n=-\infty}^{\infty}(-1)^n x^{n(3n+1)/2} = (x;x)_\infty,$$

where the terms $n(3n \pm 1)/2$ correspond to the pentagonal numbers.

Theorem 4 (Jacobi's identity). *The following remarkable identity holds:*

$$\sum_{n=0}^{\infty}(-1)^n(2n+1)x^{n(n+1)/2} = (x;x)_\infty^3.$$

Theorem 5 (q-Analogue of the Binomial Theorem). *For $|x| < 1$ and $|a| < 1$, the following identity generalizes the binomial theorem:*

$$\sum_{n=0}^{\infty} \frac{(a;x)_n}{(x;x)_n} z^n = \frac{(az;x)_\infty}{(a;x)_\infty}.$$

7 Ramanujan's Congruences

Ramanujan discovered three extraordinary congruences in 1919 related to the partition function, which are now famously referred to as **Ramanujan's Congruences**. These congruences unveiled hidden regularities in the behavior of partition numbers, which were previously considered to lack a clear structure. Although Ramanujan did not provide rigorous mathematical proofs for these congruences, his groundbreaking observations spurred significant advancements in the study of partition theory and inspired mathematicians to develop formal proofs.

The partition function exhibits the following remarkable properties:

Theorem 6 (Ramanujan's Congruences). *For any non-negative integer n,*

$$p(nk + \delta_n) \equiv 0 \pmod{n}.$$

where $n \in \{5, 7, 11\}$ and the corresponding values of δ_n are $\{4, 5, 6\}$.

Proof for $n = 5$. [3]

From the Binomial Theorem, it follows that:

$$((x;x)_\infty)^5 \equiv (x^5;x^5)_\infty \pmod{5}.$$

Equivalently, we can write:

$$\frac{(x^5;x^5)_\infty}{((x;x)_\infty)^5} \equiv 1 \pmod{5}.$$

Using this, we obtain:

$$x \cdot ((x;x)_\infty)^4 \cdot \frac{(x^5;x^5)_\infty}{(x;x)_\infty^5} = x \cdot \frac{(x^5;x^5)_\infty}{(x;x)_\infty}.$$

The generating function for the partition function $p(n)$ is given by:

$$x \cdot \frac{(x^5;x^5)_\infty}{(x;x)_\infty} = (x^5;x^5)_\infty \cdot \sum_{m=0}^{\infty} p(m) \cdot x^{m+1}.$$

Thus, we can express:

$$x \cdot ((x;x)_\infty)^4 \equiv (x^5;x^5)_\infty \cdot \sum_{m=0}^{\infty} p(m) \cdot x^{m+1}.$$

Rewriting, we have:

$$x((x;x)_\infty)^4 = x(x;x)_\infty((x;x)_\infty)^3.$$

By applying Euler's Pentagonal Number Theorem and Jacobi's Identity, we derive:

$$\begin{aligned} x((x;x)_\infty)^4 &= x \sum_{r=-\infty}^{\infty} (-1)^r x^{r(3r+1)/2} \sum_{s=0}^{\infty} (-1)^s (2s+1) x^{s(s+1)/2} \\ &= \sum_{r=-\infty}^{\infty} \sum_{s=0}^{\infty} (-1)^{(r+s)} (2s+1) x^{(1+r(3r+1)/2+s(s+1)/2)}. \end{aligned}$$

Analyzing the exponents, we observe:

$$1 + r\frac{3r+1}{2} + s\frac{s+1}{2} = \frac{2(r+1)^2 + (2s+1)^2 + 10r^2 + 5}{8}.$$

This implies:

$$\text{The L.H.S is a multiple of 5} \iff 2(r+1)^2 + (2s+1)^2 \equiv 0 \pmod 5.$$

Further analysis shows:

$$2(r+1)^2 \equiv 0, 2, \text{ or } 3 \pmod 5, \quad \text{and} \quad (2s+1)^2 \equiv 0, 1, \text{ or } 4 \pmod 5.$$

This condition holds only if:

$$2(r+1)^2 \equiv 0 \pmod 5, \quad \text{and} \quad (2s+1)^2 \equiv 0 \pmod 5.$$

Thus, the coefficient of x^{5n} in $x((x;x)_\infty)^4$ is divisible by 5. This is equivalent to the coefficient of x^{5n} in $(x^5;x^5)_\infty \cdot \sum_{m=0}^{\infty} p(m)x^{m+1}$. To satisfy the equality x^{5n}, we require:

$$5r + m + 1 = 5n \implies m \equiv 4 \pmod 5.$$

Therefore, m must be of the form $5n+4$. This implies:

$$p(5n+4) \equiv 0 \pmod 5.$$

□

Similarly, we can prove it for n = 7 and 11. Ramanujan's congruences were a groundbreaking discovery in number theory. While partition functions had been studied before, Ramanujan's insights revealed deep, hidden patterns that were completely unexpected. He did not provide formal proofs for these congruences, but they were later rigorously proven by mathematicians like G. N. Watson and F. J. Dyson..

8 Rank

This section is referenced from [2]

In 1944, Freeman J. Dyson introduced the notion of the *rank* of a partition as a tool to investigate Ramanujan's congruences for the partition function $p(n)$ modulo 5, 7, and 11. Dyson's work focused on understanding the distribution of ranks, aiming to offer a combinatorial justification for these remarkable congruences, specifically in the following cases:

- $p(5n+4) \equiv 0 \pmod 5$
- $p(7n+5) \equiv 0 \pmod 7$

8.1 Definition

The rank of a partition is defined as the largest part of the partition minus the number of parts. For example, for the partition $(5,3,1)$ of $n = 9$, the rank is:

$$\text{Rank} = 5 - 3 = 2.$$

The rank provides a way to categorize partitions into different classes or "residue classes" modulo an integer, which can then be analyzed for congruences.

Let $R(r,n)$ denote The count of paritions of$n$ with rank exactly r. Also, let $R(r,q,n)$ denote The count of paritions of$n$ with rank congruent to r modulo q. Thus, we define:

$$R(r,q,n) = \sum_{r=-\infty}^{\infty} R(r+tq,n).$$

This expression sums over all integer shifts t of r by multiples of q to capture all ranks congruent to r modulo q.

8.2 Symmetry Properties

The following identities hold for $R(r,n)$ and $R(r,q,n)$:

$$R(r,n) = R(-r,n), \tag{1}$$

$$R(r,q,n) = R(q-r,q,n). \tag{2}$$

The first identity, $R(r,n) = R(-r,n)$, states that any partitions has rank equal and opposite to that of its conjugate partition. The second identity, $R(r,q,n) = R(q-r,q,n)$, implies a symmetry in the residue classes modulo q.

8.3 Generating Function

Dyson provides the generating functions for $R(r,n)$ and $R(r,q,n)$, given as follows:

Generating Function for $R(r,n)$:

$$\sum_{n=0}^{\infty} R(r,n)x^n = \sum_{n=1}^{\infty}(-1)^{n-1}x^{\frac{1}{2}n(3n-1)+rn}(1-x)^n \prod_{r=1}^{\infty}(1-x^r)^{-1},$$

where $r > 0$.

Generating Function for $R(r,q,n)$:

$$\sum_{n=0}^{\infty} R(r,q,n)x^n = \sum_{n=-\infty}^{\infty}{}'(-1)^{n-1}x^{\frac{1}{2}n(3n+1)\frac{x^{rn}+x^{n(q-r)}}{1-x^{qn}}\prod_{r=1}^{\infty}(1-x^r)^{-1}},$$

where the dash in the second product denotes that the term $n = 0$ is to be omitted. This expression holds for $0 < r < q$ which ensures that the modular conditions for ranks are appropriately applied.

These generating functions provide insight into the distribution of partitions by their rank r, particularly in modular arithmetic contexts.

8.4 Dyson's Motivation and Role of Rank in Ramanujan's congruences

Ramanujan discovered striking congruences for the partition function $p(n)$ modulo 5, 7, and 11, which revealed intriguing divisibility properties. Despite their significance, the underlying mathematical reasons for these patterns were not initially understood. To address this gap, Dyson introduced the concept of the rank to provide a combinatorial framework for interpreting these congruences. This innovative idea sought to clarify why Ramanujan's congruences hold true.

Dyson proposed that the partitions of $5n+4$ could be evenly distributed into 5 distinct groups according to their rank modulo 5. Similarly, the partitions of $7n + 5$ could be divided equally into 7 categories based on their rank modulo 7. This division offered a combinatorial justification for the congruences and supported Ramanujan's observations.

8.5 Conjecture for Modulo 5

Dyson conjectured that the partitions of $5n + 4$ could be divided equally among the five possible rank residues modulo 5.

Formally, this is expressed as:

$$R(0, 5, 5n + 4) = R(1, 5, 5n + 4) = R(2, 5, 5n + 4) = R(3, 5, 5n + 4) = R(4, 5, 5n + 4).$$

In words, this means that the partitions of $5n + 4$ can be categorized into five equally numerous classes based on the least positive residue of their ranks modulo 5. This would account for the divisibility of $p(5n + 4)$ by 5, as each residue class would contribute an equal number of partitions.

	0 (mod 5)	**1 (mod 5)**	**2 (mod 5)**	**3 (mod 5)**	**4 (mod 5)**
4	$2+2$	$3+1$	$1+1+1+1$	4	$2+1+1$
9	$7+2$, $5+1+1+1+1$, $4+3+1+1$, $4+2+2+1$, $3+3+3$, $2+2+1+1+1+1+1$	$8+1$, $5+2+1+1$, $4+4+1$, $4+3+2$, $3+1+1+1+1+1+1$, $2+2+2+1+1+1$	$6+1+1+1$, $5+3+1$, $5+2+2$, $3+2+1+1+1+1$, $2+2+2+2+1$, $1+1+1+1+1+1+1+1+1$	9, $6+2+1$, $5+4$, $3+3+1+1+1$, $4+1+1+1+1+1$, $3+2+2+1+1$	$7+1+1$, $6+3$, $4+2+1+1+1$ $3+3+2+1$, $3+2+2+2$, $2+1+1+1+1+1+1+1$

Table 1: Partitions of 4 & 9 grouped by rank modulo 5

8.6 Conjecture for Modulo 7

Similarly, Dyson extended this reasoning to the case where $p(7n + 5) \equiv 0 \pmod 7$. He conjectured that the partitions of $7n+5$ could be evenly divided among the seven possible values of the rank modulo 7.

This is written as:

$$R(0, 7, 7n + 5) = R(1, 7, 7n + 5) = R(2, 7, 7n + 5) = \cdots = R(6, 7, 7n + 5).$$

This implies that the partitions of $7n+5$ can be classified into seven equally numerous groups according to their ranks modulo 7, thereby explaining why $p(7n + 5)$ is divisible by 7.

	0 (mod 7)	**1 (mod 7)**	**2 (mod 7)**	**3 (mod 7)**	**4 (mod 7)**	**5 (mod 7)**	**6 (mod 7)**
5	$3+1+1$	$3+2$	$4+1$	$1+1+1+1+1$	5	$2+1+1+1$	$2+2+1$
	$9,3$	$4,4,4$	$5,4,3$	$6,3,3$	12	$7,5$	$8,4$
	$4,3,3,2$	$10,2$	$5,5,2$	$6,4,2$	$6,6$	$8,2,2$	$3,3,3,3$
	$4,4,2,2$	$5,3,2,2$	$6,2,2,2$	$2,2,2,2,2,2$	$7,3,2$	$3,3,2,2,2$	$4,2,2,2,2$
	$4,4,3,1$	$5,3,3,1$	$11,1$	$6,5,1$	$7,4,1$	$8,3,1$	$9,2,1$
12	$5,2,2,2,1$	$5,4,2,1$	$6,3,2,1$	$7,2,2,1$	$3,2,2,2,2,1$	$3,3,3,2,1$	$4,3,2,2,1$
	$10,1,1$	$5,5,1,1$	$6,4,1,1$	$7,3,1,1$	$8,2,1,1$	$4,2,2,2,1,1$	$4,3,3,1,1$
	$5,3,2,1,1$	$6,2,2,1,1$	$2,2,2,2,2,1,1$	$3,2,2,2,1,1,1$	$3,3,2,2,1,1$	$9,1,1,1$	$4,4,2,1,1$
	$5,4,1,1,1$	$6,3,1,1,1$	$7,2,1,1,1$	$8,1,1,1,1$	$3,3,3,1,1,1$	$4,3,2,1,1,1$	$5,2,2,1,1,1$
	$6,2,1,1,1,1$	$2,2,2,2,1,1,1,1$	$3,2,2,1,1,1,1,1$	$3,3,2,1,1,1,1$	$4,2,2,1,1,1,1$	$4,4,1,1,1,1$	$5,3,1,1,1,1$
	$2,2,2,1,1,1,1,1,1$	$7,1,1,1,1,1$	$3,3,1,1,1,1,1,1$	$4,2,1,1,1,1,1,1$	$4,3,1,1,1,1,1$	$5,2,1,1,1,1,1$	$6,1,1,1,1,1,1$
	$3,1,1,1,1,1,1,1,1,1$	$3,2,1,1,1,1,1,1,1$	$4,1,1,1,1,1,1,1,1$	$1,1,1,1,1,1,1,1,1,1,1,1$	$5,1,1,1,1,1,1,1$	$2,1,1,1,1,1,1,1,1,1,1$	$2,2,1,1,1,1,1,1,1,1$

Table 2: Partitions of 5 & 12 grouped by rank modulo 7

8.7 Failure of Explanation for the Third Ramanujan Congruence

While Dyson's rank conjecture successfully explained the divisibility of $p(5n+4)$ by 5 and $p(7n+5)$ by 7, it failed to explain the third Ramanujan congruence, $p(11n+6) \equiv 0 \pmod{11}$. Dyson's approach assumed that partitions of $5n+4$ and $7n+5$ could be evenly divided into residue classes based on their ranks modulo 5 and 7, respectively. However, for $p(11n+6)$, no viable combinatorial structure was found to divide the partitions into 11 equally numerous classes based on rank residues modulo 11.

Rank (mod 11)	**Partitions**
0	3 + 2 + 1
1	4 + 1 + 1, 3 + 3
2	4 + 2
3	5 + 1
5	6
6	1 + 1 + 1 + 1 + 1 + 1
8	2 + 1 + 1 + 1 + 1
9	2 + 2 + 1 + 1
10	3 + 1 + 1 + 1, 2 + 2 + 2

Table 3: Partitions of 6 and ranks

We can clearly observe from the table that number of partition of numbers of the form $11n+6$ could not be neatly divided into 11 equally numerous classes based on the rank residues modulo 11. In other words, no straightforward partitioning scheme based on ranks modulo 11 was found that would give an equal number of partitions in each of the 11 classes.

9 Crank

This section is refernced from [1]

The failure of Dyson's rank to explain the third Ramanujan congruence led George Andrews and Frank Garvan to introduce the *crank* in the 1980s. The crank extends the concept of the rank by incorporating additional structural details, enabling the partitions of $11n + 6$ to be evenly distributed into 11 classes. This provides a combinatorial explanation for the divisibility of $p(11n + 6)$ by 11. Unlike the rank, the crank captures a symmetry that ensures this balanced classification and resolves the limitations faced in the case of modulo 11.

9.1 Vector Partitions

Let $\#(\xi)$ be total parts of ξ, and let $a(\xi)$ be their sum, with $\#(\emptyset) = a(\emptyset) = 0$ for the empty partition $\emptyset$ of 0. Define the set

$V = \{(\xi_1, \xi_2, \xi_3) \mid \xi_1$ is a partition into distinct parts, ξ_2, ξ_3 are partitions without restriction$\}$.

Vector partitions are denoted by V . For $\xi = (\xi_1, \xi_2, \xi_3) \in V$, we define α as sum, *beta* as weight, and γ as crank of the partition:

$$\alpha(\xi) = a(\xi_1) + a(\xi_2) + a(\xi_3),$$

$$\beta(\xi) = (-1)^{\#(\xi_1)},$$

$$\gamma(\xi) = \#(\xi_2) - \#(\xi_3).$$

We say ξ is a **vector partition** of n if $\alpha(\xi) = n$. For example, if

$$\xi = (5 + 3 + 2,\ 2 + 2 + 1,\ 2 + 1 + 1),$$

then $\alpha(\xi) = 19$, $\beta(\xi) = -1$, $\gamma(\xi) = 0$, and ξ is a vector partition of 19.

9.2 Definitions

Vector partitions of n with weight β and crank c is denoted by $C_v(c, n)$, so that:

$$C_v(c, n) = \sum_{\xi \in V,\, \gamma(\xi) = c} \beta(\xi).$$

The number of vector partitions of n with weight β and crank, c $\equiv k$(mod t) is denoted by $C_v(k, t, n)$, so that:

$$C_v(k, t, n) = \sum_{c=-\infty}^{\infty} C_v(ct + k, n) = \sum_{\xi \in V,\, \gamma(\xi) \equiv k \pmod{t}} \beta(\xi).$$

9.3 Symmetry Properties

By considering the transformation that interchanges ξ_2 and ξ_3, we have:

$$C_v(c, n) = C_v(-c, n),$$

and thus:

$$C_v(t - c, t, n) = C_v(c, t, n).$$

9.4 Generating Function

The generating function for $C_v(c, n)$ is given by:

$$\sum_{c=-\infty}^{\infty} \sum_{n=0}^{\infty} C_v(c, n) z^c x^n = \prod_{n=1}^{\infty} \frac{(1 - x^n)}{(1 - zq^n)(1 - z^{-1}x^n)}.$$

By setting $z = 1$ in the generating function, we recover:

$$\sum_{n=0}^{\infty} \sum_{c=-\infty}^{\infty} C_v(c, n) x^n = \prod_{n=1}^{\infty} \frac{1}{(1 - x^n)} = \sum_{n=0}^{\infty} p(n) x^n,$$

so that:

$$\sum_{c=-\infty}^{\infty} C_v(c, n) = p(n),$$

where $p(n)$ is the partition function.

9.5 Crank for Ordinary Partitions

For a partition ξ, let:

- $L(\xi)$:largest part ,
- $O(\xi)$: count of ones,
- $G(\xi)$of parts greater than $O(\xi)$.

The *crank* $c(\xi)$ of ξ is defined as:

$$c(\xi) = \begin{cases} L(\xi) & \text{if } O(\xi) = 0, \\ G(\xi) - O(\xi) & \text{if } O(\xi) > 0. \end{cases}$$

Theorem 7. *The number of partitions $p(n)$ with crank $\gamma(\xi) = m$ is $C(m, n)$ for all $n > 1$.*

9.6 Vector Crank Theorem

The Vector-Crank Theorem provides a combinatorial explanation for Ramanujan's partition congruences by partitioning the partitions of $nk + \delta_n$ into n equinumerous subsets based on a vector-crank statistic.

$$C_v(0, 5, 5n + 4) = C_v(1, 5, 5n + 4) = \cdots = C_v(4, 5, 5n + 4) = \frac{p(5n + 4)}{5},$$

$$C_v(0, 7, 7n + 5) = C_v(1, 7, 7n + 5) = \cdots = C_v(6, 7, 7n + 5) = \frac{p(7n + 5)}{7},$$

$$C_v(0, 11, 11n + 6) = C_v(1, 11, 11n + 6) = \cdots = C_v(10, 11, 11n + 6) = \frac{p(11n + 6)}{11}.$$

This theorem is taken from [4] [5]

Here, $C_v(i, n, nk+\delta_n)$ represents The count of paritions of $nk+\delta_n$ where the vector-crank is congruent to $i \pmod{n}$, and $p(nk + \delta_n)$ is the partition function. The theorem shows that the partitions of $nk + \delta_n$ are equidistributed among these n subsets.

Partition	**Crank**	**Crank (mod 5)**
3 + 1	0	0
1 + 1 + 1 + 1	-4	1
2 + 2	2	2
2 + 1 + 1	-2	3
4	4	4

Table 4: Partitions of 4 with cranks

Partition	**Crank**	**Crank (mod 7)**
4 + 1	0	0
2 + 2 + 1	1	1
1 + 1 + 1 + 1 + 1	-5	2
3 + 2	3	3
2 + 1 + 1 + 1	-3	4
5	5	5
3 + 1 + 1	-1	6

Table 5: Partitions of 7 with cranks

By examining these tables, it is evident that the crank function effectively addresses the challenge encountered with partitions of numbers of the form $11n + 6$, as it divides them into groups of equal size.

Partition	Crank	Crank (mod 11)
5 + 1	0	0
3 + 2 + 1	1	1
2 + 2 + 2	2	2
3 + 3	3	3
4 + 2	4	4
1 + 1 + 1 + 1 + 1 + 1	-6	5
6	6	6
2 + 1 + 1 + 1 + 1	-4	7
3 + 1 + 1 + 1	-3	8
2 + 2 + 1 + 1	-2	9
4 + 1 + 1	-1	10

Table 6: Partitions of 6 with cranks

10 Conclusion

This report provides a detailed exploration of the arithmetic properties of partition functions, emphasizing their theoretical significance and practical implications in mathematics. It traces the historical development of this fascinating area, starting with Euler's foundational work on generating functions and progressing to Ramanujan's remarkable congruences. These congruences, which highlight surprising divisibility properties of partition numbers, were initially discovered through intuition but lacked rigorous proof. Over time, they have been extensively studied and proven using a variety of approaches, demonstrating the richness of mathematical tools and perspectives available to researchers.

The study presents multiple proofs of Ramanujan's congruences, using both classical and modern techniques. Traditional methods, such as modular arithmetic and infinite product expansions, provide a firm foundation, while combinatorial tools like Dyson's rank and Andrews and Garvan's crank add depth and versatility to the analysis. These approaches offer unique insights into the structure of partition numbers, revealing symmetries and patterns that connect various areas of mathematics. For instance, Dyson's rank effectively categorizes partitions modulo 5 and 7, while the crank resolves the challenges associated with modulo 11, bridging gaps in understanding that earlier methods could not address. These tools demonstrate the combinatorial elegance and mathematical interconnectedness inherent in partition functions.

Beyond their theoretical interest, partition functions have broader implications, serving as a link between discrete and continuous mathematics. Their generating functions, with their elegant infinite product representations, reveal deep relationships between combinatorics, number theory, and modular forms. This report not only revisits historical results but also highlights their relevance to contemporary mathematical problems, opening doors to new areas of exploration. The symmetry properties and combinatorial interpretations of ranks and cranks, for example, can inspire innovative solutions to problems in algebra, analysis, and computational mathematics.

This work underscores the enduring impact of Ramanujan's contributions to partition theory and emphasizes the importance of studying these functions from multiple perspectives. While much progress has been made in explaining their congruences, there remain open questions about their behavior in other modular contexts and in higher dimensions. Modern computational tools and advanced algorithms offer opportunities to discover new patterns and deepen our understanding of partition functions.

In summary, this report serves as a comprehensive study of partition functions, presenting clear and accessible proofs of key results while exploring their broader significance. It celebrates the historical journey of mathematical discovery while pointing toward future research opportunities. The field of partition theory exemplifies how simple, intuitive questions about numbers can lead to profound insights and connections, cementing its place as a cornerstone of mathematical inquiry and innovation.

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