SOME PROPERTIES OF PARTITIONS

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1. WE denote by $p(n)$ the number of unrestricted partitions of a positive integer n. Ramanujan discovered, and later proved, three striking arithmetical properties of $p(n)$, namely:

$$p(5n+4) \equiv 0 \pmod 5, \tag{1.1}$$

$$p(7n+5) \equiv 0 \pmod 7, \tag{1.2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.3}$$

All existing proofs of these results appeal to the theory of generating functions, and provide no method of actually separating the partitions concerned into q equal classes ($q = 5$, 7, or 11). Dyson (**1**) discovered empirically a remarkable combinatorial method of dividing the partitions of $5n+4$ and $7n+5$ into 5 and 7 equal classes respectively. Defining the *rank* of a partition as the largest part minus the number of parts, he divided the partitions of any number into 5 classes according to their ranks modulo 5. For numbers of the form $5n+4$, these 5 classes are all equal, while for numbers of other forms some but not all of the classes are equal; similar results hold for 7 but definitely not for 11.

The main object of the present paper is to prove these conjectures of Dyson. The results form part of Theorems 4 and 5. It is noteworthy that we have to obtain at the same time all the results stated in these theorems—we cannot simplify the working so as merely to obtain Dyson's identities.

Theorems 1 to 3 give some simple congruence properties of partitions which we obtained in the course of this work. In fact, each series

$$\sum_{n=0}^{\infty} p(qn+b)y^n \quad (q = 5,\ 7,\ \text{or}\ 11;\ \ 0 \leqslant b < q)$$

is congruent modulo q to a simple infinite product. Theorems 1 and 2 follow immediately from Theorems 4 and 5 respectively, but we have given direct proofs also.

In order to reduce the apparent complexity of our formulae, we have had to introduce a great deal of shorthand notation. This is done in the relevant parts of the text, and a complete list is given at the end of the paper.

We should like to express our deep indebtedness to Professor J. E. Littlewood for his constant help and encouragement, and for much valuable advice during the writing of this paper.

2. We shall now give a fuller account of Dyson's paper, since it is not readily available. The number of partitions of n with rank m is denoted by $N(m,n)$, and the number of partitions of n with rank congruent to m modulo q by $N(m,q,n)$. Then

$$N(m,q,n) = \sum_{r=-\infty}^{\infty} N(m+rq,n). \tag{2.1}$$

It is clear from the definition of rank that any partition has rank equal and opposite to that of its conjugate† partition, so that

$$N(m,n) = N(-m,n),$$
$$N(m,q,n) = N(q-m,q,n).$$

Dyson now states the following conjectural identities:

$$N(1,5,5n+1) = N(2,5,5n+1), \tag{2.2}$$
$$N(0,5,5n+2) = N(2,5,5n+2), \tag{2.3}$$
$$N(0,5,5n+4) = N(1,5,5n+4) = N(2,5,5n+4); \tag{2.4}$$
$$N(2,7,7n) = N(3,7,7n), \tag{2.5}$$
$$N(1,7,7n+1) = N(2,7,7n+1) = N(3,7,7n+1), \tag{2.6}$$
$$N(0,7,7n+2) = N(3,7,7n+2), \tag{2.7}$$
$$N(0,7,7n+3) = N(2,7,7n+3), \qquad N(1,7,7n+3) = N(3,7,7n+3), \tag{2.8}$$
$$N(0,7,7n+4) = N(1,7,7n+4) = N(3,7,7n+4), \tag{2.9}$$
$$N(0,7,7n+5) = N(1,7,7n+5) = N(2,7,7n+5) = N(3,7,7n+5), \tag{2.10}$$
$$N(0,7,7n+6)+N(1,7,7n+6) = N(2,7,7n+6)+N(3,7,7n+6). \tag{2.11}$$

These we prove in Theorems 4 and 5.‡ (2.4) and (2.10) lead immediately to the Ramanujan congruences (1.1) and (1.2) respectively.

Dyson next gives the generating functions of $N(m,n)$ and $N(m,q,n)$, namely

$$\sum_{n=0}^{\infty} N(m,n)x^n = \sum_{n=1}^{\infty} (-)^{n-1}x^{\frac{1}{2}n(3n-1)+mn}(1-x^n)\prod_{r=1}^{\infty}(1-x^r)^{-1}, \tag{2.12}$$

$$\sum_{n=0}^{\infty} N(m,q,n)x^n = \sum_{n=-\infty}^{\infty}{}' (-)^n x^{\frac{1}{2}n(3n+1)}\frac{x^{mn}+x^{n(q-m)}}{1-x^{qn}}\prod_{r=1}^{\infty}(1-x^r)^{-1}, \tag{2.13}$$

where the dash in (2.13) denotes as usual that the term given by $n=0$ is to be omitted. (2.12) holds for $m \geqslant 0$; (2.13) for $0 \leqslant m \leqslant q$. Clearly (2.13) follows at once from (2.1) and (2.12), which latter is proved in

† For a definition of this term, cf. Hardy and Wright (**2**), p. 272.

‡ They are respectively (6.14), (6.15), (6.19) and (6.20), (6.23), (6.27) and (6.28), (6.29), (6.32) and (6.33), (6.35) and (6.36), (6.38) to (6.40), (6.43).

Lemma 1 below. The generating functions enable us to transform (2.2)–(2.11) into statements of vanishing coefficients in certain power series. They also provide an effective method of calculating numerical values of the $N(m,q,n)$.

It is convenient to explain here a difficulty about $p(0)$. This is normally taken to be unity; on the other hand (2.12) gives us no partitions of 0 of any rank. It is therefore natural to take $p(0) = 1$ in Theorems 1 to 3, and $p(0) = 0$ in Theorems 4 and 5. We do not now have to put an extra term in our formulae to allow for our convention; but there is of course an anomaly in comparing Theorems 1 and 4, or Theorems 2 and 5.

LEMMA 1. (2.12) *holds for* $m \geqslant 0$.

We require the two well-known identities†

$$\prod_{u=1}^{j}(1+ax^u) = \sum_{u=0}^{j} a^u x^{\frac{1}{2}u(u+1)} \prod_{t=1}^{u}\left(\frac{1-x^{j-t+1}}{1-x^t}\right), \tag{2.14}$$

$$\prod_{u=1}^{j}(1-ax^u)^{-1} = \sum_{u=0}^{\infty} a^u x^u \prod_{t=1}^{u}\left(\frac{1-x^{j+t-1}}{1-x^t}\right), \tag{2.15}$$

where an empty product is as usual taken to be unity.

The number of partitions of n into exactly s parts, the largest of which is $s+m$, is equal to the number of partitions of $n-s-m$ into exactly $s-1$ parts, none of which exceeds $s+m$. It is therefore the coefficient of $x^{n-s-m}z^{s-1}$ in

$$\prod_{u=1}^{s+m}(1-zx^r)^{-1},$$

which by (2.15) is the coefficient of x^n in

$$x^{2s+m-1}\prod_{t=1}^{s-1}\left(\frac{1-x^{s+m+t-1}}{1-x^t}\right).$$

Now by (2.14), with $a = -x^{s+m-1}$, $j = s-1$, this expression is

$$x^{2s+m-1}\sum_{u=0}^{s-1}(-)^u x^{\frac{1}{2}u(u+1)+u(s+m-1)}\prod_{r=1}^{u}(1-x^r)^{-1}\prod_{t=1}^{s-u-1}(1-x^t)^{-1}.$$

Hence

$$\sum_{n=0}^{\infty} N(m,n)x^n \prod_{u=1}^{\infty}(1-x^u)$$

$$= \sum_{s=1}^{\infty} x^{2s+m-1}\sum_{u=0}^{s-1}(-)^u x^{\frac{1}{2}u(u+1)+u(s+m-1)}\cdot\prod_{r=u+1}^{\infty}(1-x^r)\prod_{t=1}^{s-u-1}(1-x^t)^{-1}$$

$$= \sum_{u=1}^{\infty}(-)^{u-1}x^{\frac{1}{2}u(u-1)+mu}(1-x^u)\sum_{s=u}^{\infty}x^{su+s-u}\prod_{r=u+1}^{\infty}(1-x^r)\prod_{t=1}^{s-u}(1-x^t)^{-1}, \tag{2.16}$$

† Cf. Hardy and Wright (2), Theorems 348 and 349.

writing $u-1$ for u and interchanging the order of summation. But

$$\sum_{s=u}^{\infty} x^{su+s-u} \prod_{t=1}^{s-u} (1-x^t)^{-1} = x^{u^2} \sum_{s=0}^{\infty} x^{s(u+1)} \prod_{t=1}^{s} (1-x^t)^{-1} = x^{u^2} \prod_{r=u+1}^{\infty} (1-x^r)^{-1},$$

by (2.15) with $a = x^u$ and $j \to \infty$. Thus the $\sum_{s=u}^{\infty}$ in the last line of (2.16) reduces to x^{u^2} and (2.16) becomes (2.12).

3. We now introduce some notation. $q > 1$ is a positive integer prime to 6; and we write $q = 6\lambda+\mu$, where λ is a positive integer and $\mu = \pm 1$. In the applications we take $q = 5$ in Theorems 1 and 4, $q = 7$ in Theorems 2 and 5, and $q = 11$ in Theorem 3; but the lemmas are all proved for general q. Small latin letters, except x, y, z, w, denote integers (positive, negative, or zero); and we further demand that a is not a multiple of q.

x, y, z, w are variables in the complex plane; and to ensure the convergence of our series we shall require $|x| < 1$, $|w| < 1$. The variables x and y are always related by the equation

$$y = x^q.$$

We shall regard any power series in x as a polynomial of degree $q-1$ in x, whose coefficients are power series in y. Thus any identity or congruence between two power series in x can be regarded, on equating coefficients of powers of x, as equivalent to q identities or congruences between power series in y. These latter will usually be much simpler to prove; indeed this technique is fundamental to the whole of this paper.

If no specific variable or range is indicated, $\prod$ denotes a product from $r = 1$ to $r = \infty$, and $\sum$ a sum from $n = -\infty$ to $n = \infty$. $\sum'$ denotes a sum over the same range, but with the term given by $n = 0$ omitted.

We write $$P(z,w) = \prod (1-zw^{r-1})(1-z^{-1}w^r) \tag{3.1}$$

so that $P(z,w)$ is a single-valued analytic function of z in any ring-shaped region $0 < z_1 \leqslant |z| \leqslant z_2$, and satisfies

$$P(z^{-1}w,w) = P(z,w), \qquad P(zw,w) = -z^{-1}P(z,w). \tag{3.2}$$

We also write

$$P(a) = P_q(a) = P(y^a, y^q) = \prod (1-y^{qr-q+a})(1-y^{qr-a}),$$
$$P(0) = P_q(0) = \prod (1-y^{qr}).$$

It should be noted that $P(0)$ is *not* the expression that would be obtained by writing 0 instead of a in the definition of $P(a)$. From (3.2) we have

$$P(q-a) = P(a), \qquad P(-a) = P(q+a) = -y^{-a}P(a), \tag{3.3}$$

which we shall use without explicit mention below.

It is clear that the $P(z, w)$ are closely akin to the theta-functions. In Lemmas 3 to 6 we shall establish those of their properties which we shall need below. Most of them are well-known results lightly disguised—Lemma 4 in particular is the addition theorem—but we prove them here for the sake of completeness. The proofs are based, after Weber (6), on Lemma 2, which we shall also need for the deeper Lemmas 7 and 8.

LEMMA 2. *Let $f(z)$ be a single-valued analytic function of z, except possibly for a finite number of poles, in every region $0 < z_1 \leqslant |z| \leqslant z_2$; and suppose that for some constants A and w, with $0 < |w| < 1$, and some (positive, zero, or negative) integer n, we have*

$$f(zw) = Az^n f(z) \tag{3.4}$$

identically in z. Then either $f(z)$ has just n more poles than zeros in

$$|w| < |z| \leqslant 1,$$

or $f(z)$ vanishes identically.

Suppose that $f(z)$ does not vanish identically. Then it must have only a finite number of zeros in any region $0 < z_1 \leqslant |z| < z_2$. Now choose $\epsilon > 0$ so that $f(z)$ has no poles or zeros in $1 < |z| \leqslant 1+\epsilon$, and so also none in $|w| < |z| \leqslant |w|(1+\epsilon)$, since clearly $A \neq 0$. Let C, C' be the circles $|z| = 1+\epsilon$, $|z| = |w|(1+\epsilon)$ respectively. Then $f(z)$ has the same poles and zeros in $|w| < |z| \leqslant 1$ as it has in the region between C and C'. Thus the excess of the number of poles over the number of zeros of $f(z)$ in $|w| < |z| \leqslant 1$ is

$$-\frac{1}{2\pi i}\int\limits_C \frac{f'(z)}{f(z)}\,dz + \frac{1}{2\pi i}\int\limits_{C'} \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int\limits_C \frac{n}{z} = n,$$

where we have used (3.4) to transform the integral over C' into an integral over C. This proves the lemma.

Let us call two points z_1, z_2 equivalent if $z_1 = w^m z_2$ for some integer m. Clearly if z_1 is a pole or zero of $f(z)$, so is z_2; and there is just one point in $|w| < |z| \leqslant 1$ equivalent to any given point (other than the origin). Hence we may replace the conclusion of the lemma by

Then either $f(z)$ has just n more inequivalent poles than inequivalent zeros, or $f(z)$ vanishes identically.

This is the form in which we shall use the lemma.

LEMMA 3. *We have*

$$\prod (1-x^r)^3 \equiv P(0) \sum_{c=0}^{(q-3)/2} (-)^c (2c+1) x^{\frac{1}{2}c(c+1)} P\left(\frac{q-1}{2} - c\right) \pmod{q}.$$

We need the results†

$$P(0)P(a) = \sum (-)^n y^{\frac{1}{2}qn(n+1)-an}, \tag{3.5}$$

$$\prod (1-x^r)^3 = \sum_{s=0}^{\infty} (-)^s(2s+1)x^{\frac{1}{2}s(s+1)}. \tag{3.6}$$

Writing $s = qr+b$ $(0 \leqslant b \leqslant q-1,\ 0 \leqslant r < \infty)$ in (3.6) we find

$$\prod (1-x^r)^3 = \sum_{b=0}^{q-1} \sum_{r=0}^{\infty} (-)^{r+b}(2qr+2b+1)x^{\frac{1}{2}(qr+b)(qr+b+1)}$$

$$\equiv \sum_{b=0}^{(q-3)/2} (-)^b(2b+1)x^{\frac{1}{2}b(b+1)} \sum_{r=0}^{\infty} (-)^r y^{\frac{1}{2}r(qr+2b+1)} +$$

$$+ \sum_{c=0}^{(q-3)/2} (-)^c(2c+1)x^{\frac{1}{2}c(c+1)} \sum_{t=-\infty}^{-1} (-)^t y^{\frac{1}{2}t(qt+2c+1)} \pmod q$$

(writing $q-c-1$ for b and $-t-1$ for r when $b > \frac{1}{2}(q-1)$)

$$= \sum_{c=0}^{(q-3)/2} (-)^c(2c+1)x^{\frac{1}{2}c(c+1)} \sum (-)^n y^{\frac{1}{2}n(qn+2c+1)}$$

$$= \sum_{c=0}^{(q-3)/2} (-)^c(2c+1)x^{\frac{1}{2}c(c+1)} P(0)P\left(\frac{q-1}{2}-c\right),$$

by (3.5). This is Lemma 3.

LEMMA 4. *Suppose that none of b, c, d, $b \pm c$, $c \pm d$, $b \pm d$ is divisible by q. Then we have*

$$P^2(b)P(c+d)P(c-d) - P^2(c)P(b+d)P(b-d) + $$
$$+ y^{c-d}P^2(d)P(b+c)P(b-c) = 0.$$

We shall prove the more general result

$$P^2(z,w)P(\zeta t,w)P(\zeta t^{-1},w) - P^2(\zeta,w)P(zt,w)P(zt^{-1},w) +$$
$$+ \zeta t^{-1}P^2(t,w)P(z\zeta,w)P(z\zeta^{-1},w) = 0 \tag{3.7}$$

which reduces to the lemma on writing $w = y^q$, $z = y^b$, $\zeta = y^c$, $t = y^d$.

Suppose first that 1, ζ, t are all inequivalent. Let us write $f(z)$ for the left-hand side of (3.7); then by (3.2) we easily obtain

$$f(wz) = z^{-2}f(z). \tag{3.8}$$

But it is clear that $f(z)$ has no poles and the three inequivalent zeros $z = 1$, $z = \zeta$, and $z = t$. Hence, by Lemma 2, $f(z)$ must vanish identically. For the values of ζ, t which do not satisfy the condition above, we may prove (3.7) either by analytic continuation or directly from (3.2). They are in any case irrelevant for the lemma.

† First stated by Jacobi, though (3.5) was known to Gauss. Cf. Hardy and Wright (2), Theorem 357 and formula (19.9.1).

LEMMA 5. *We have*

$$\sum (-)^n w^{\frac{1}{2}n(3n-1)}(z^{3n}+z^{1-3n}) = \frac{P(z^2,w)}{P(z,w)} \prod (1-w^r).$$

Let us write $f_1(z)$, $f_2(z)$ for the left- and right-hand sides of this equation. We have

$$f_1(wz) = \sum (-)^n w^{\frac{1}{2}n(3n-1)}(wz)^{3n} + \sum (-)^n w^{\frac{1}{2}n(3n-1)}(wz)^{1-3n}$$
$$= -\sum (-)^n w^{\frac{1}{2}(n-1)(3n-4)}(wz)^{3n-3} - \sum (-)^n w^{\frac{1}{2}(n+1)(3n+2)}(wz)^{-3n-2}$$

(writing $n-1$ for n in the first sum and $n+1$ for n in the second)

$$= -w^{-1}z^{-3} \sum (-)^n w^{\frac{1}{2}n(3n-1)} z^{3n} - w^{-1}z^{-3} \sum (-)^n w^{\frac{1}{2}n(3n-1)} z^{1-3n}$$
$$= -w^{-1}z^{-3} f_1(z);$$

and, as in the proof of Lemma 4,

$$f_2(wz) = -w^{-1}z^{-3} f_2(z).$$

Thus $f(z) = f_1(z) - f_2(z)$ satisfies the conditions of Lemma 2 with $n = -3$; and clearly $f(z)$ has no poles. But $f(z)$ has the 4 inequivalent zeros ± 1, $\pm w^{\frac{1}{2}}$, for†

$$f_1(-1) = f_2(-1) = 0,$$
$$f_1(\pm w^{\frac{1}{2}}) = f_2(\pm w^{\frac{1}{2}}) = 0,$$
$$f_1(1) = f_2(1) = 2 \prod (1-w^r);$$

so that, by Lemma 2, $f(z)$ must vanish identically. This proves the lemma.

LEMMA 6.‡ *We have*

$$\prod (1-x^r) = (-)^\lambda x^{\frac{1}{2}\lambda(3\lambda+\mu)} P(0) \left[1 + \sum_{c=1}^{(q-1)/2} (-)^c x^{\frac{1}{2}c(3c-q)} \frac{P(2c)}{P(c)}\right].$$

We may write Euler's formula§ in the form

$$\prod (1-x^r) = \sum_{m=-\infty}^{\infty} (-)^m x^{\frac{1}{2}m(3m-\mu)}. \tag{3.9}$$

Putting $m = qn - c - \lambda$ ($c = 0, \pm 1, \ldots, \pm \frac{1}{2}(q-1)$; $-\infty < n < \infty$) this gives

$$\prod (1-x^r) = \sum_c \sum (-)^{n+c+\lambda} y^{\frac{1}{2}(3qn^2-nq-6nc+c)} x^{\frac{1}{2}(3c^2+3\lambda^2+\lambda\mu)}$$
$$= \sum_{c=1}^{(q-1)/2} (-)^{c+\lambda} x^{\frac{1}{2}(3c^2-qc+3\lambda^2+\lambda\mu)} \sum (-)^n y^{\frac{1}{2}nq(3n-1)} (y^{c(1-3n)} + y^{3cn}) +$$
$$+ (-)^\lambda x^{\frac{1}{2}\lambda(3\lambda+\mu)} \sum (-)^n y^{\frac{1}{2}qn(3n-1)},$$

† We leave the verification of these results, and of subsequent similar ones, to the reader. The results for $f_1(z)$ follow from (3.9) and suitable changes of the variable of summation; those for $f_2(z)$ from the definition of $P(z, w)$.

‡ The cases $q = 5$ and $q = 7$ are formulae (2.1) and (4.1) respectively in Watson (**4**). The proof below is substantially the same as his. We would remind the reader that $q = 6\lambda + \mu$, with λ an integer and $\mu = \pm 1$.

§ Cf. Hardy and Wright (**2**), Theorem 353.

combining the two terms with equal and opposite values of c. If we evaluate the sums over n by Lemma 5 and (3.9) respectively, we obtain the result in the lemma.

4. We now prove Theorems 1 to 3, the congruence properties of $p(n)$ referred to in section 1. We write

$$\Phi(b) = \Phi_q(b) = \sum_{n=0}^{\infty} p(qn+b)y^n \quad (0 \leqslant b < q),$$

so that we have
$$\sum_{n=0}^{\infty} p(n)x^n = \sum_{b=0}^{q-1} \Phi(b)x^b.$$

THEOREM 1. *We have, for* $q = 5$,

$$\Phi(0) \equiv P(0)P(2)/P^2(1) \pmod 5,$$
$$\Phi(1) \equiv P(0)/P(1) \pmod 5,$$
$$\Phi(2) \equiv 2P(0)/P(2) \pmod 5,$$
$$\Phi(3) \equiv 3P(0)P(1)/P^2(2) \pmod 5,$$
$$\Phi(4) \equiv 0 \pmod 5.$$

The last of these is Ramanujan's (1.1). In the same way (1.2) and (1.3) appear as part of Theorems 2 and 3 respectively.

By Lemma 3, with $q = 5$,

$$\prod (1-x^r)^3 \equiv P(0)\{P(2)-3xP(1)\} \pmod 5. \tag{4.1}$$

Now
$$(1-x^r)^5 \equiv 1-y^r \pmod 5,$$

and thus

$$\sum_{b=0}^{4} \Phi(b)x^b = \prod (1-x^r)^{-1} \equiv \{\prod (1-x^r)^3\}^3/\prod (1-y^r)^2$$
$$\equiv P(0)\left\{\frac{P(2)}{P^2(1)}+\frac{x}{P(1)}+\frac{2x^2}{P(2)}+3x^3\frac{P(1)}{P^2(2)}\right\} \pmod 5, \tag{4.2}$$

by (4.1) and
$$\prod (1-y^r) = P(0)P(1)P(2).$$

Equating coefficients of powers of x in (4.2) we obtain the theorem.

THEOREM 2. *For* $q = 7$ *we have*

$$\Phi(0) \equiv P(0)P(3)/P(1)P(2) \pmod 7,$$
$$\Phi(1) \equiv P(0)/P(1) \pmod 7,$$
$$\Phi(2) \equiv 2P(0)P(2)/P(1)P(3) \pmod 7,$$
$$\Phi(3) \equiv 3P(0)/P(2) \pmod 7,$$
$$\Phi(4) \equiv 5P(0)/P(3) \pmod 7,$$
$$\Phi(5) \equiv 0 \pmod 7,$$
$$\Phi(6) \equiv 4P(0)P(1)/P(2)P(3) \pmod 7.$$

By Lemma 3 with $q = 7$,

$$\prod (1-x^r)^3 \equiv P(0)\{P(3)-3xP(2)+5x^3P(1)\} \pmod 7. \tag{4.3}$$

Now
$$(1-x^r)^7 \equiv 1-y^r \pmod 7,$$
and thus

$$\sum_{b=0}^{6} \Phi(b)x^b = \prod (1-x^r)^{-1} \equiv \{\prod (1-x^r)^3\}^2/\prod (1-y^r)$$
$$\equiv P(0)\left\{\frac{P(3)}{P(1)P(2)}+\frac{x}{P(1)}+\frac{2x^2P(2)}{P(1)P(3)}+\frac{3x^3}{P(2)}+\frac{5x^4}{P(3)}+ +\frac{4x^6P(1)}{P(2)P(3)}\right\} \pmod 7, \tag{4.4}$$

by (4.3) and
$$\prod (1-y^r) = P(0)P(1)P(2)P(3).$$

Equating coefficients of powers of x in (4.4) we obtain the theorem.

THEOREM 3. *For $q = 11$ we have*

$$\Phi(0) \equiv P(0)/P(1) \pmod{11},$$
$$\Phi(1) \equiv P(0)P(5)/P(2)P(3) \pmod{11},$$
$$\Phi(2) \equiv 2P(0)P(3)/P(1)P(4) \pmod{11},$$
$$\Phi(3) \equiv 3P(0)P(2)/P(1)P(3) \pmod{11},$$
$$\Phi(4) \equiv 5P(0)/P(2) \pmod{11},$$
$$\Phi(5) \equiv 7P(0)P(4)/P(2)P(5) \pmod{11},$$
$$\Phi(6) \equiv 0 \pmod{11},$$
$$\Phi(7) \equiv 4P(0)/P(3) \pmod{11},$$
$$\Phi(8) \equiv 6yP(0)P(1)/P(4)P(5) \pmod{11},$$
$$\Phi(9) \equiv 8P(0)/P(4) \pmod{11},$$
$$\Phi(10) \equiv 9P(0)/P(5) \pmod{11}.$$

It would be possible to prove these results in the same way as Theorems 1 and 2: we write

$$\prod (1-x^r)^{-1} \equiv \{\prod (1-x^r)^3\}^3\{\prod (1-x^r)\}/\prod (1-y^r) \pmod{11}$$

and substitute for the expressions inside the curly brackets by means of Lemmas 3 and 6. Using the 10 identities between the $P(a)$ given by Lemma 4 to simplify the resulting expressions, we eventually obtain the theorem. This method would be extremely tedious, and we therefore give a purely synthetic proof.

Let us for convenience write the congruences in Theorem 3 in the form

$$\Phi(b) \equiv \Phi'(b) \pmod{11};$$

then it is clearly sufficient to prove

$$\{\prod (1-x^r)^3\} \sum_{b=0}^{10} \Phi'(b)x^b - \{\prod (1-x^r)\}^2 \equiv 0 \pmod{11}. \tag{4.5}$$

Expressing the infinite products as polynomials in x from Lemmas 3 and 6 respectively and equating coefficients of powers of x in the usual way, we have now 11 congruences to prove.

The coefficient of x^{10} vanishes trivially (mod 11), all its terms being numerical multiples of $P^2(0)$. The remaining 10 congruences fall naturally into two equal classes, one being given by the coefficients of x^0, x^2, x^3, x^4, and x^8, and the other by those of x, x^5, x^6, x^7, and x^9. We shall prove one congruence in each class, the proofs of the others being exactly similar.

If in Lemma 4 we take $(b,c,d) = (5,3,1)$, $(5,4,3)$, $(5,2,1)$, $(4,3,2)$, and $(4,2,1)$, we obtain respectively

$$P(2)P(4)P^2(5)-P^2(3)P(4)P(5)+y^2P^2(1)P(2)P(3) = 0, \tag{4.6}$$

$$P(1)P(4)P^2(5)-P(2)P(3)P^2(4)+yP(1)P(2)P^2(3) = 0, \tag{4.7}$$

$$P(1)P(3)P^2(5)-P^2(2)P(4)P(5)+yP^2(1)P(3)P(4) = 0, \tag{4.8}$$

$$P(1)P^2(4)P(5)-P(2)P^2(3)P(5)+yP(1)P^2(2)P(4) = 0, \tag{4.9}$$

$$P(1)P(3)P^2(4)-P^2(2)P(3)P(5)+yP^2(1)P(2)P(5) = 0. \tag{4.10}$$

Now the coefficient of x^0 on the left-hand side of (4.5) is

$$P^2(0)\left[\frac{P(5)}{P(1)}-\frac{P^2(4)}{P^2(2)}-76y\,\frac{P(4)}{P(5)}+9y\,\frac{P(1)P(5)}{P(2)P(3)}+32y^2\,\frac{P(1)P(3)}{P(4)P(5)}\right]. \tag{4.11}$$

If we divide (4.7) by $y^{-1}P(2)P(3)P(4)P(5)$ and (4.10) by $P(1)P^2(2)P(3)$, and add the results, we obtain

$$\frac{P^2(4)}{P^2(2)}-\frac{P(5)}{P(1)}-y\,\frac{P(4)}{P(5)}+2y\,\frac{P(1)P(5)}{P(2)P(3)}+y^2\,\frac{P(1)P(3)}{P(4)P(5)} = 0;$$

thus the expression (4.11) vanishes (mod 11). Similarly the coefficient of x on the left-hand side of (4.5) is

$$P^2(0)\left[\frac{P^2(5)}{P(2)P(3)}-\frac{P(4)}{P(1)}+56y\,\frac{P(3)}{P(4)}\right] = 55y\,P^2(0)\,\frac{P(3)}{P(4)}$$

by (4.7). With the similar arguments for the other coefficients, this proves (4.5), which is equivalent to the theorem.

This is a more elementary proof of (1.3) than any previously given. If we replace our proof of Lemma 4 by Jacobi's original proof of the addition theorem, and prove Lemma 5 algebraically (which is not difficult), we should be led to a strictly elementary proof of (1.3).

5. We now define

$$\Sigma(z,\zeta,w)=\sum\frac{(-)^n\zeta^n w^{\frac{1}{2}n(n+1)}}{1-zw^n},$$

which is obviously an analytic function of z, ζ in any region

$$0<z_1\leqslant |z|\leqslant z_2,$$

except for simple poles at the points $z=w^n$. We also write†

$$\Sigma(a,b)=\Sigma_q(a,b)=\Sigma(y^a,y^b,y^q)=\sum\frac{(-)^n y^{bn+\frac{1}{2}qn(n+1)}}{1-y^{qn+a}},$$

$$\Sigma(0,b)=\Sigma_q(0,b)=\sum{}'\frac{(-)^n y^{bn+\frac{1}{2}qn(n+1)}}{1-y^{qn}}.$$

It will be seen that the functions we have just introduced include as special cases the mock theta-functions of the third order, in the form given by Watson (5).

Theorems 4 and 5 are simple (though laborious) corollaries of identities connecting these sums with the products $P(a)$ and $P(0)$. The crucial step is contained in Lemmas 7 and 8, which are general identities in three and two variables respectively; Lemma 8 is essentially a limiting case of Lemma 7. It appears impossible to prove these results by simple algebraic manipulation; instead we proceed as in Lemmas 4 and 5.

LEMMA 7.‡ *Suppose that none of a, b, $a\pm b$ is divisible by q. Then*

$$y^{3a}\Sigma(a+b,3a)+\Sigma(b-a,-3a)-y^a\frac{P(2a)}{P(a)}\Sigma(b,0)-\frac{P^2(0)P(a)P(2a)}{P(b+a)P(b)P(b-a)}=0.$$

We shall prove the more general result

$$\zeta^3\Sigma(z\zeta,\zeta^3,w)+\Sigma(z\zeta^{-1},\zeta^{-3},w)-\zeta\frac{P(\zeta^2,w)}{P(\zeta,w)}\Sigma(z,1,w)-\frac{P(\zeta,w)P(\zeta^2,w)\prod(1-w^r)^2}{P(z\zeta^{-1},w)P(z,w)P(z\zeta,w)}=0, \quad (5.1)$$

which reduces to the lemma on writing $\zeta=y^a$, $z=y^b$, $w=y^q$.

† We would remind the reader that, by definition, $a\not\equiv 0 \pmod q$.

‡ Lemma 7 has also been proved by Miss Jackson (3), to whom we communicated it, using basic hypergeometric series. It is really a special case of the expansion of $e^{(2n-1)iz}/\vartheta_2(z-\alpha)\vartheta_2(z-\beta)\vartheta_2(z-\gamma)$ in partial fractions obtained by Watson (5).

We need a preliminary result. We have

$$z^3\Sigma(z,\zeta,w)+\zeta\Sigma(zw,\zeta,w) = \sum(-)^n\frac{z^3\zeta^n w^{\frac12 n(n+1)}}{1-zw^n}+\sum(-)^n\frac{\zeta^{n+1}w^{\frac12 n(n+1)}}{1-zw^{n+1}}$$

$$=\sum(-)^n\zeta^n w^{\frac12 n(n-1)}\left(\frac{z^3w^{3n}-1}{1-zw^n}\right)$$

(writing $n-1$ for n in the second sum)

$$= -\sum(-)^n\zeta^n w^{\frac12 n(n-1)}(1+zw^n+z^2w^{2n}). \tag{5.2}$$

These last sums are most conveniently left in this form, though they are also expressible in terms of the P.

Now for convenience let us write (5.1) in the form

$$f(z) = f_1(z)+f_2(z)-f_3(z)-f_4(z) = 0.$$

After (5.2) we have

$$z^3\zeta^3 f_1(z)+\zeta^3 f_1(zw) = -\zeta^3\sum(-)^n\zeta^{3n}w^{\frac32 n(n-1)}(1+z\zeta w^n+z^2\zeta^2w^{2n})$$

and

$$z^3\zeta^{-3}f_2(z)+\zeta^{-3}f_2(zw) = -\sum(-)^n\zeta^{-3n}w^{\frac32 n(n-1)}(1+z\zeta^{-1}w^n+z^2\zeta^{-2}w^{2n});$$

with suitable changes of variables of summation these give

$$z^3\{f_1(z)+f_2(z)\}+\{f_1(zw)+f_2(zw)\}$$

$$= -\zeta(z+z^2)\sum(-)^n w^{\frac12 n(3n-1)}(\zeta^{3n}+\zeta^{1-3n})$$

$$= -\zeta(z+z^2)\frac{P(\zeta^2,w)}{P(\zeta,w)}\prod(1-w^r),$$

by Lemma 5. Again, (5.2) gives

$$z^3\Sigma(z,1,w)+\Sigma(zw,1,w) = -\sum(-)^n w^{\frac12 n(n-1)}-(z+z^2)\sum(-)^n w^{\frac12 n(3n-1)}$$

$$= -(z+z^2)\prod(1-w^r), \tag{5.3}$$

combining the terms with n, $1-n$ in the first sum, and using (3.9) in the second. Thus

$$z^3f_3(z)+f_3(zw) = -\zeta(z+z^2)\frac{P(\zeta^2,w)}{P(\zeta,w)}\prod(1-w^r).$$

By (3.2) we have immediately

$$z^3f_4(z)+f_4(zw) = 0.$$

Combining all these results, we obtain

$$z^3f(z)+f(zw) = 0.$$

Now, by Lemma 2, either $f(z)$ vanishes identically or it has just three more inequivalent poles than inequivalent zeros. But the only possible poles of $f(z)$ are equivalent to 1, ζ or ζ^{-1}; and it is easy to show that $z = 1$ is not in fact a pole. Thus $f(z) = 0$, which proves the theorem.

We now write

$$g(z,w) = z\frac{P(z^2,w)}{P(z,w)}\Sigma(z,1,w)-z^3\Sigma(z^2,z^3,w)- \\ -\sum{}' \frac{(-)^n z^{-3n}w^{\frac{3}{2}n(n+1)}}{1-w^n} \tag{5.4}$$

which is obviously an analytic function of z in any region $0 < z_1 \leqslant |z| \leqslant z_2$, except for possible simple poles at the points given by $z^2 = w^n$. We also write

$$g(a) = g(y^a,y^q) = y^a\frac{P(2a)}{P(a)}\Sigma(a,0)-y^{3a}\Sigma(2a,3a)-\Sigma(0,-3a). \tag{5.5}$$

LEMMA 8. *We have*

$$2g(a)-g(2a) = \frac{P^2(0)P^2(3a)}{P^2(a)P(4a)}-1$$

and

$$g(a)+g(q-a) = 1.$$

By means of this lemma, we can express the $g(a)$ in terms of $P(0)$ and the $P(a)$, for any fixed q.

As usual, we shall prove the more general results

$$2g(z,w)-g(z^2,w) = \frac{P^2(z^3,w)}{P^2(z,w)P(z^4,w)}\prod(1-w^r)^2-1 \tag{5.6}$$

and

$$g(z,w)+g(z^{-1}w,w) = 1. \tag{5.7}$$

We could obtain (5.6) from (5.1) by letting $\zeta \to z$ and using results about the derivatives of theta-functions; but it is less tedious to argue as follows. For convenience, write (5.4) in the form

$$g(z,w) = f_1(z)-f_2(z)-f_3(z).$$

(3.2) and (5.3) together give

$$f_1(zw)-f_1(z) = (1+z^{-1})\frac{P(z^2,w)}{P(z,w)}\prod(1-w^r). \tag{5.8}$$

For $f_2(z)$, $f_3(z)$ we argue as in (5.2):

$$f_2(zw)-f_2(z) = z^3w^3\sum(-)^n\frac{z^{3n}w^{\frac{3}{2}n(n+3)}}{1-z^2w^{n+2}}-z^3\sum(-)^n\frac{z^{3n}w^{\frac{3}{2}n(n+1)}}{1-z^2w^n}$$
$$= \sum(-)^n z^{3n-3}w^{\frac{3}{2}n(n-1)}\left(\frac{1-z^6w^{3n}}{1-z^2w^n}\right)$$

(writing $n-2$ for n in the first sum)

$$= \sum(-)^n z^{3n-3}w^{\frac{3}{2}n(n-1)}(1+z^2w^n+z^4w^{2n})$$
$$= -\sum(-)^n z^{3n}w^{\frac{3}{2}n(n+1)}+z^{-1}\sum(-)^n z^{3n}w^{\frac{1}{2}n(3n-1)}+ \\ +\sum(-)^n z^{1-3n}w^{\frac{1}{2}n(3n-1)}, \tag{5.9}$$

by suitable changes of the variable of summation. Similarly

$$
\begin{aligned}
f_3(zw)-f_3(z) &= \sum{}' (-)^n z^{-3n} w^{\frac{3}{2}n(n+1)} \left(\frac{w^{-3n}-1}{1-w^n}\right) \\
&= \sum{}' (-)^n z^{-3n} w^{\frac{3}{2}n(n-1)} (1+w^n+w^{2n}) \\
&= -3+\sum (-)^n z^{3n} w^{\frac{3}{2}n(n+1)} + z^{-1} \sum (-)^n z^{1-3n} w^{\frac{1}{2}n(3n-1)} + \\
&\qquad + \sum (-)^n z^{3n} w^{\frac{1}{2}n(3n-1)}, \qquad (5.10)
\end{aligned}
$$

by suitable changes of the variable of summation. If we add (5.9) and (5.10), and subtract their sum from (5.8), we obtain

$$g(z,w)-g(zw,w) = -3, \qquad (5.11)$$

the other terms on the right-hand side disappearing by Lemma 5. If further we write

$$f(z) = 2g(z,w)-g(z^2,w)-\frac{P^2(z^3,w)}{P^2(z,w)P(z^4,w)} \prod (1-w^r)^2+1,$$

we find, from (5.11) and (3.2),

$$f(zw,w)-f(z,w) = 0.$$

To show that $f(z)$ vanishes identically, it is now sufficient, in view of Lemma 2, to show that it has no poles and at least one zero. The poles of $g(z,w)$ are simple ones at the points equivalent to $z = 1, -1, \pm w^{\frac{1}{2}}$; and the residues of $g(z,w)$ at these four points are respectively $-\frac{3}{2}, -\frac{1}{2}, \mp\frac{1}{2}w^{\frac{1}{2}}$. Thus the only possible poles of $f(z)$ are simple ones at points equivalent to those given by $z^4 = 1, w, w^2$, or w^3; and the reader may now easily verify that none of these points is in fact a pole.

Consider next the points at which $z^2+z+1 = 0$. This gives $z^2 = z^{-1}$, and so by (3.2), $f_1(z) = -\Sigma(z,1,w)$. Again

$$
\begin{aligned}
f_2(z) = \Sigma(z^{-1},1,w) &= \sum (-)^n \frac{w^{\frac{3}{2}n(n+1)}}{1-z^{-1}w^n} \\
&= -z\sum (-)^n \frac{w^{\frac{1}{2}n(3n-1)}}{1-zw^n}
\end{aligned}
$$

(writing $-n$ for n); thus

$$
\begin{aligned}
f_1(z)-f_2(z) &= -\sum (-)^n w^{\frac{1}{2}n(3n-1)} \left(\frac{w^{2n}-z}{1-zw^n}\right) \\
&= z\sum (-)^n w^{\frac{1}{2}n(3n-1)} (1+zw^n) \\
&= (z+z^2) \prod (1-w^r)
\end{aligned}
$$

by (3.9). Also

$$f_3(z) = \sum{}' (-)^n \frac{w^{\frac{3}{2}n(n+1)}}{1-w^n} = \sum_{n=1}^{\infty} (-)^n \frac{w^{\frac{3}{2}n(n+1)}}{1-w^n} - \sum_{n=1}^{\infty} (-)^n \frac{w^{\frac{1}{2}n(3n-1)}}{1-w^n}$$

(writing $-n$ for n when $n < 0$)

$$= -\sum_{n=1}^{\infty}(-)^n w^{\frac{1}{2}n(3n-1)}(1+w^n) = 1 - \sum (-)^n w^{\frac{1}{2}n(3n-1)}$$

$$= 1 - \prod (1-w^r)$$

by (3.9). Thus for these particular values of z, $g(z,w) = -1$ and similarly $g(z^2,w) = -1$. Hence these points are zeros of $f(z)$, which must therefore vanish identically.

This is (5.6); and to prove (5.7) we need only show, after (5.11),

$$g(z^{-1},w)+g(z,w) = -2. \tag{5.12}$$

Now

$$\Sigma(z,1,w)+z^{-3}\Sigma(z^{-1},1,w) = \sum (-)^n \frac{w^{\frac{3}{2}n(n+1)}}{1-zw^n} - z^{-2}\sum (-)^n \frac{w^{\frac{1}{2}n(3n-1)}}{1-zw^n}$$

(writing $-n$ for n in the second sum)

$$= -z^{-2}\sum (-)^n w^{\frac{1}{2}n(3n-1)}\left(\frac{1-z^2w^{2n}}{1-zw^n}\right)$$

$$= -z^{-2}\sum (-)^n w^{\frac{1}{2}n(3n-1)}(1+zw^n)$$

$$= -(z^{-1}+z^{-2})\prod (1-w^r)$$

by (3.9). Thus, by (3.2),

$$f_1(z)+f_1(z^{-1}) = -(1+z^{-1})\frac{P(z^2,w)}{P(z,w)}\prod (1-w^r). \tag{5.13}$$

Again

$$f_2(z)+f_2(z^{-1}) = z^3\sum (-)^n \frac{z^{3n}w^{\frac{3}{2}n(n+1)}}{1-z^2w^n} - z^{-1}\sum (-)^n \frac{z^{3n}w^{\frac{1}{2}n(3n-1)}}{1-z^2w^n}$$

(writing $-n$ for n in the second sum)

$$= -z^{-1}\sum (-)^n z^{3n} w^{\frac{1}{2}n(3n-1)}\left(\frac{1-z^4w^{2n}}{1-z^2w^n}\right)$$

$$= -z^{-1}\sum (-)^n z^{3n} w^{\frac{1}{2}n(3n-1)}(1+z^2w^n)$$

$$= -z^{-1}\sum (-)^n z^{3n} w^{\frac{1}{2}n(3n-1)} - \sum (-)^n z^{1-3n} w^{\frac{1}{2}n(3n-1)}, \tag{5.14}$$

writing $-n$ for n in the second sum. Also

$$f_3(z)+f_3(z^{-1}) = -\sum{}' (-)^n \frac{z^{3n}w^{\frac{1}{2}n(3n-1)}}{1-w^n} + \sum{}' (-)^n \frac{z^{3n}w^{\frac{3}{2}n(n+1)}}{1-w^n}$$

(writing $-n$ for n in the first sum)

$$= -\sum{}' (-)^n z^{3n} w^{\frac{1}{2}n(3n-1)} \left(\frac{1-w^{2n}}{1-w^n}\right)$$

$$= -\sum{}' (-)^n z^{3n} w^{\frac{1}{2}n(3n-1)} (1+w^n)$$

$$= 2 - \sum (-)^n z^{3n} w^{\frac{1}{2}n(3n-1)} - z^{-1} \sum (-)^n z^{1-3n} w^{\frac{1}{2}n(3n-1)} \tag{5.15}$$

writing $-n$ for n in the second sum. Adding (5.14) to (5.15) and subtracting the sum from (5.13), we obtain (5.12), since the other terms on the right-hand side vanish by Lemma 5. This proves (5.7) and so completes the proof of the lemma.

6. We can now return to Dyson's generating function (2.13). We shall have to consider sums of the form

$$S(b) = S_q(b) = \sum{}' (-)^n \frac{x^{bn+\frac{1}{2}n(3n+1)}}{1-x^{qn}}. \tag{6.1}$$

Writing $-n$ for n in (6.1) we find

$$S(b) = -S(q-1-b). \tag{6.2}$$

Again

$$S(b)-S(b+q) = \sum{}' (-)^n x^{\frac{1}{2}n(3n+1)} \left(\frac{x^{bn}-x^{(b+q)n}}{1-x^{qn}}\right)$$

$$= \sum (-)^n x^{bn+\frac{1}{2}n(3n+1)} - 1. \tag{6.3}$$

In view of (3.9), it may be shown by a suitable change of the variable of summation that the sum on the right-hand side of (6.3) reduces to

$$\begin{array}{ll} (-)^b x^{-b(b+1)/6} \prod (1-x^r) & \text{if } b \equiv 0 \pmod 3, \\ 0 & \text{if } b \equiv 1 \pmod 3, \\ (-)^{b-1} x^{-b(b+1)/6} \prod (1-x^r) & \text{if } b \equiv 2 \pmod 3. \end{array}$$

When we apply (6.3) we shall assume these results included in it.

We note that, by (6.2),

$$S(\tfrac{1}{2}(q-1)) = 0. \tag{6.4}$$

We now consider $S(\frac{1}{2}(3q-1)-3m)$, where $m \not\equiv 0 \pmod q$. This apparently curious form is chosen so that our final result (6.7) appears as simple as possible. If we write

$$n = qr+m+b \quad (-\infty < r < \infty), \tag{6.5}$$

we have, after simplification,

$$\tfrac{1}{2}n(3q-1)-3mn+\tfrac{1}{2}n(3n+1) = \tfrac{3}{2}q^2r(r+1)+3bqr+\tfrac{3}{2}(b+m)(b-m+q).$$

We make the substitution (6.5) in (6.1), giving b the values 0, $\pm a$, $\pm m$, where a runs through the values 1, 2,..., $\frac{1}{2}(q-1)$, omitting the value congruent to $\pm m$. For convenience we shall use $\sum\limits_a''$ to denote a sum over these values of a. We thus obtain

$$S(\tfrac{1}{2}(3q-1)-3m) = \sum{}' (-)^n \frac{x^{\frac{1}{2}n(3q-1)-3mn+\frac{1}{2}n(3n+1)}}{1-x^{qn}}$$

$$= \sum_{b,r}\sum{}' (-)^{r+m+b} x^{\frac{3}{2}(b+m)(b-m+q)} \frac{y^{\frac{3}{2}qr(r+1)+3br}}{1-y^{qr+m+b}},$$

where the double sum is taken over $-\infty < r < \infty$, $b = 0$, $\pm a$, $\pm m$, the term omitted being given by $r = 0$, $b = -m$. Thus

$$S(\tfrac{1}{2}(3q-1)-3m) = (-)^m x^{\frac{3}{2}m(q-m)}\Sigma(m,0)+\Sigma(0,-3m)+y^{3m}\Sigma(2m,3m)+$$

$$+\sum_a{}'' (-)^{a+m} x^{\frac{3}{2}(a+m)(a-m+q)}\{\Sigma(a+m,3a)+y^{-3a}\Sigma(m-a,-3a)\}, \quad (6.6)$$

the first three terms corresponding to $b = 0$, $b = -m$, and $b = m$ respectively. We can now simplify this by means of Lemmas 7 and 8. By Lemma 7 the sum of the two terms inside the curly brackets is

$$y^{-2a}\frac{P(2a)}{P(a)}\Sigma(m,0)+y^{-3a}\frac{P^2(0)P(a)P(2a)}{P(m+a)P(m)P(m-a)}.$$

Similarly by (5.5) the sum of the second and third terms on the right-hand side of (6.6) is

$$y^m \frac{P(2m)}{P(m)}\Sigma(m,0)-g(m).$$

If we substitute these in (6.6) we have a number of terms which are multiples of $\Sigma(m,0)$. Their total contribution is

$$\Sigma(m,0)\left\{(-)^m x^{\frac{3}{2}m(q-m)}+y^m\frac{P(2m)}{P(m)}+\sum_a{}''(-)^{a+m}x^{\frac{3}{2}(a^2-m^2)+\frac{1}{2}q(3m-a)}\frac{P(2a)}{P(a)}\right\};$$

and by Lemma 6 the expression inside the curly brackets is

$$(-)^{m+\lambda}x^{\frac{3}{2}m(q-m)-\frac{1}{2}\lambda(3\lambda+\mu)}\frac{\prod(1-x^r)}{P(0)}.$$

Assembling all these results, we have

$$S(\tfrac{1}{2}(3q-1)-3m) = -g(m)+(-)^{m+\lambda}x^{\frac{3}{2}m(q-m)-\frac{1}{2}\lambda(3\lambda+\mu)}\frac{\Sigma(m,0)}{P(0)}\prod(1-x^r)+$$

$$+\sum_a{}''(-)^{a+m}x^{\frac{3}{2}(a-m)(a+m-q)}\frac{P^2(0)P(a)P(2a)}{P(m+a)P(m)P(m-a)}. \quad (6.7)$$

By means of (6.2), (6.3), (6.4), and (6.7) we can now express all the $S(b)$ with which we shall be concerned in our standard form—that is to say, as polynomials in x whose coefficients are power series in y. In virtue of Lemma 6 we can already do this with $\prod(1-x^r)$.

We are now in a position to state and prove Theorems 4 and 5. For convenience we shall write

$$r_b(d) = r_b(d,q) = \sum_{n=0}^{\infty} N(b,q,qn+d)y^n \tag{6.8}$$

and

$$r_{bc}(d) = r_{bc}(d,q) = r_b(d)-r_c(d). \tag{6.9}$$

Evidently

$$\sum_{n=0}^{\infty} N(b,q,n)x^n = \sum_{n=0}^{q-1} r_b(s)x^s. \tag{6.10}$$

Theorems 4 and 5 give (for the special cases $q = 5$, $q = 7$ respectively) the values of the $r_{bc}(d)$ in simple form. The method of proof runs as follows: We can express

$$\sum_{n=0}^{\infty} \{N(b,q,n)-N(c,q,n)\}x^n \prod(1-x^r)$$

as a polynomial in x whose coefficients are power series in y, by means of (2.13) and the results obtained earlier in this section. If the assertions of the theorem are true, we can also express it in this form by means of the theorem and Lemma 6. To prove the theorem, it is now sufficient to prove that, for each pair of values of b, c, these two polynomials are the same.

THEOREM 4. *For $q = 5$ we have*

$$r_{12}(0) = y\Sigma(1,0)/P(0), \tag{6.11}$$

$$r_{02}(0)+2r_{12}(0) = P(0)P(2)/P^2(1)-1, \tag{6.12}$$

$$r_{02}(1) = P(0)/P(1), \tag{6.13}$$

$$r_{12}(1) = 0, \tag{6.14}$$

$$r_{02}(2) = 0, \tag{6.15}$$

$$r_{12}(2) = P(0)/P(2), \tag{6.16}$$

$$r_{02}(3) = -y\Sigma(2,0)/P(0), \tag{6.17}$$

$$r_{01}(3)+r_{02}(3) = P(0)P(1)/P^2(2), \tag{6.18}$$

$$r_{02}(4) = 0, \tag{6.19}$$

$$r_{12}(4) = 0. \tag{6.20}$$

By (2.13) and (6.1)–(6.4) we find

$$\sum_{n=0}^{\infty} N(0,5,n)x^n \prod (1-x^r) = S(0)+S(5)$$
$$= 2S(0)+1-\prod (1-x^r) = 1-\prod (1-x^r)-2S(4),$$
$$\sum_{n=0}^{\infty} N(1,5,n)x^n \prod (1-x^r) = S(1)+S(4),$$
$$\sum_{n=0}^{\infty} N(2,5,n)x^n \prod (1-x^r) = S(2)+S(3) = -S(1);$$

and by (6.7) and (3.3),

$$S(1) = -g(2)-x^3\frac{\Sigma(2,0)}{P(0)}\prod (1-x^r)-x^3\frac{P^2(0)}{P(2)}, \tag{6.21}$$

$$S(4) = -g(1)+y\frac{\Sigma(1,0)}{P(0)}\prod (1-x^r)+x^2\frac{P^2(0)}{P(1)}. \tag{6.22}$$

We now use the method of proof sketched above. It is sufficient to consider the two pairs of values $(b,c) = (0,2)$, $(1,2)$. Thus we have to prove

$$1-\prod (1-x^r)+S(1)-2S(4)$$
$$= \left\{\frac{P(0)P(2)}{P^2(1)}-2y\frac{\Sigma(1,0)}{P(0)}-1+x\frac{P(0)}{P(1)}-x^3y\frac{\Sigma(2,0)}{P(0)}\right\}\prod (1-x^r)$$

and

$$2S(1)+S(4)$$
$$= \left\{y\frac{\Sigma(1,0)}{P(0)}+x^2\frac{P(0)}{P(2)}-2x^3y\frac{\Sigma(2,0)}{P(0)}-x^3\frac{P(0)P(1)}{P^2(2)}\right\}\prod (1-x^r).$$

By (6.21) and (6.22) these are respectively equivalent to

$$2g(1)-g(2)-2x^2\frac{P^2(0)}{P(1)}-x^3\frac{P^2(0)}{P(2)}+1 = \left\{\frac{P(0)P(2)}{P^2(1)}+x\frac{P(0)}{P(1)}\right\}\prod (1-x^r)$$

and

$$-g(1)-2g(2)+x^2\frac{P^2(0)}{P(1)}-2x^3\frac{P^2(0)}{P(2)} = \left\{x^2\frac{P(0)}{P(2)}-x^3\frac{P(0)P(1)}{P^2(2)}\right\}\prod (1-x^r).$$

Since by Lemma 6 we have

$$\prod (1-x^r) = P(0)\left\{\frac{P(2)}{P(1)}-x-x^2\frac{P(1)}{P(2)}\right\},$$

these last two equations are respectively equivalent to

$$2g(1)-g(2)+1 = \frac{P^2(0)P^2(2)}{P^3(1)}$$

and
$$2g(2)+g(1) = -y\frac{P^2(0)P^2(1)}{P^3(2)};$$

and by (3.3) these are merely Lemma 8 with $a = 1$, $a = 2$ respectively. This proves the theorem.

Theorem 5. *For $q = 7$ we have*

$$r_{23}(0) = 0, \tag{6.23}$$

$$r_{03}(0)+r_{13}(0) = y\Sigma(1,0)/P(0), \tag{6.24}$$

$$r_{03}(0)+2r_{13}(0) = P(0)P(3)/P(1)P(2)-1, \tag{6.25}$$

$$r_{03}(1) = P(0)/P(1), \tag{6.26}$$

$$r_{13}(1) = 0, \tag{6.27}$$

$$r_{23}(1) = 0, \tag{6.28}$$

$$r_{03}(2) = 0, \tag{6.29}$$

$$r_{23}(2) = -y^2\Sigma(3,0)/P(0), \tag{6.30}$$

$$r_{13}(2)+r_{23}(2) = P(0)P(2)/P(1)P(3), \tag{6.31}$$

$$r_{02}(3) = 0, \tag{6.32}$$

$$r_{13}(3) = 0, \tag{6.33}$$

$$r_{03}(3) = P(0)/P(2), \tag{6.34}$$

$$r_{03}(4) = 0, \tag{6.35}$$

$$r_{13}(4) = 0, \tag{6.36}$$

$$r_{23}(4) = -P(0)/P(3), \tag{6.37}$$

$$r_{03}(5) = 0, \tag{6.38}$$

$$r_{13}(5) = 0, \tag{6.39}$$

$$r_{23}(5) = 0, \tag{6.40}$$

$$r_{03}(6) = y\Sigma(2,0)/P(0), \tag{6.41}$$

$$r_{01}(6) = -P(0)P(1)/P(2)P(3), \tag{6.42}$$

$$r_{03}(6)+r_{12}(6) = 0. \tag{6.43}$$

By (2.13) and (6.1)–(6.4) we find

$$\sum_{n=0}^{\infty} N(0,7,n)x^n \prod (1-x^r) = S(0)+S(7) = -1+\prod (1-x^r)+2S(7),$$

$$\begin{aligned}\sum_{n=0}^{\infty} N(1,7,n)x^n \prod (1-x^r) = S(1)+S(6) &= S(1)-S(0)\\ &= 1-\prod (1-x^r)+S(1)-S(7),\end{aligned}$$

$$\sum_{n=0}^{\infty} N(2,7,n)x^n \prod (1-x^r) = S(2)+S(5) = -S(1)-S(4),$$

$$\sum_{n=0}^{\infty} N(3,7,n)x^n \prod (1-x^r) = S(3)+S(4) = S(4);$$

and by (6.7) and (3.3),

$$S(1) = -g(3)+x^2y^2\frac{\Sigma(3,0)}{P(0)}\prod(1-x^r)+x^2y\frac{P^2(0)P(1)}{P^2(3)}-x^3\frac{P^2(0)}{P(1)}, \tag{6.44}$$

$$S(4) = -g(2)-x^6y\frac{\Sigma(2,0)}{P(0)}\prod(1-x^r)-x^6\frac{P^2(0)}{P(3)}+x^4\frac{P^2(0)P(3)}{P^2(2)}, \tag{6.45}$$

$$S(7) = -g(1)+y\frac{\Sigma(1,0)}{P(0)}\prod(1-x^r)+x\frac{P^2(0)P(2)}{P^2(1)}-x^5\frac{P^2(0)}{P(2)}. \tag{6.46}$$

We now proceed as in Theorem 4. We have to consider the three pairs of values $(b,c) = (0,3), (1,3), (2,3)$. Thus we have to prove

$$-1+\prod(1-x^r)-S(4)+2S(7)$$
$$= \left\{2y\frac{\Sigma(1,0)}{P(0)}-\frac{P(0)P(3)}{P(1)P(2)}+1+x\frac{P(0)}{P(1)}+x^3\frac{P(0)}{P(2)}+x^6y\frac{\Sigma(2,0)}{P(0)}\right\}\prod(1-x^r), \tag{6.47}$$

$$1-\prod(1-x^r)+S(1)-S(4)-S(7) = \left\{\frac{P(0)P(3)}{P(1)P(2)}-1-y\frac{\Sigma(1,0)}{P(0)}+\right.$$
$$\left.+x^2\frac{P(0)P(2)}{P(1)P(3)}+x^2y^2\frac{\Sigma(3,0)}{P(0)}+x^6\frac{P(0)P(1)}{P(2)P(3)}+x^6y\frac{\Sigma(2,0)}{P(0)}\right\}\prod(1-x^r) \tag{6.48}$$

and

$$-S(1)-2S(4) = \left\{-x^2y^2\frac{\Sigma(3,0)}{P(0)}+x^3\frac{P(0)}{P(2)}-x^4\frac{P(0)}{P(3)}+\right.$$
$$\left.+2x^6y\frac{\Sigma(2,0)}{P(0)}+x^6\frac{P(0)P(1)}{P(2)P(3)}\right\}\prod(1-x^r). \tag{6.49}$$

If we add (6.47) and (6.48), and subtract (6.49) from the sum, we obtain

$$2S(1)+S(7)$$
$$= \left\{y\frac{\Sigma(1,0)}{P(0)}+x\frac{P(0)}{P(1)}+x^2\frac{P(0)P(2)}{P(1)P(3)}+2x^2y^2\frac{\Sigma(3,0)}{P(0)}+x^4\frac{P(0)}{P(3)}\right\}\prod(1-x^r). \tag{6.50}$$

To prove the theorem it is now sufficient to prove the three equations (6.47), (6.49), and (6.50). By (6.44), (6.45), and (6.46) they are respectively equivalent to

$$-2g(1)+g(2)-1+2x\frac{P^2(0)P(2)}{P^2(1)}-x^4\frac{P^2(0)P(3)}{P^2(2)}-2x^5\frac{P^2(0)}{P(2)}+x^6\frac{P^2(0)}{P(3)}$$
$$= \left\{-\frac{P(0)P(3)}{P(1)P(2)}+x\frac{P(0)}{P(1)}+x^3\frac{P(0)}{P(2)}\right\}\prod(1-x^r), \tag{6.51}$$

$$2g(2)+g(3)-x^2y\frac{P^2(0)P(1)}{P^2(3)}+x^3\frac{P^2(0)}{P(1)}-2x^4\frac{P^2(0)P(3)}{P^2(2)}+2x^6\frac{P^2(0)}{P(3)}$$
$$= \left\{x^3\frac{P(0)}{P(2)}-x^4\frac{P(0)}{P(3)}+x^6\frac{P(0)P(1)}{P(2)P(3)}\right\}\prod(1-x^r) \tag{6.52}$$

and

$$-g(1)-2g(3)+x\frac{P^2(0)P(2)}{P^2(1)}+2x^2y\frac{P^2(0)P(1)}{P^2(3)}-2x^3\frac{P^2(0)}{P(1)}-x^5\frac{P^2(0)}{P(2)}$$
$$=\left\{x\frac{P(0)}{P(1)}+x^2\frac{P(0)P(2)}{P(1)P(3)}+x^4\frac{P(0)}{P(3)}\right\}\prod(1-x^r). \quad (6.53)$$

Now by Lemma 6 we have

$$\prod(1-x^r) = P(0)\left\{\frac{P(2)}{P(1)}-x\frac{P(3)}{P(2)}-x^2+x^5\frac{P(1)}{P(3)}\right\}.$$

Substituting this in each of (6.51)–(6.53) and equating coefficients of powers of x, we now have 21 equations to prove. The coefficients of x^0 give us respectively

$$-2g(1)+g(2)-1 = -\frac{P^2(0)P(3)}{P^2(1)},$$
$$2g(2)+g(3) = -y\frac{P^2(0)P(1)}{P^2(2)},$$

and
$$-g(1)-2g(3) = y\frac{P^2(0)P(2)}{P^2(3)};$$

and by (3.3) these are simply Lemma 8 with $a = 1, 2$, and 3 respectively. All the other equations are trivially satisfied except for the coefficient of x in (6.51), the coefficient of x^4 in (6.52) and the coefficient of x^2 in (6.53). These are respectively

$$P^2(0)\left\{\frac{P(2)}{P^2(1)}-\frac{P^2(3)}{P(1)P^2(2)}-y\frac{P(1)}{P(2)P(3)}\right\} = 0,$$
$$P^2(0)\left\{-\frac{P(3)}{P^2(2)}+\frac{P(2)}{P(1)P(3)}-y\frac{P^2(1)}{P(2)P^2(3)}\right\} = 0,$$

and
$$P^2(0)\left\{y\frac{P(1)}{P^2(3)}-\frac{P^2(2)}{P^2(1)P(3)}+\frac{P(3)}{P(1)P(2)}\right\} = 0;$$

and each of them reduces to

$$P(1)P^3(3)-P(3)P^3(2)+yP^3(1)P(2) = 0,$$

which is simply Lemma 4 with $b = 3$, $c = 2$, and $d = 1$. This proves the theorem.

Notation

$p(n)$ is the number of unrestricted partitions of n.

$N(m,n)$ is the number of partitions of n with rank m.

$N(m,q,n)$ is the number of partitions of n whose rank is congruent to m modulo q.

$q = 6\lambda+\mu$ where λ is a positive integer and $\mu = \pm 1$.

$a \not\equiv 0 \pmod q$.

$y = x^q$, $|x| < 1$, $|w| < 1$.

$\prod$, $\sum$, $\sum'$, unless otherwise specified, denote $\prod\limits_{r=1}^{\infty}$, $\sum\limits_{n=-\infty}^{\infty}$, $\sum\limits_{n=-\infty}^{\infty}{}'$ (the term given by $n = 0$ being omitted).

The points z_1, z_2 are called equivalent if $z_1 = z_2 w^n$ for some integer n.

$P(z,w) = \prod (1-zw^{r-1})(1-z^{-1}w^r)$.

$P(a) = P(y^a, y^q)$, $P(0) = \prod (1-y^{qr})$.

$\Phi(b) = \sum\limits_{n=0}^{\infty} p(qn+b)y^n$.

$$\Sigma(z,\zeta,w) = \sum (-)^n \frac{\zeta^n w^{\frac{1}{2}n(n+1)}}{1-zw^n}.$$

$$\Sigma(a,b) = \Sigma(y^a, y^b, y^q),\ \Sigma(0,b) = \sum{}' (-)^n \frac{y^{\frac{1}{2}qn(n+1)+bn}}{1-y^{qn}}.$$

$$g(z,w) = z\frac{P(z^2,w)}{P(z,w)}\Sigma(z,1,w) - z^3\Sigma(z^2,z^3,w) - \sum{}' (-)^n z^{-3n}\frac{w^{\frac{1}{2}n(n+1)}}{1-w^n},$$

$g(a) = g(y^a, y^q)$.

$$S(b) = \sum{}' (-)^n \frac{x^{\frac{1}{2}n(3n+1)+bn}}{1-x^{qn}}.$$

$r_b(d) = \sum\limits_{n=0}^{\infty} N(b,q,qn+d)y^n$, $r_{bc}(d) = r_b(d) - r_c(d)$.

$f(z)$ and $f_i(z)$ ($i = 1, 2, 3, 4$) are defined in the proofs of Lemmas 4, 5, 7, and 8; but the definitions last only for the length of the relevant proof.

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