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Ranks of partitions modulo 10

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ABSTRACT

In 1954, A.O.L. Atkin and H.P.F. Swinnerton-Dyer established the generating functions for rank differences modulo 5 and 7 for partition functions. In this paper, we derive formulas for the generating functions of ranks of partitions modulo 10 and some inequalities between them.

1. Introduction

Let $p(n)$ denote the number of unrestricted partitions of n. Ramanujan discovered and later proved the following three famous congruences:

$$p(5n+4) \equiv 0 \pmod 5, \tag{1.1}$$

$$p(7n+5) \equiv 0 \pmod 7, \tag{1.2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{1.3}$$

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In order to find a combinatorial interpretation for Ramanujan's congruences, in 1944, F.J. Dyson [6] defined the rank of a partition to be the largest part minus the number of parts. If $N(m,n)$ is defined to be the number of partitions of n with rank m, then the generating function for $N(m,n)$ is given by

$$\sum_{n=0}^{\infty}\sum_{m=-\infty}^{\infty} N(m,n)z^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq;q)_n(q/z;q)_n}. \tag{1.4}$$

Here and throughout, we use the notations

$$\begin{aligned}
(x_1,x_2,\ldots,x_k;q)_m &:= \prod_{n=0}^{m-1}\left(1-x_1q^n\right)\left(1-x_2q^n\right)\cdots\left(1-x_kq^n\right),\\
(x_1,x_2,\ldots,x_k;q)_\infty &:= \prod_{n=0}^{\infty}\left(1-x_1q^n\right)\left(1-x_2q^n\right)\cdots\left(1-x_kq^n\right),\\
[x_1,x_2,\ldots,x_k;q]_\infty &:= (x_1,q/x_1,x_2,q/x_2,\ldots,x_k,q/x_k;q)_\infty,\\
J_{a,b} &:= \left(q^a,q^{b-a},q^b;q^b\right),\\
\overline{J}_{a,b} &:= \left(-q^a,-q^{b-a},q^b;q^b\right),\\
J_b &:= \left(q^b;q^b\right)_\infty,\\
\overline{J}_b &:= \left(-q^b;q^b\right)_\infty,
\end{aligned}$$

and we require $|q|<1$ for absolute convergence.

Let $N(s,l,n)$ denote the number of partitions of n whose rank is congruent to s modulo l. Dyson then conjectured

$$N(k,5,5n+4) = \frac{p(5n+4)}{5}, \quad 0 \leqslant k \leqslant 4 \tag{1.5}$$

and

$$N(k,7,7n+5) = \frac{p(7n+5)}{5}, \quad 0 \leqslant k \leqslant 6. \tag{1.6}$$

It is easy to see that the above two conjectures imply Ramanujan's congruences $p(5n+4)\equiv 0 \pmod 5$ and $p(7n+5)\equiv 0 \pmod 7$, respectively. Dyson's conjectures were first proved by A.O.L. Atkin and H.P.F. Swinnerton-Dyer [3] in 1954. In fact, they established the generating functions for every rank difference $N(s,l,ln+d)-N(t,l,ln+d)$ with $l=5$ or 7 and $0\leqslant d,s,t\leqslant l$. Although Dyson's rank fails to explain Ramanujan's congruence (1.3) combinatorially, the method developed by Atkin and Swinnerton-Dyer [3] is widely used to get rank differences for other types of partitions ranks (see [16–18], for example). Besides (1.5) and (1.6), more relations between ranks of partitions modulo 5 and 7 have been obtained. As examples, we have

$$N(1,5,5n) > N(2,5,5n) \quad \text{for } n \geqslant 1, \tag{1.7}$$

$$N(1,5,5n+1) = N(2,5,5n+1) \quad \text{for } n \geqslant 0, \tag{1.8}$$

$$N(0,5,5n+1) > N(1,5,5n+1) \quad \text{for } n \geqslant 0, \tag{1.9}$$

$$N(0,5,5n+1) > N(2,5,5n+1) \quad \text{for } n \geqslant 0, \tag{1.10}$$

$$N(0,5,5n+2) = N(2,5,5n+2) \quad \text{for } n \geqslant 0, \tag{1.11}$$

$$N(1,5,5n+2) \geqslant N(2,5,5n+2) \quad \text{for } n \geqslant 0, \tag{1.12}$$

$$N(1,5,5n+2) \geqslant N(0,5,5n+2) \quad \text{for } n \geqslant 0. \tag{1.13}$$

Equalities (1.8) and (1.11) follow from the equalities (6.14) and (6.15) in [3], respectively. F.G. Garvan proved (1.7), (1.9) and (1.12) in [7]. Inequality (1.10) follows from (1.8) and (1.9). While (1.13) follows from (1.11) and (1.12).

The first objective of this article is to prove the following generating functions of ranks of partitions modulo 10:

Theorem 1.1. *We have*

$$\begin{aligned}
&\sum_{n\geqslant 0}\big(N(0,10,n)+N(1,10,n)-N(4,10,n)-N(5,10,n)\big)q^n \\
&\quad= \frac{J_{25}J_{20,50}^2J_{50}^5}{J_{10,50}^4J_{15,50}^3} + \frac{1}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{75n(n+1)/2+5}}{1+q^{25n+5}} \\
&\qquad+ \frac{qJ_{25}J_{50}^5}{J_{5,50}J_{10,50}^2J_{15,50}^2} + \frac{q^2J_{25}J_{50}^5}{J_{5,50}^2J_{15,50}J_{20,50}^2} + \frac{q^3J_{25}J_{10,50}^2J_{50}^5}{J_{5,50}^3J_{20,50}^4} \\
&\qquad- \frac{1}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{75n(n+1)/2+8}}{1+q^{25n+10}} + \frac{2q^4J_{50}^6}{J_{25}J_{5,50}J_{10,50}J_{15,50}J_{20,50}}
\end{aligned} \tag{1.14}$$

and

$$\begin{aligned}
&\sum_{n\geqslant 0}\big(N(1,10,n)+N(2,10,n)-N(3,10,n)-N(4,10,n)\big)q^n \\
&\quad= \frac{2q^5J_{50}^6}{J_{25}J_{10,50}^2J_{15,50}^2} - \frac{1}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{75n(n+1)/2+5}}{1+q^{25n+5}} \\
&\qquad+ \frac{2q^6J_{50}^6}{J_{25}J_{5,50}J_{15,50}J_{20,50}^2} + \frac{q^2J_{25}J_{20,50}J_{50}^5}{J_{10,50}^3J_{15,50}^3} + \frac{q^3J_{25}J_{50}^5}{J_{5,50}J_{10,50}J_{15,50}^2J_{20,50}} \\
&\qquad+ \frac{J_{25}J_{20,50}^2J_{25,50}J_{50}^5}{2qJ_{10,50}^4J_{15,50}^4} - \frac{1}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(75n^2+25n)/2-1}}{1+q^{25n}}.
\end{aligned} \tag{1.15}$$

With the generating functions for $N(0,10,5n+l)+N(1,10,5n+l)-N(4,10,5n+l)-N(5,10,5n+l)$ and $N(1,10,5n+l)+N(2,10,5n+l)-N(3,10,5n+l)-N(4,10,5n+l)$

($0 \leqslant l \leqslant 4$) established in Theorem 1.1, we discover the following inequalities between the ranks of partitions modulo 10.

Corollary 1.2. *For $n \geqslant 0$, $i = 1,2,3,4$ and $j = 1,2,3,4$, we have*

$$N(0,10,5n+i) + N(1,10,5n+i) > N(4,10,5n+i) + N(5,10,5n+i), \tag{1.16}$$

$$N(1,10,5n+j) + N(2,10,5n+j) \geqslant N(3,10,5n+j) + N(4,10,5n+j), \tag{1.17}$$

$$N(0,10,5n+1) > N(4,10,5n+1), \tag{1.18}$$

$$N(1,10,5n+1) > N(5,10,5n+1), \tag{1.19}$$

$$N(1,10,5n+1) \geqslant N(3,10,5n+1), \tag{1.20}$$

$$N(2,10,5n+1) \geqslant N(4,10,5n+1), \tag{1.21}$$

$$N(1,10,5n+2) > N(5,10,5n+2), \tag{1.22}$$

$$N(1,10,5n+2) \geqslant N(3,10,5n+2), \tag{1.23}$$

$$N(0,10,5n+4) > N(4,10,5n+4), \tag{1.24}$$

$$N(1,10,5n+4) > N(5,10,5n+4), \tag{1.25}$$

$$N(2,10,5n+4) \geqslant N(4,10,5n+4), \tag{1.26}$$

$$N(1,10,5n+4) \geqslant N(3,10,5n+4). \tag{1.27}$$

For (1.17), (1.20), (1.21), (1.23), (1.26) and (1.27), we have a strict inequality except in the following cases: $(j,n) = (1,0), (2,1)$ and $(4,0)$ in (1.17), $n = 0$ in (1.20), (1.21), (1.26) and (1.27), $n = 1$ in (1.23).

Remark.

(1) Relations between ranks of partitions were also discussed by Lewis and Santa-Gadea (see [10,13,20], for example). Dyson also conjectured the existence of a "crank" which separates the partitions of $11n+6$ into 11 equal classes. This was found in [1] and [7]. Numerous relations between ranks and cranks of partitions are found over the years (see [2,9,11,12,14,15], for example). The relations between cranks of partitions modulo 10 are also discussed in [8].
(2) Computer evidence suggests that (1.16) and (1.17) might still hold when $i = j = 0$. We failed to prove them and so we leave them in the following conjecture.

Conjecture 1.3.

$$N(0,10,5n) + N(1,10,5n) > N(4,10,5n) + N(5,10,5n) \quad \text{for } n \geqslant 0,$$

$$N(1,10,5n) + N(2,10,5n) \geqslant N(3,10,5n) + N(4,10,5n) \quad \text{for } n \geqslant 1.$$

The paper is organized as follows. We first give some lemmas in Section 2, then prove Theorem 1.1 in Section 3. In Section 4 we prove the inequalities.

2. Some lemmas

First, we derive some theta identities which will be used repeatedly in the proof of Theorem 1.1. Let

$$H(a,b,c,q) := \frac{(ab, q/(ab), bc, q/(bc), ca, q/(ca); q)_\infty (q;q)_\infty^2}{(a, q/a, b, q/b, c, q/c, abc, q/(abc); q)_\infty}.$$

We have the following equivalent version of Corollary 4.4 in [5]:

Lemma 2.1.

$$H(a,b,c,q) - H(a,b,d,q) = H(c,1/d,abd,q), \tag{2.1}$$

$$H\left(a,a,q^{25}/a,q^{50}\right) + H\left(b,b,q^{25}/b,q^{50}\right) = 2H\left(a,q^{25}/a,b,q^{50}\right), \tag{2.2}$$

$$H\left(a,a,q^{25}/a,q^{50}\right) - H\left(b,b,q^{25}/b,q^{50}\right) = 2H\left(a,q^{25}/a,b/q^{25},q^{50}\right). \tag{2.3}$$

We need the following special cases. First, replacing q by q^{50} and setting $(a,b,c,d) = (q^{10}, q^{25}, q^{10}, q^{-5})$ in (2.1), we find that

$$\frac{J_{15,50}^2 J_{20,50}}{J_{5,50} J_{10,50}^2 J_{25,50}} + \frac{q^5 J_{15,50}}{J_{10,50} J_{25,50}} = \frac{J_{15,50}^2}{J_{5,50}^2 J_{20,50}}. \tag{2.4}$$

Replacing q by q^{50} and setting $(a,b,c,d) = (q^{15}, q^{15}, q^{10}, q^{-10})$ in (2.1), we find that

$$\frac{J_{20,50} J_{25,50}^2}{J_{10,50}^2 J_{15,50}^2} + \frac{q^{10} J_{5,50}^2}{J_{10,50} J_{15,50}^2} = \frac{J_{20,50}^2}{J_{10,50}^3}.$$

Multiplying by $\frac{J_{10,50} J_{50}^6}{J_{5,50}^2 J_{20,50}^2}$ throughout and rearranging, the above equation becomes

$$\frac{J_{25,50}^2 J_{50}^6}{J_{5,50}^2 J_{10,50} J_{15,50}^2 J_{20,50}} = \frac{J_{50}^6}{J_{5,50}^2 J_{10,50}^2} - \frac{q^{10} J_{50}^6}{J_{15,50}^2 J_{20,50}^2}. \tag{2.5}$$

Replacing q by q^{50} and setting $(a,b,c,d) = (q^{15}, q^{10}, q^{10}, q^{-5})$ in (2.1), we find that

$$\frac{J_{20,50} J_{25,50}^2}{J_{10,50}^2 J_{15,50}^2} + \frac{q^5 J_{25,50}}{J_{15,50} J_{20,50}} = \frac{J_{25,50}}{J_{5,50} J_{10,50}}. \tag{2.6}$$

Next, replacing q by q^{25} and setting $(a,b,c,d) = (-1, -q^{10}, -q^5, q^5)$, in (2.1), we find that

$$\frac{J_{5,25} J_{10,25}}{\overline{J}_{5,25} \overline{J}_{10,25}} - \frac{\overline{J}_{5,25} \overline{J}_{10,25}}{J_{5,25} J_{10,25}} = -\frac{4q^5 \overline{J}_{25}^4 J_{25}^2}{J_{5,25} J_{10,25}}. \tag{2.7}$$

Replacing q by q^{25} and setting $(a,b,c,d)=(-q^5,-q^5,-q^5,q^5)$ in (2.1), we find that

$$\frac{J_{10,25}^2}{\overline{J}_{5,25}\overline{J}_{10,25}}-\frac{\overline{J}_{10,25}^2}{J_{5,25}J_{10,25}}=-\frac{2q^5\overline{J}_{25}^2\overline{J}_{5,25}^2J_{25}}{J_{5,25}J_{10,25}\overline{J}_{10,25}}. \tag{2.8}$$

Setting $a=q^5, b=q^{10}$ in (2.2) and (2.3), respectively, we find that

$$\frac{J_{10,50}}{J_{5,50}^2J_{20,50}^2}+\frac{J_{20,50}}{J_{10,50}^2J_{15,50}^2}=\frac{2}{J_{5,50}J_{10,50}J_{25,50}} \tag{2.9}$$

and

$$\frac{J_{10,50}}{J_{5,50}^2J_{20,50}^2}-\frac{J_{20,50}}{J_{10,50}^2J_{15,50}^2}=\frac{2q^5}{J_{15,50}J_{20,50}J_{25,50}}. \tag{2.10}$$

Then multiplying (2.10) by (2.9), we get

$$\frac{J_{10,50}^2}{J_{5,50}^4J_{20,50}^4}-\frac{J_{20,50}^2}{J_{10,50}^4J_{15,50}^4}=\frac{4q^5}{J_{5,50}J_{10,50}J_{15,50}J_{20,50}J_{25,50}^2}. \tag{2.11}$$

We also need [7, Lemma (3.18)] as follows:

Lemma 2.2.

$$(q;q)_\infty=J_{25}\times\left\{\frac{J_{10,25}}{J_{5,25}}-q-q^2\frac{J_{5,25}}{J_{10,25}}\right\}. \tag{2.12}$$

The following lemma is a special case with $r=0$, $s=3$ of Theorem 2.1 in [4].

Lemma 2.3. *For* $|q|<|\frac{1}{b_1b_2b_3}|<1$,

$$\begin{aligned}\frac{(q;q)_\infty^2}{[b_1,b_2,b_3;q]_\infty}&=\frac{1}{[b_2/b_1,b_3/b_1;q]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n(n+1)/2}}{1-b_1q^n}\left(\frac{b_1^2}{b_2b_3}\right)^n\\&\quad+\frac{1}{[b_1/b_2,b_3/b_2;q]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n(n+1)/2}}{1-b_2q^n}\left(\frac{b_2^2}{b_1b_3}\right)^n\\&\quad+\frac{1}{[b_1/b_3,b_2/b_3;q]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n(n+1)/2}}{1-b_3q^n}\left(\frac{b_3^2}{b_2b_2}\right)^n.\end{aligned} \tag{2.13}$$

We also require a result (see [19, Theorem 1]) on non-negativity of coefficients of certain products.

Lemma 2.4. *If p and r are positive integers with $p \geqslant 2$ and $r < p$ and*

$$L_{p,r}(q) := \sum_{n=0}^{\infty} b_{p,r}(n)q^n := \frac{(q^p;q^p)_\infty}{(q^r;q^p)_\infty (q^{p-r};q^p)_\infty},$$

then $b_{p,r}(n) \geqslant 0$ for all n. Moreover, if we let

$$L_{p,r}(q) + q^p := \sum_{n=0}^{\infty} c_{p,r}(n)q^n := \sum_0 + \sum_1 + \cdots + \sum_{r-1}, \tag{2.14}$$

where

$$\sum_i = \sum_{n=0}^{\infty} c_{p,r}(nr+i)q^{nr+i},$$

then for each i, the sequence $\{c_{p,r}(nr+i)\}_{n \geqslant 0}$ is non-decreasing.

3. Proof of Theorem 1.1

Lemma 3.1. *Let*

$$P(a,b) := \frac{[a,a^2;q^{25}]_\infty (q^{25};q^{25})_\infty^2}{[b/a,ab,b;q^{25}]_\infty}.$$

We have

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1+q^{5n}} = P\big(q^5,-q^5\big) - \frac{P(q^{10},-q^5)}{q^3} + \frac{(q;q)_\infty}{J_{25}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{75n(n+1)/2+5}}{1+q^{25n+5}}, \tag{3.1}$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2+n}}{1+q^{5n}} = P\big(q^{10},-q^{10}\big) - q^3 P\big(q^5,-q^{10}\big) - \frac{(q;q)_\infty}{J_{25}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{75n(n+1)/2+8}}{1+q^{25n+10}}, \tag{3.2}$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2+2n}}{1+q^{5n}} = \frac{P(q^5,-1)}{q^6} - \frac{P(q^{10},-1)}{q^9} - \frac{(q;q)_\infty}{J_{25}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{(75n^2+25n)/2-1}}{1+q^{25n}}. \tag{3.3}$$

Proof. Since the proofs of the above three equations are similar, we only show (3.1). First, replacing n with $-n$ in the summation index of the series on the left side of (3.1), we find that

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1+q^{5n}}=\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+9)/2}}{1+q^{5n}}. \tag{3.4}$$

Splitting the series on the right side of (3.4) into five series according to the summation index n modulo 5, we find that

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1+q^{5n}}\\
&\quad=\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+9)/2}}{1+q^{5n}}\\
&\quad=\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(75n^2+45n)/2}}{1+q^{25n}}-\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(75n^2+75n)/2+6}}{1+q^{25n+5}}\\
&\qquad+\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(75n^2+105n)/2+15}}{1+q^{25n+10}}-\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(75n^2+135n)/2+27}}{1+q^{25n+15}}\\
&\qquad+\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(75n^2+165n)/2+45}}{1+q^{25n+20}}\\
&\quad=: T_0-T_1+T_2-T_3+T_4.
\end{aligned} \tag{3.5}$$

We apply identity (2.13) twice. First, replacing q, b_1, b_2 and b_3 by $q^{25}, -1, -q^{10}$ and $-q^5$, respectively, in (2.13), we find that

$$\begin{aligned}
P\big(q^5,-q^5\big)&=\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(75n^2+45n)/2}}{1+q^{25n}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(75n^2+105n)/2+15}}{1+q^{25n+10}}\\
&\quad-\frac{J_{10,25}}{J_{5,25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(75n^2+75n)/2+5}}{1+q^{25n+5}},
\end{aligned}$$

which is equivalent to

$$T_0+T_2=P\big(q^5,-q^5\big)+\frac{J_{10,25}}{qJ_{5,25}}T_1. \tag{3.6}$$

Next, replacing q, b_1, b_2 and b_3 by q^{25}, $-q^{15}$, $-q^{-5}$ and $-q^5$, respectively, in (2.13), we find that

$$P(q^{10},-q^5)=\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(75n^2+135n)/2+30}}{1+q^{25n+15}}-\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(75n^2+15n)/2}}{1+q^{25n-5}}$$
$$-\frac{J_{5,25}}{J_{10,25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(75n^2+75n)/2+10}}{1+q^{25n+5}},$$

which is equivalent

$$T_3-T_4=\frac{P(q^{10},-q^5)}{q^3}+\frac{qJ_{5,25}}{J_{10,25}}T_1. \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.5), we find that

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1+q^{5n}}&=P(q^5,-q^5)-\frac{P(q^{10},-q^5)}{q^3}+\frac{J_{10,25}}{qJ_{5,25}}T_1-T_1-\frac{qJ_{5,25}}{J_{10,25}}T_1\\&=P(q^5,-q^5)-\frac{P(q^{10},-q^5)}{q^3}+\left\{\frac{J_{10,25}}{J_{5,25}}-q-\frac{q^2J_{5,25}}{J_{10,25}}\right\}\frac{T_1}{q}.\end{aligned}$$

We complete the proof after invoking (2.12) on the right side of the last equality. □

Next, we obtain the 5-dissection of

$$\frac{1}{(q;q)_\infty}\left\{P(q^5,-q^5)-\frac{P(q^{10},-q^5)}{q^3}+P(q^{10},-q^{10})-q^3P(q^5,-q^{10})\right\}$$

and

$$\frac{1}{(q;q)_\infty}\left\{\frac{P(q^{10},-q^5)}{q^3}-P(q^5,-q^5)+\frac{P(q^5,-1)}{q^6}-\frac{P(q^{10},-1)}{q^9}\right\}$$

via the next two lemmas.

Lemma 3.2. *Let*

$$\begin{aligned}A_0&:=\frac{(q^{25};q^{50})_\infty J_{50}^6J_{20,50}^2}{J_{10,50}^4J_{15,50}^3},\\A_1&:=\frac{q(q^{25};q^{50})_\infty J_{50}^6}{J_{5,50}J_{10,50}^2J_{15,50}^2},\\A_2&:=\frac{q^2(q^{25};q^{50})_\infty J_{50}^6}{J_{5,50}^2J_{15,50}J_{20,50}^2},\\A_3&:=\frac{q^3(q^{25};q^{50})_\infty J_{50}^6J_{10,50}^2}{J_{5,50}^3J_{20,50}^4},\\A_4&:=\frac{2q^4J_{50}^5}{(q^{25};q^{50})_\infty J_{5,50}J_{10,50}J_{15,50}J_{20,50}}.\end{aligned}$$

We have

$$P(q^5,-q^5)-\frac{P(q^{10},-q^5)}{q^3}+P(q^{10},-q^{10})-q^3P(q^5,-q^{10})$$
$$=(q;q)_\infty\times\{A_0+A_1+A_2+A_3+A_4\}. \tag{3.8}$$

Proof. By Lemma 2.2, we have

$$(q;q)_\infty=\frac{J_{10,25}J_{25}}{J_{5,25}}-qJ_{25}-\frac{q^2J_{5,25}J_{25}}{J_{10,25}}.$$

Expanding the right side of (3.8) and comparing both sides according to the powers of q modulo 5, we find that it suffices to prove the five identities:

$$\begin{aligned}&P(q^5,-q^5)+P(q^{10},-q^{10})\\&\quad=\frac{(q^{25};q^{50})_\infty J_{50}^6J_{20,50}^2}{J_{10,50}^4J_{15,50}^3}\times\frac{J_{10,25}J_{25}}{J_{5,25}}-qJ_{25}\times\frac{2q^4J_{50}^5}{(q^{25};q^{50})_\infty J_{5,50}J_{10,50}J_{15,50}J_{20,50}}\\&\quad-\frac{q^2J_{5,25}J_{25}}{J_{10,25}}\times\frac{q^3(q^{25};q^{50})_\infty J_{50}^6J_{10,50}^2}{J_{5,50}^3J_{20,50}^4},\end{aligned}\tag{3.9}$$

$$\begin{aligned}0&=\frac{J_{10,25}J_{25}}{J_{5,25}}\times\frac{q(q^{25};q^{50})_\infty J_{50}^6}{J_{5,50}J_{10,50}^2J_{15,50}^2}-qJ_{25}\times\frac{(q^{25};q^{50})_\infty J_{50}^6J_{20,50}^2}{J_{10,50}^4J_{15,50}^3}\\&\quad-\frac{q^2J_{5,25}J_{25}}{J_{10,25}}\times\frac{2q^4J_{50}^5}{(q^{25};q^{50})_\infty J_{5,50}J_{10,50}J_{15,50}J_{20,50}},\end{aligned}\tag{3.10}$$

$$\begin{aligned}-\frac{P(q^{10},-q^5)}{q^3}&=\frac{J_{10,25}J_{25}}{J_{5,25}}\times\frac{q^2(q^{25};q^{50})_\infty J_{50}^6}{J_{5,50}^2J_{15,50}J_{20,50}^2}-qJ_{25}\times\frac{q(q^{25};q^{50})_\infty J_{50}^6}{J_{5,50}J_{10,50}^2J_{15,50}^2}\\&\quad-\frac{q^2J_{5,25}J_{25}}{J_{10,25}}\times\frac{(q^{25};q^{50})_\infty J_{50}^6J_{20,50}^2}{J_{10,50}^4J_{15,50}^3},\end{aligned}\tag{3.11}$$

$$\begin{aligned}-P(q^5,-q^{10})q^3&=\frac{J_{10,25}J_{25}}{J_{5,25}}\times\frac{q^3(q^{25};q^{50})_\infty J_{50}^6J_{10,50}^2}{J_{5,50}^3J_{20,50}^4}-qJ_{25}\times\frac{q^2(q^{25};q^{50})_\infty J_{50}^6}{J_{5,50}^2J_{15,50}J_{20,50}^2}\\&\quad-\frac{q^2J_{5,25}J_{25}}{J_{10,25}}\times\frac{q(q^{25};q^{50})_\infty J_{50}^6}{J_{5,50}J_{10,50}^2J_{15,50}^2},\end{aligned}\tag{3.12}$$

$$\begin{aligned}0&=\frac{J_{10,25}J_{25}}{J_{5,25}}\times\frac{2q^4J_{50}^5}{(q^{25};q^{50})_\infty J_{5,50}J_{10,50}J_{15,50}J_{20,50}}-qJ_{25}\times\frac{q^3(q^{25};q^{50})_\infty J_{50}^6J_{10,50}^2}{J_{5,50}^3J_{20,50}^4}\\&\quad-\frac{q^2J_{5,25}J_{25}}{J_{10,25}}\times\frac{q^2(q^{25};q^{50})_\infty J_{50}^6}{J_{5,50}^2J_{15,50}J_{20,50}^2}.\end{aligned}\tag{3.13}$$

Simplifying each of these five identities and noting that

$$P(q^5,-q^5)=P(q^{10},-q^{10})=\frac{J_{5,50}^2J_{10,50}J_{15,50}^2J_{20,50}J_{25,50}^2}{2J_{50}^6},$$

we see that to prove (3.9), it suffices to show that

$$\begin{aligned}
&\frac{J_{5,50}^2 J_{10,50} J_{15,50}^2 J_{20,50} J_{25,50}^2}{J_{50}^6} \\
&\quad = \frac{J_{20,50} J_{25,50} J_{50}^6}{J_{5,50} J_{10,50}^3 J_{15,50}^2} - \frac{2q^5 J_{50}^6}{J_{5,50} J_{10,50} J_{15,50} J_{20,50}} - \frac{q^5 J_{10,50} J_{25,50} J_{50}^6}{J_{5,50}^2 J_{15,50} J_{20,50}^3}.
\end{aligned} \tag{3.14}$$

Multiplying by $\frac{q^5 J_{10,50} J_{25,50} J_{50}^6}{J_{15,50}^3 J_{20,50}^2}$ on both sides of (2.4) and rearranging, we have

$$\frac{q^5 J_{50}^6}{J_{5,50} J_{10,50} J_{15,50} J_{20,50}} - \frac{q^5 J_{10,50} J_{25,50} J_{50}^6}{J_{5,50}^2 J_{15,50} J_{20,50}^3} = -\frac{q^{10} J_{50}^6}{J_{15,50}^2 J_{20,50}^2}. \tag{3.15}$$

Next, we multiply by $\frac{J_{50}^6}{J_{5,50} J_{10,50} J_{25,50}}$ on both sides of (2.6) and get

$$\frac{J_{20,50} J_{25,50} J_{50}^6}{J_{5,50} J_{10,50}^3 J_{15,50}^2} + \frac{q^5 J_{50}^6}{J_{5,50} J_{10,50} J_{15,50} J_{20,50}} = \frac{J_{50}^6}{J_{5,50}^2 J_{10,50}^2}. \tag{3.16}$$

Adding (3.15) to (3.16), we find that

$$\begin{aligned}
&\frac{2q^5 J_{50}^6}{J_{5,50} J_{10,50} J_{15,50} J_{20,50}} - \frac{q^5 J_{10,50} J_{25,50} J_{50}^6}{J_{5,50}^2 J_{15,50} J_{20,50}^3} + \frac{J_{20,50} J_{25,50} J_{50}^6}{J_{5,50} J_{10,50}^3 J_{15,50}^2} \\
&\quad = \frac{J_{50}^6}{J_{5,50}^2 J_{10,50}^2} - \frac{q^{10} J_{50}^6}{J_{15,50}^2 J_{20,50}^2} \\
&\quad = \frac{J_{25,50}^2 J_{50}^6}{J_{5,50}^2 J_{10,50} J_{15,50}^2 J_{20,50}},
\end{aligned} \tag{3.17}$$

where the second equality follows from (2.5). Adding (3.17) to (3.14) and simplifying, we find that (3.14) is equivalent to

$$\frac{J_{5,50}^2 J_{10,50} J_{15,50}^2 J_{20,50} J_{25,50}^2}{J_{50}^6} = \frac{J_{25,50}^2 J_{50}^6}{J_{5,50}^2 J_{10,50} J_{15,50}^2 J_{20,50}} - \frac{4q^5 J_{50}^6}{J_{5,50} J_{10,50} J_{15,50} J_{20,50}}. \tag{3.18}$$

Dividing both sides by $J_{25,50}^2$ and noting that

$$\begin{aligned}
\frac{J_{5,50}^2 J_{10,50} J_{15,50}^2 J_{20,50}}{J_{50}^6} &= \frac{J_{5,25} J_{10,25}}{\overline{J}_{5,25} \overline{J}_{10,25}}, \\
\frac{J_{50}^6}{J_{5,50}^2 J_{10,50} J_{15,50}^2 J_{20,50}} &= \frac{\overline{J}_{5,25} \overline{J}_{10,25}}{J_{5,25} J_{10,25}}
\end{aligned}$$

and

$$\frac{4q^5 J_{50}^6}{J_{5,50}J_{10,50}J_{15,50}J_{20,50}J_{25,50}^2}=\frac{4q^5 \overline{J}_{25}^4 J_{25}^2}{J_{5,25}J_{10,25}},$$

we find that (3.18) is equivalent to

$$\frac{J_{5,25}J_{10,25}}{\overline{J}_{5,25}\overline{J}_{10,25}}=\frac{\overline{J}_{5,25}\overline{J}_{10,25}}{J_{5,25}J_{10,25}}-\frac{4q^5 \overline{J}_{25}^4 J_{25}^2}{J_{5,25}J_{10,25}}.$$

This is precisely (2.7), completing the proof of (3.9).

To prove (3.10), it suffices to show that

$$0=\frac{J_{25,50}J_{50}^6}{J_{5,50}^2J_{10,50}J_{15,50}J_{20,50}}-\frac{J_{20,50}^2J_{25,50}J_{50}^6}{J_{10,50}^4J_{15,50}^3}-\frac{2q^5 J_{50}^6}{J_{10,50}^2J_{15,50}^2},$$

which follows from multiplying by $\frac{J_{20,50}J_{25,50}J_{50}^6}{J_{10,50}^2J_{15,50}}$ on both sides of (2.10).

To prove (3.11), it suffices to show that

$$\begin{aligned}&-\frac{J_{5,50}^3J_{15,50}^2J_{20,50}^2J_{25,50}}{J_{50}^6}\\&\quad=\frac{J_{10,50}J_{25,50}J_{50}^6}{J_{5,50}^3J_{20,50}^3}-\frac{J_{25,50}J_{50}^6}{J_{5,50}J_{10,50}^2J_{15,50}^2}-\frac{J_{5,50}J_{20,50}^3J_{25,50}J_{50}^6}{J_{10,50}^5J_{15,50}^4}.\end{aligned}\tag{3.19}$$

Multiplying by $\frac{J_{25,50}J_{50}^6}{J_{5,50}J_{20,50}}$ on both sides of (2.10), we have

$$\frac{J_{10,50}J_{25,50}J_{50}^6}{J_{5,50}^3J_{20,50}^3}-\frac{J_{25,50}J_{50}^6}{J_{5,50}J_{10,50}^2J_{15,50}^2}=\frac{2q^5 J_{50}^6}{J_{5,50}J_{15,50}J_{20,50}^2}.$$

Substituting the above identity into (3.19), we find that (3.19) is equivalent to

$$-\frac{J_{5,50}^3J_{15,50}^2J_{20,50}^2J_{25,50}}{J_{50}^6}=\frac{2q^5 J_{50}^6}{J_{5,50}J_{15,50}J_{20,50}^2}-\frac{J_{5,50}J_{20,50}^3J_{25,50}J_{50}^6}{J_{10,50}^5J_{15,50}^4}.\tag{3.20}$$

Multiplying by $\frac{J_{10,50}}{J_{5,50}J_{25,50}J_{50}}$ and noting that

$$\begin{aligned}\frac{J_{5,50}^2J_{10,50}J_{15,50}^2J_{20,50}^2}{J_{50}^7}&=\frac{J_{5,25}J_{10,25}^2}{\overline{J}_{5,25}J_{25}^2},\\ \frac{2q^5 J_{10,50}J_{50}^5}{J_{5,50}^2J_{15,50}J_{20,50}^2J_{25,50}}&=\frac{2q^5 \overline{J}_{5,25}^2\overline{J}_{25}^2}{J_{10,25}J_{25}}\end{aligned}$$

and

$$\frac{J_{20,50}^3 J_{50}^5}{J_{10,50}^4 J_{15,50}^4} = \frac{\overline{J}_{10,25}^3}{J_{10,25} J_{25}^2},$$

we find that (3.20) is equivalent to

$$-\frac{J_{5,25} J_{10,25}^2}{\overline{J}_{5,25} J_{25}^2} = \frac{2q^5 \overline{J}_{5,25}^2 \overline{J}_{25}^2}{J_{10,25} J_{25}} - \frac{\overline{J}_{10,25}^3}{J_{10,25} J_{25}^2}.$$

This follows from multiplying by $\frac{J_{5,25}\overline{J}_{10,25}}{J_{25}^2}$ on both sides of (2.8). So the proof of (3.11) is completed.

To prove (3.12), it suffices to show that

$$\begin{aligned} &-\frac{J_{5,50}^2 J_{10,50}^2 J_{15,50}^3 J_{25,50}}{J_{50}^6} \\ &\quad = \frac{J_{10,50}^3 J_{15,50} J_{25,50} J_{50}^6}{J_{5,50}^4 J_{20,50}^5} - \frac{J_{25,50} J_{50}^6}{J_{5,50}^2 J_{15,50} J_{20,50}^2} - \frac{J_{20,50} J_{25,50} J_{50}^6}{J_{10,50}^3 J_{15,50}^3}. \end{aligned} \tag{3.21}$$

Multiplying by $\frac{J_{10,50}^2 J_{15,50} J_{25,50} J_{50}^6}{J_{5,50}^2 J_{20,50}^3}$ on both sides of (2.10), we find that

$$\frac{J_{10,50}^3 J_{15,50} J_{25,50} J_{50}^6}{J_{5,50}^4 J_{20,50}^5} - \frac{J_{25,50} J_{50}^6}{J_{5,50}^2 J_{15,50} J_{20,50}^2} = \frac{2q^5 J_{10,50}^2 J_{50}^6}{J_{5,50}^2 J_{20,50}^4}.$$

Substituting the above identity into (3.21), we find that (3.21) is equivalent to

$$-\frac{J_{5,50}^2 J_{10,50}^2 J_{15,50}^3 J_{25,50}}{J_{50}^6} = \frac{2q^5 J_{10,50}^2 J_{50}^6}{J_{5,50}^2 J_{20,50}^4} - \frac{J_{20,50} J_{25,50} J_{50}^6}{J_{10,50}^3 J_{15,50}^3}. \tag{3.22}$$

This follows from multiplying by $\frac{J_{10,50}^2 J_{15,50}}{J_{5,50} J_{20,50}^2}$ on both sides of (3.20). This completes the proof of (3.12).

To prove (3.13), it suffices to show that

$$0 = \frac{2J_{50}^6}{J_{5,50}^2 J_{20,50}^2} - \frac{J_{10,50}^2 J_{25,50} J_{50}^6}{J_{5,50}^3 J_{20,50}^4} - \frac{J_{25,50} J_{50}^6}{J_{5,50} J_{10,50} J_{15,50}^2 J_{20,50}},$$

which follows from multiplying by $\frac{J_{10,50} J_{25,50} J_{50}^6}{J_{5,50} J_{20,50}^2}$ on both sides of (2.9). $\square$

Lemma 3.3. *Let*

$$B_0 := \frac{2q^5 J_{50}^5}{(q^{25};q^{50})_\infty J_{10,50}^2 J_{15,50}^2},$$

$$B_1 := \frac{2q^6 J_{50}^5}{(q^{25};q^{50})_\infty J_{5,50} J_{15,50} J_{20,50}^2},$$

$$B_2 := \frac{q^2 (q^{25};q^{50})_\infty J_{50}^6 J_{20,50}}{J_{10,50}^3 J_{15,50}^3},$$

$$B_3 := \frac{q^3 (q^{25};q^{50})_\infty J_{50}^6}{J_{5,50} J_{10,50} J_{15,50}^2 J_{20,50}},$$

$$B_4 := \frac{(q^{25};q^{50})_\infty J_{50}^6 J_{20,50}^2 J_{25,50}}{2q J_{10,50}^4 J_{15,50}^4}.$$

We have

$$\begin{aligned} &\frac{P(q^{10},-q^5)}{q^3} - P\big(q^5,-q^5\big) + \frac{P(q^5,-1)}{q^6} - \frac{P(q^{10},-1)}{q^9} \\ &\quad = (q;q)_\infty \times \{B_0 + B_1 + B_2 + B_3 + B_4\}. \end{aligned} \tag{3.23}$$

Proof. Expanding the right side of (3.23) and comparing both sides according to the powers of q modulo 5, we find that it suffices to prove the five identities,

$$\begin{aligned} -P\big(q^5,-q^5\big) = {}& \frac{J_{10,25}J_{25}}{J_{5,25}} \times \frac{2q^5 J_{50}^5}{(q^{25};q^{50})_\infty J_{10,50}^2 J_{15,50}^2} - qJ_{25} \times \frac{(q^{25};q^{50})_\infty J_{50}^6 J_{20,50}^2 J_{25,50}}{2q J_{10,50}^4 J_{15,50}^4} \\ & - \frac{q^2 J_{5,25}J_{25}}{J_{10,25}} \times \frac{q^3 (q^{25};q^{50})_\infty J_{50}^6}{J_{5,50} J_{10,50} J_{15,50}^2 J_{20,50}}, \end{aligned} \tag{3.24}$$

$$\begin{aligned} -\frac{P(q^{10},-1)}{q^9} = {}& \frac{J_{10,25}J_{25}}{J_{5,25}} \times \frac{2q^6 J_{50}^5}{(q^{25};q^{50})_\infty J_{5,50} J_{15,50} J_{20,50}^2} - qJ_{25} \times \frac{2q^5 J_{50}^5}{(q^{25};q^{50})_\infty J_{10,50}^2 J_{15,50}^2} \\ & - \frac{q^2 J_{5,25}J_{25}}{J_{10,25}} \times \frac{(q^{25};q^{50})_\infty J_{50}^6 J_{20,50}^2 J_{25,50}}{2q J_{10,50}^4 J_{15,50}^4}, \end{aligned} \tag{3.25}$$

$$\begin{aligned} \frac{P(q^{10},-q^5)}{q^3} = {}& \frac{J_{10,25}J_{25}}{J_{5,25}} \times \frac{q^2 (q^{25};q^{50})_\infty J_{50}^6 J_{20,50}}{J_{10,50}^3 J_{15,50}^3} - qJ_{25} \times \frac{2q^6 J_{50}^5}{(q^{25};q^{50})_\infty J_{5,50} J_{15,50} J_{20,50}^2} \\ & - \frac{q^2 J_{5,25}J_{25}}{J_{10,25}} \times \frac{2q^5 J_{50}^5}{(q^{25};q^{50})_\infty J_{10,50}^2 J_{15,50}^2}, \end{aligned} \tag{3.26}$$

$$\begin{aligned} 0 = {}& \frac{J_{10,25}J_{25}}{J_{5,25}} \times \frac{q^3 (q^{25};q^{50})_\infty J_{50}^6}{J_{5,50} J_{10,50} J_{15,50}^2 J_{20,50}} - qJ_{25} \times \frac{q^2 (q^{25};q^{50})_\infty J_{50}^6 J_{20,50}}{J_{10,50}^3 J_{15,50}^3} \\ & - \frac{q^2 J_{5,25}J_{25}}{J_{10,25}} \times \frac{2q^6 J_{50}^5}{(q^{25};q^{50})_\infty J_{5,50} J_{15,50} J_{20,50}^2}, \end{aligned} \tag{3.27}$$

$$\begin{aligned}\frac{P(q^5,-1)}{q^6} &= \frac{J_{10,25}J_{25}}{J_{5,25}} \times \frac{(q^{25};q^{50})_\infty J_{50}^6 J_{20,50}^2 J_{25,50}}{2qJ_{10,50}^4 J_{15,50}^4} - qJ_{25} \times \frac{q^3(q^{25};q^{50})_\infty J_{50}^6}{J_{5,50}J_{10,50}J_{15,50}^2 J_{20,50}} \\ &\quad - \frac{q^2 J_{5,25}J_{25}}{J_{10,25}} \times \frac{q^2(q^{25};q^{50})_\infty J_{50}^6 J_{20,50}}{J_{10,50}^3 J_{15,50}^3}. \end{aligned} \tag{3.28}$$

Simplifying each of these five identities, we see that to prove (3.24), it suffices to show that

$$\begin{aligned}&-\frac{J_{5,50}^2 J_{10,50} J_{15,50}^2 J_{20,50} J_{25,50}^2}{2J_{50}^6} \\ &\quad = \frac{2q^5 J_{50}^6}{J_{5,50}J_{10,50}J_{15,50}J_{20,50}} - \frac{J_{50}^6 J_{20,50}^2 J_{25,50}^2}{2J_{10,50}^4 J_{15,50}^4} - \frac{q^5 J_{50}^6 J_{25,50}}{J_{10,50}^2 J_{15,50}^3}. \end{aligned} \tag{3.29}$$

Multiplying by $\frac{q^5 J_{25,50} J_{50}^6}{J_{15,50} J_{20,50}}$ on both sides of (2.9) and rearranging, we find that

$$\frac{2q^5 J_{50}^6}{J_{5,50}J_{10,50}J_{15,50}J_{20,50}} - \frac{q^5 J_{50}^6 J_{25,50}}{J_{10,50}^2 J_{15,50}^3} = \frac{q^5 J_{50}^6 J_{10,50} J_{25,50}}{J_{5,50}^2 J_{15,50} J_{20,50}^3}.$$

Substituting the above equality into (3.29), we find that (3.29) is equivalent to

$$-\frac{J_{5,50}^2 J_{10,50} J_{15,50}^2 J_{20,50} J_{25,50}^2}{2J_{50}^6} = \frac{q^5 J_{50}^6 J_{10,50} J_{25,50}}{J_{5,50}^2 J_{15,50} J_{20,50}^3} - \frac{J_{50}^6 J_{20,50}^2 J_{25,50}^2}{2J_{10,50}^4 J_{15,50}^4}. \tag{3.30}$$

This follows from multiplying by $\frac{J_{10,50}J_{25,50}}{2J_{5,50}J_{20,50}}$ on both sides of (3.20). This completes the proof of (3.24).

To prove (3.25), it suffices to show that

$$-\frac{J_{5,50} J_{10,50}^3 J_{15,50}^3 J_{25,50}^2}{2J_{50}^6 J_{20,50}} = \frac{2q^5 J_{10,50} J_{50}^6}{J_{5,50}^2 J_{20,50}^3} - \frac{2q^5 J_{50}^6}{J_{10,50}^2 J_{15,50}^2} - \frac{J_{5,50} J_{20,50}^3 J_{25,50}^2 J_{50}^6}{2J_{10,50}^5 J_{15,50}^5}. \tag{3.31}$$

Multiplying by $\frac{J_{5,50} J_{20,50} J_{25,50}^2 J_{50}^6}{2J_{10,50} J_{15,50}}$ on both sides of (2.11) and rearranging, we find that

$$-\frac{2q^5 J_{50}^6}{J_{10,50}^2 J_{15,50}^2} - \frac{J_{5,50} J_{20,50}^3 J_{25,50}^2 J_{50}^6}{2J_{10,50}^5 J_{15,50}^5} = -\frac{J_{10,50} J_{25,50}^2 J_{50}^6}{2J_{5,50}^3 J_{15,50} J_{20,50}^3}.$$

Substituting the above equality into (3.31), we find that (3.31) is equivalent to

$$-\frac{J_{5,50} J_{10,50}^3 J_{15,50}^3 J_{25,50}^2}{2J_{50}^6 J_{20,50}} = \frac{2q^5 J_{10,50} J_{50}^6}{J_{5,50}^2 J_{20,50}^3} - \frac{J_{10,50} J_{25,50}^2 J_{50}^6}{2J_{5,50}^3 J_{15,50} J_{20,50}^3}.$$

This follows from multiplying by $-\frac{J_{10,50}^2 J_{15,50}}{2J_{5,50}J_{20,50}^2}$ on both sides of (3.18). This completes the proof of (3.25).

To prove (3.26), it suffices to show that

$$\frac{J_{5,50}^3 J_{15,50}^2 J_{20,50}^2 J_{25,50}}{J_{50}^6} = \frac{J_{25,50} J_{50}^6}{J_{5,50} J_{10,50}^2 J_{15,50}^2} - \frac{2q^5 J_{50}^6}{J_{5,50} J_{15,50} J_{20,50}^2} - \frac{2q^5 J_{5,50} J_{20,50} J_{50}^6}{J_{10,50}^3 J_{15,50}^3}. \tag{3.32}$$

Multiplying by $\frac{J_{5,50} J_{20,50}^2 J_{25,50} J_{50}^6}{J_{10,50}^3 J_{15,50}^2}$ on both sides of (2.10) and rearranging, we find that

$$-\frac{2q^5 J_{5,50} J_{20,50} J_{50}^6}{J_{10,50}^3 J_{15,50}^3} + \frac{J_{25,50} J_{50}^6}{J_{5,50} J_{10,50}^2 J_{15,50}^2} = \frac{J_{5,50} J_{20,50}^3 J_{25,50} J_{50}^6}{J_{10,50}^5 J_{15,50}^4}.$$

Substituting the above equality into (3.32), we find that (3.32) is equivalent to

$$\frac{J_{5,50}^3 J_{15,50}^2 J_{20,50}^2 J_{25,50}}{J_{50}^6} = \frac{J_{5,50} J_{20,50}^3 J_{25,50} J_{50}^6}{J_{10,50}^5 J_{15,50}^4} - \frac{2q^5 J_{50}^6}{J_{5,50} J_{15,50} J_{20,50}^2}. \tag{3.33}$$

This follows from (3.20). This completes the proof of (3.26).

To prove (3.27), it suffices to show that

$$0 = \frac{J_{25,50} J_{50}^6}{J_{5,50}^2 J_{15,50} J_{20,50}^2} - \frac{J_{20,50} J_{25,50} J_{50}^6}{J_{10,50}^3 J_{15,50}^3} - \frac{2q^5 J_{50}^6}{J_{10,50} J_{15,50}^2 J_{20,50}},$$

which follows from multiplying by $\frac{J_{25,50} J_{50}^6}{J_{10,50} J_{15,50}}$ on both sides of (2.10).

To prove (3.28), it suffices to show that

$$\frac{J_{5,50}^3 J_{15,50} J_{20,50}^3 J_{25,50}^2}{2 J_{10,50} J_{50}^6} = \frac{J_{20,50} J_{25,50}^2 J_{50}^6}{2 J_{5,50} J_{10,50}^3 J_{15,50}^3} - \frac{q^5 J_{25,50} J_{50}^6}{J_{5,50} J_{10,50} J_{15,50}^2 J_{20,50}} - \frac{q^5 J_{5,50} J_{20,50}^2 J_{25,50} J_{50}^6}{J_{10,50}^4 J_{15,50}^4}. \tag{3.34}$$

Multiplying by $\frac{J_{5,50} J_{20,50}^3 J_{25,50}^2 J_{50}^6}{2 J_{10,50}^4 J_{15,50}^3}$ on both sides of (2.10) and rearranging, we find that

$$\frac{J_{20,50} J_{25,50}^2 J_{50}^6}{2 J_{5,50} J_{10,50}^3 J_{15,50}^3} - \frac{q^5 J_{5,50} J_{20,50}^2 J_{25,50} J_{50}^6}{J_{10,50}^4 J_{15,50}^4} = \frac{J_{5,50} J_{20,50}^4 J_{25,50}^2 J_{50}^6}{2 J_{10,50}^6 J_{15,50}^5}.$$

Substituting the above equality into (3.34), we find that (3.34) is equivalent to

$$\frac{J_{5,50}^3 J_{15,50} J_{20,50}^3 J_{25,50}^2}{2 J_{10,50} J_{50}^6} = \frac{J_{5,50} J_{20,50}^4 J_{25,50}^2 J_{50}^6}{2 J_{10,50}^6 J_{15,50}^5} - \frac{q^5 J_{25,50} J_{50}^6}{J_{5,50} J_{10,50} J_{15,50}^2 J_{20,50}}. \tag{3.35}$$

This follows from multiplying by $\frac{J_{20,50} J_{25,50}}{2 J_{10,50} J_{15,50}}$ on both sides of (3.20). $\square$

Now we prove Theorem 1.1.

Proof. Replacing z by $\xi = e^{\frac{\pi i}{5}}$ in (1.4), we have

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(-\xi q, q/\xi; q)_n} = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N(m,n)\xi^m q^n = \sum_{n=0}^{\infty} \sum_{t=0}^{9} \sum_{m=-\infty}^{\infty} N(10m+t,n)\xi^t q^n.$$

By the definition of $N(s,l,n)$, we know that

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(-\xi q, q/\xi; q)_n} = \sum_{n=0}^{\infty} \sum_{t=0}^{9} N(t,10,n)\xi^t q^n = \sum_{t=0}^{9} \sum_{n=0}^{\infty} N(t,10,n)\xi^t q^n.$$

Expanding the last series in the above equation according to the summation index t, noting that $N(t,10,n) = N(10-t,10,n)$, $\xi + \xi^3 + \xi^7 + \xi^9 = 1$ and $\xi^5 = -1$, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^{n^2}}{(-\xi q, q/\xi; q)_n} \\
&\quad = \sum_{n\geqslant 0} \big(N(0,10,n) + N(1,10,n) - N(4,10,n) - N(5,10,n)\big) q^n \\
&\qquad + \big(\xi^2 - \xi^3\big) \sum_{n\geqslant 0} \big(N(1,10,n) + N(2,10,n) - N(3,10,n) - N(4,10,n)\big) q^n. \qquad (3.36)
\end{aligned}$$

Next, by [7, Eq. (7.6)], we know that

$$(q;q)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq, q/z; q)_n} = 1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(3n+1)/2}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)}. \qquad (3.37)$$

Replacing the summation index n by $-n$, we find that

$$\sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(3n+1)/2+n}}{(1-zq^n)(1-z^{-1}q^n)} = \sum_{n=-\infty}^{-1} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(3n+1)/2}}{(1-zq^n)(1-z^{-1}q^n)}.$$

Substituting the above equality into (3.37), we get

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(zq, q/z; q)_n} = \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(3n+1)/2}}{(1-zq^n)(1-z^{-1}q^n)}.$$

Setting $z = \xi$ in the above equation and simplifying, we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{q^{n^2}}{(\xi q, q/\xi; q)_n} \\
&\quad = \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(1-\xi)(1-1/\xi)(-1)^n q^{n(3n+1)/2}}{(1-\xi q^n)(1-1/\xi q^n)} \\
&\quad = \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(1+\xi^6)(1+\xi^4)(-1)^n q^{n(3n+1)/2}}{(1+\xi^6 q^n)(1+\xi^4 q^n)} \\
&\quad = \frac{(1+\xi^6)(1+\xi^4)}{(q;q)_\infty} \times \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1+q^n)(1+\xi^2 q^n)(1+\xi^8 q^n)}{(1+\xi^6 q^n)(1+\xi^4 q^n)(1+q^n)(1+\xi^2 q^n)(1+\xi^8 q^n)} \\
&\quad = \frac{(\xi^2+\xi^8)}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(q^n+q^{2n}-1-q^{3n})}{1+q^{5n}} \\
&\qquad + \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1+q^{3n})}{1+q^{5n}}. \qquad (3.38)
\end{aligned}$$

Replacing n with $-n$ in the summation index, we find that

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2+n}}{1+q^{5n}} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2+3n}}{1+q^{5n}},$$

and hence the first sum on the right side of the last equality in (3.38) becomes

$$\frac{(\xi^2-\xi^3)}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(q^{2n}-1)}{1+q^{5n}}.$$

Substituting this into (3.38), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(\xi q, q/\xi; q)_n} &= \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1+q^n)}{1+q^{5n}} \\
&\quad + \frac{(\xi^2-\xi^3)}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(q^{2n}-1)}{1+q^{5n}},
\end{aligned}$$

where

$$F_1(q) := \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(1+q^n)}{1+q^{5n}} \qquad (3.39)$$

and

$$F_2(q) := \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}(q^{2n}-1)}{1+q^{5n}}. \qquad (3.40)$$

By (3.39), (3.40) and (3.36), we have

$$\begin{aligned}
&\sum_{n\geqslant 0}\bigl(N(0,10,n)+N(1,10,n)-N(4,10,n)-N(5,10,n)\bigr)q^n+\bigl(\xi^2-\xi^3\bigr)\\
&\qquad\times\sum_{n\geqslant 0}\bigl(N(1,10,n)+N(2,10,n)-N(3,10,n)-N(4,10,n)\bigr)q^n\\
&\quad=F_1(q)+\bigl(\xi^2-\xi^3\bigr)F_2(q).
\end{aligned}\tag{3.41}$$

Since the coefficients of $F_1(q)$ and $F_2(q)$ are all integers and $[\mathbb{Q}(\xi):\mathbb{Q}]=4$, we equate the coefficient of ξ^k on both sides of (3.41) and find that

$$\sum_{n=0}^{\infty}\bigl(N(0,10,n)+N(1,10,n)-N(4,10,n)-N(5,10,n)\bigr)q^n=F_1(q)$$

and

$$\sum_{n=0}^{\infty}\bigl(N(1,10,n)+N(2,10,n)-N(3,10,n)-N(4,10,n)\bigr)q^n=F_2(q).$$

Substituting (3.1) and (3.2) into the series on the right side of (3.39), we find that

$$\begin{aligned}
F_1(q)&=\frac{1}{(q;q)_\infty}\left\{P\bigl(q^5,-q^5\bigr)-\frac{P(q^{10},-q^5)}{q^3}+\frac{(q;q)_\infty}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{75n(n+1)/2+5}}{1+q^{25n+5}}\right\}\\
&\quad+\frac{1}{(q;q)_\infty}\left\{P\bigl(q^{10},-q^{10}\bigr)-q^3P\bigl(q^5,-q^{10}\bigr)-\frac{(q;q)_\infty}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{75n(n+1)/2+8}}{1+q^{25n+10}}\right\}\\
&=\frac{1}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{75n(n+1)/2+5}}{1+q^{25n+5}}-\frac{1}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{75n(n+1)/2+8}}{1+q^{25n+10}}\\
&\quad+\frac{1}{(q;q)_\infty}\left\{P\bigl(q^5,-q^5\bigr)-\frac{P(q^{10},-q^5)}{q^3}+P\bigl(q^{10},-q^{10}\bigr)-q^3P\bigl(q^5,-q^{10}\bigr)\right\}.
\end{aligned}$$

This completes our proof of (1.14) after invoking Lemma 3.2. Similarly, substituting (3.1) and (3.3) into the series on the right side of (3.40), we find that

$$\begin{aligned}
F_2(q)&=\frac{1}{(q;q)_\infty}\left\{\frac{P(q^5,-1)}{q^6}-\frac{P(q^{10},-1)}{q^9}-\frac{(q;q)_\infty}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(75n^2+25n)/2-1}}{1+q^{25n}}\right\}\\
&\quad-\frac{1}{(q;q)_\infty}\left\{P\bigl(q^5,-q^5\bigr)-\frac{P(q^{10},-q^5)}{q^3}+\frac{(q;q)_\infty}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{75n(n+1)/2+5}}{1+q^{25n+5}}\right\}\\
&=-\frac{1}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{75n(n+1)/2+5}}{1+q^{25n+5}}-\frac{1}{J_{25}}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(75n^2+25n)/2-1}}{1+q^{25n}}
\end{aligned}$$

$$+\frac{1}{(q;q)_\infty}\left\{\frac{P(q^5,-1)}{q^6}-\frac{P(q^{10},-1)}{q^9}-P\left(q^5,-q^5\right)+\frac{P(q^{10},-q^5)}{q^3}\right\}.$$

This completes our proof of (1.15) after invoking Lemma 3.3. □

4. Proof of Corollary 1.2

Proof Corollary 1.2. First, we prove (1.16). By (1.14), we find that when $i=1$, we have

$$\sum_{n=0}^{\infty}\big(N(0,10,5n+1)+N(1,10,5n+1)-N(4,10,5n+1)-N(5,10,5n+1)\big)q^n$$
$$=\frac{J_5J_{10}^5}{J_{1,10}J_{2,10}^2J_{3,10}^2}=\frac{(q^5,q^{10};q^{10})_\infty}{[q;q^{10}]_\infty[q^2,q^3;q^{10}]_\infty^2}=\frac{(q^5;q^5)_\infty}{[q^2;q^5]_\infty^2[q;q^{10}]_\infty}=\frac{L_{5,2}(q)}{[q^2;q^5]_\infty[q;q^{10}]_\infty}.$$

It is clear that a product of the type $\frac{1}{1-q^m}$ has non-negative coefficients and the term $\frac{1}{1-q}$ appearing on the right side of the last equality has positive coefficients. Since, by Lemma 2.4, $L_{5,2}(q)$ (whose constant term is 1) has non-negative nth coefficient for all $n\geqslant 1$, our inequality follows.

When $i=2$, we have

$$\sum_{n=0}^{\infty}\big(N(0,10,5n+2)+N(1,10,5n+2)-N(4,10,5n+2)-N(5,10,5n+2)\big)q^n$$
$$=\frac{J_5J_{10}^5}{J_{1,10}^2J_{3,10}J_{4,10}^2}=\frac{(q^5,q^{10};q^{10})_\infty}{[q,q^4;q^{10}]_\infty^2[q^3;q^{10}]_\infty}=\frac{(q^5;q^5)_\infty}{[q;q^5]_\infty^2[q^3;q^{10}]_\infty}=\frac{L_{5,1}(q)}{[q;q^5]_\infty[q^3;q^{10}]_\infty}.$$

Since the term $\frac{1}{1-q}$ appearing on the right side of the last equality has positive coefficients, and by Lemma 2.4, $L_{5,1}(q)$ (whose constant term is 1) has non-negative nth coefficient for all $n\geqslant 1$, our inequality follows.

When $i=3$, we have

$$\sum_{n=0}^{\infty}\big(N(0,10,5n+3)+N(1,10,5n+3)-N(4,10,5n+3)-N(5,10,5n+3)\big)q^n$$
$$=\frac{J_5J_{2,10}^2J_{10}^5}{J_{1,10}^3J_{4,10}^4}-\frac{1}{J_5}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(15n^2+15n)/2+1}}{1+q^{5n+2}}$$
$$=\frac{[-q;q^5]_\infty}{[q^2;q^5]_\infty J_5}\sum_{n=-\infty}^{\infty}\frac{q^{(15n^2+9n)/2}}{1-q^{5n+1}}-\frac{[-q;q^5]_\infty}{[q^2;q^5]_\infty J_5}\sum_{n=-\infty}^{\infty}\frac{q^{(15n^2+21n)/2+3}}{1-q^{5n+3}},\tag{4.1}$$

where the second equality follows from replacing b_1, b_2, b_3 and q by $-q^2$, q^1, q^3 and q^5, respectively, in (2.13). We see that to prove our inequality, it suffices to show that the first sum on the right side of the second equality in (4.1) has positive coefficients and the second sum on the right side of the second equality in (4.1) has non-negative coefficients. We examine the first sum and find that

$$
\begin{aligned}
&\frac{[-q;q^5]_\infty}{[q^2;q^5]_\infty J_5}\sum_{n=-\infty}^{\infty}\frac{q^{(15n^2+9n)/2}}{1-q^{5n+1}}\\
&\quad=\frac{1}{[q;q^5]_\infty[q^3;q^{10}]_\infty J_5}\left(\sum_{n=0}^{\infty}\frac{q^{(15n^2+9n)/2}}{1-q^{5n+1}}-\sum_{n=0}^{\infty}\frac{q^{(15n^2+31n)/2+7}}{1-q^{5n+4}}\right)\\
&\quad=\frac{1}{[q;q^5]_\infty[q^3;q^{10}]_\infty J_5}\sum_{n=0}^{\infty}\left(\frac{q^{(15n^2+9n)/2}}{1-q^{5n+1}}-\frac{q^{(15n^2+31n)/2+7}}{1-q^{5n+4}}\right)\\
&\quad=\frac{1}{[q;q^5]_\infty[q^3;q^{10}]_\infty J_5}\sum_{n=0}^{\infty}q^{(15n^2+9n)/2}\left(\frac{1}{1-q^{5n+1}}-\frac{q^{11n+7}}{1-q^{5n+4}}\right)\\
&\quad=\frac{1}{[q;q^5]_\infty[q^3;q^{10}]_\infty J_5}\sum_{n=0}^{\infty}\frac{q^{(15n^2+9n)/2}}{(1-q^{5n+1})(1-q^{5n+4})}\left\{1-q^{5n+4}-q^{11n+7}\left(1-q^{5n+1}\right)\right\}\\
&\quad=\frac{1}{[q;q^5]_\infty[q^3;q^{10}]_\infty J_5}\sum_{n=0}^{\infty}\frac{q^{(15n^2+9n)/2}}{(1-q^{5n+1})(1-q^{5n+4})}\\
&\qquad\times\left\{\left(1-q^{5n+4}\right)\left(1-q^{11n+7}\right)+q^{16n+8}\left(1-q^3\right)\right\}\\
&\quad=\frac{1}{(q^6,q^4;q^5)_\infty[q^3;q^{10}]_\infty J_5}\sum_{n=0}^{\infty}\frac{q^{(15n^2+9n)/2}}{1-q^{5n+1}}\\
&\qquad\times\left\{\frac{(1-q^{5n+4})(1-q^{11n+7})+q^{16n+8}(1-q^3)}{(1-q)(1-q^{5n+4})}\right\}\\
&\quad=\frac{1}{(q^6,q^4;q^5)_\infty[q^3;q^{10}]_\infty J_5}\sum_{n=0}^{\infty}\frac{q^{(15n^2+9n)/2}}{1-q^{5n+1}}\left\{\sum_{k=0}^{11l+6}q^k+\frac{q^{16n+8}(1+q+q^2)}{1-q^{5n+4}}\right\}.
\end{aligned}
$$

Since the term $\frac{1}{1-q}$ appearing in sum of the last equality has positive coefficients, we see that $\frac{[-q;q^5]_\infty}{[q^2;q^5]_\infty J_5}\sum_{n=-\infty}^{\infty}\frac{q^{15n^2/2+9n/2}}{1-q^{5n+1}}$ has positive coefficients of q^n for all $n \geqslant 0$. Similarly, since we have

$$
\begin{aligned}
&-\frac{[-q;q^5]_\infty}{[q^2;q^5]_\infty J_5}\sum_{n=-\infty}^{\infty}\frac{q^{(15n^2+21n)/2+3}}{1-q^{5n+3}}\\
&\quad=\frac{[-q;q^5]_\infty}{[q^2;q^5]_\infty J_5}\sum_{n=-\infty}^{\infty}\frac{q^{(15n^2+19n)/2+2}}{1-q^{5n+2}}\\
&\quad=\frac{1}{[q;q^5]_\infty[q^3;q^{10}]_\infty J_5}\sum_{n=0}^{\infty}\left(\frac{q^{(15n^2+19n)/2+2}}{1-q^{5n+2}}-\frac{q^{15n^2/2+21n/2+3}}{1-q^{5n+3}}\right)\\
&\quad=\frac{1}{[q;q^5]_\infty[q^3;q^{10}]_\infty J_5}\sum_{n=0}^{\infty}q^{(15n^2+19n)/2+2}\left(\frac{1}{1-q^{5n+2}}-\frac{q^{n+1}}{1-q^{5n+3}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{[q;q^5]_\infty[q^3;q^{10}]_\infty J_5}\sum_{n=0}^{\infty}\frac{q^{(15n^2+19n)/2+2}}{(1-q^{5n+2})(1-q^{5n+3})}\left\{1-q^{5n+3}-q^{n+1}\left(1-q^{5n+2}\right)\right\}\\
&= \frac{1}{[q;q^5]_\infty[q^3;q^{10}]_\infty J_5}\sum_{n=0}^{\infty}\frac{q^{(15n^2+19n)/2+2}}{(1-q^{5n+3})(1-q^{5n+2})}\\
&\quad\times\left\{\left(1-q^{5n+3}\right)\left(1-q^{n+1}\right)+q^{6n+3}(1-q)\right\}\\
&= \frac{1}{[q;q^5]_\infty[q^3;q^{10}]_\infty J_5}\sum_{n=0}^{\infty}\frac{q^{(15n^2+19n)/2+2}}{1-q^{5n+2}}\left\{\left(1-q^{n+1}\right)+\frac{q^{6n+3}(1-q)}{1-q^{5n+3}}\right\}\\
&= \frac{1}{(q^4,q^6;q^5)_\infty[q^3;q^{10}]_\infty J_5}\sum_{n=0}^{\infty}\frac{q^{(15n^2+19n)/2+2}}{1-q^{5n+2}}\left\{\sum_{k=0}^{n}q^k+\frac{q^{6n+3}}{1-q^{5n+3}}\right\},
\end{aligned}
$$

it is easy to see that $-\frac{[-q;q^5]_\infty}{[q^2;q^5]_\infty J_5}\sum_{n=-\infty}^{\infty}\frac{q^{15n^2/2+21n/2+3}}{1-q^{5n+3}}$ has non-negative coefficients. This completes the proof of the case $i=3$.

When $i=4$, we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\big(N(0,10,5n+4)+N(1,10,5n+4)-N(4,10,5n+4)-N(5,10,5n+4)\big)q^n\\
&\quad= \frac{2J_{10}^6}{J_5J_{1,10}J_{2,10}J_{3,10}J_{4,10}}=\frac{2(q^{10};q^{10})_\infty}{(q^5;q^{10})_\infty[q,q^2,q^3,q^4;q^{10}]_\infty}=\frac{2L_{10,3}(q)}{(q^5;q^{10})_\infty[q,q^2,q^4;q^{10}]_\infty}.
\end{aligned}
$$

Since the term $\frac{1}{1-q}$ appearing on the right side of the last equality has positive coefficients, and by Lemma 2.4, we know that $L_{10,3}(q)$ (whose constant term is 1) has non-negative coefficients of q^n for all $n \geqslant 1$, our inequality follows. This completes the proof of (1.16).

Next we prove (1.17). By (1.15), we find that when $j=1$, we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\big(N(1,10,5n+1)+N(2,10,5n+1)-N(3,10,5n+1)-N(4,10,5n+1)\big)q^n\\
&\quad= \frac{2qJ_{10}^6}{J_5J_{1,10}J_{3,10}J_{4,10}^2}=\frac{2q(q^{10};q^{10})_\infty}{(q^5;q^{10})_\infty[q,q^3,q^4,q^4;q^{10}]_\infty}=\frac{2qL_{10,3}(q)}{(q^5;q^{10})_\infty[q,q^4,q^4;q^{10}]_\infty}.
\end{aligned}
$$

Since the term $\frac{1}{1-q}$ appearing on the right side of the last equality has positive coefficients, and by Lemma 2.4, we know that $L_{10,3}(q)$ (whose constant term is 1) has non-negative coefficients of q^n for all $n \geqslant 1$, the inequality follows.

When $j=2$, we have

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\big(N(1,10,5n+2)+N(2,10,5n+2)-N(3,10,5n+2)-N(4,10,5n+2)\big)q^n\\
&\quad= \frac{J_5J_{4,10}J_{10}^6}{J_{2,10}^3J_{3,10}^3}=\frac{(q^5,q^{10};q^{10})_\infty[q^4;q^{10}]_\infty}{[q^2,q^3;q^{10}]_\infty^3}=\frac{(q^5;q^5)_\infty[q^4;q^{10}]_\infty}{[q^2;q^5]_\infty^3}=\frac{L_{5,2}(q)[-q^2;q^5]_\infty}{[q^2;q^5]_\infty}.
\end{aligned}
$$

Since the term $\frac{1+q^3}{1-q^2}$ appearing on the right side of the last equality has positive coefficients of q^n for all $n \geqslant 2$, and by Lemma 2.4, we know that $L_{5,2}(q)$ (whose constant term is 1) has non-negative coefficients of q^n for all $n \geqslant 1$, the inequality follows.

When $j=3$, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\big(N(1,10,5n+3)+N(2,10,5n+3)-N(3,10,5n+3)-N(4,10,5n+3)\big)q^n\\
&\quad=\frac{J_5J_{10}^5}{J_{1,10}J_{2,10}J_{3,10}^2J_{4,10}}=\frac{(q^5;q^5)_\infty}{[q,q^2,q^3,q^3,q^4;q^{10}]_\infty}=\frac{(q^5;q^5)_\infty}{[q^2;q^5]_\infty[q,q^3,q^4;q^{10}]_\infty}\\
&\quad=\frac{L_{5,2}(q)}{[q,q^3,q^4;q^{10}]_\infty}.
\end{aligned}$$

Since the factor $\frac{1}{1-q}$ appears on the right side of the last equality and by Lemma 2.4, we know that $L_{5,2}(q)$ (whose constant term is 1) has non-negative coefficients of q^n for all $n \geqslant 1$, the inequality follows. Our inequality follows.

When $j=4$, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\big(N(1,10,5n+4)+N(2,10,5n+4)-N(3,10,5n+4)-N(4,10,5n+4)\big)q^n\\
&\quad=\frac{J_5J_{4,10}^2J_{5,10}J_{10}^5}{2qJ_{2,10}^4J_{3,10}^4}-\frac{1}{J_5}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(15n^2+5n)/2-1}}{1+q^{5n}}\\
&\quad=\frac{[-q^2;q^5]_\infty}{[q;q^5]_\infty J_5}\sum_{n=-\infty}^{\infty}\frac{q^{(15n^2+17n)/2+1}}{1-q^{5n+2}}-\frac{[-q^2;q^5]_\infty}{[q;q^5]_\infty J_5}\sum_{n=-\infty}^{\infty}\frac{q^{(15n^2+23n)/2+3}}{1-q^{5n+3}}\\
&\quad=2\frac{[-q^2;q^5]_\infty}{[q;q^5]_\infty J_5}\sum_{n=-\infty}^{\infty}\frac{q^{(15n^2+17n)/2+1}}{1-q^{5n+2}},
\end{aligned}$$

the second equality follows from replacing b_1, b_2, b_3 and q by -1, q^2, q^3 and q^5, respectively, in (2.13) and the last equality follows from replacing n by $-n$ in the summation index in the second series on the right side of the second equality. Since we have

$$\begin{aligned}
&\frac{[-q^2;q^5]_\infty}{[q;q^5]_\infty J_5}\sum_{n=-\infty}^{\infty}\frac{q^{(15n^2+17n)/2+1}}{1-q^{5n+2}}\\
&\quad=\frac{[-q^2;q^5]_\infty}{[q;q^5]_\infty J_5}\sum_{n=0}^{\infty}\left(\frac{q^{(15n^2+17n)/2+1}}{1-q^{5n+2}}-\frac{q^{(15n^2+23n)/2+3}}{1-q^{5n+3}}\right)\\
&\quad=\frac{[-q^2;q^5]_\infty}{[q;q^5]_\infty J_5}\sum_{n=0}^{\infty}q^{(15n^2+17n)/2+1}\left(\frac{1}{1-q^{5n+2}}-\frac{q^{3n+2}}{1-q^{5n+3}}\right)\\
&\quad=\frac{[-q^2;q^5]_\infty}{[q;q^5]_\infty J_5}\sum_{n=0}^{\infty}\frac{q^{(15n^2+17n)/2+1}}{(1-q^{5n+2})(1-q^{5n+3})}\big\{1-q^{5n+3}-q^{3n+2}\big(1-q^{5n+2}\big)\big\}
\end{aligned}$$

$$= \frac{[-q^2;q^5]_\infty}{[q;q^5]_\infty J_5} \sum_{n=0}^{\infty} \frac{q^{(15n^2+17n)/2+1}}{(1-q^{5n+2})(1-q^{5n+3})} \bigl\{ \bigl(1-q^{5n+3}\bigr)\bigl(1-q^{3n+2}\bigr) + q^{8n+4}(1-q) \bigr\}$$

$$= \frac{[-q^2;q^5]_\infty}{(q^6,q^4;q^5)_\infty J_5} \sum_{n=0}^{\infty} \frac{q^{(15n^2+17n)/2+1}}{1-q^{5n+2}}$$

$$\times \left\{ \frac{(1-q^{5n+3})(1-q^{3n+2})}{(1-q^{5n+3})(1-q)} + \frac{q^{8n+4}(1-q)}{(1-q^{5n+3})(1-q)} \right\}$$

$$= \frac{[-q^2;q^5]_\infty}{(q^6,q^4;q^5)_\infty J_5} \sum_{n=0}^{\infty} \frac{q^{(15n^2+17n)/2+1}}{1-q^{5n+2}} \left(\sum_{k=0}^{3n+1} q^k + \frac{q^{8n+4}}{1-q^{5n+3}} \right),$$

and the factor $\frac{q(1+q)}{1-q^2}$ appearing in the sum of the last expression has positive coefficient of q^n for all $n \geqslant 1$, we know that $\frac{[-q^2;q^5]_\infty}{[q;q^5]_\infty J_5} \sum_{n=-\infty}^{\infty} \frac{q^{15n^2/2+17n/2+1}}{1-q^{5n+2}}$ has positive coefficients for all $n \geqslant 1$, thus our inequality follows. This completes the proof of (1.17).

Note that, for $0 \leqslant t \leqslant 4$, we have

$$N(t,5,n) = N(t,10,n) + N(5-t,10,n). \tag{4.2}$$

Substituting (4.2) with the suitable t into the relations between ranks modulo 5, then by comparing the results with (1.16) or (1.17), we prove the rest of the inequalities in the corollary. For example, since by (4.2), we have $N(0,5,5n+4) = N(0,10,5n+4) + N(5,10,5n+4)$ and $N(1,5,5n+4) = N(1,10,5n+4) + N(4,10,5n+4)$, substitute these two equalities into (1.5) yields

$$N(0,10,5n+4) + N(5,10,5n+4) = N(1,10,5n+4) + N(4,10,5n+4). \tag{4.3}$$

On the other hand, by (1.16) (when $i=4$), we have

$$N(0,10,5n+4) + N(1,10,5n+4) \geqslant N(4,10,5n+4) + N(5,10,5n+4). \tag{4.4}$$

Adding (4.3) to (4.4), we get (1.24). We omit the proofs of other inequalities since they are similar to the above. □

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