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The M_2-rank of partitions without repeated odd parts modulo 6 and 10

Renrong Mao

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Abstract Lovejoy and Osburn (J Theor Nombres Bordeaux 21:313–334, 2009) proved formulas for the generating functions for M_2-rank differences modulo 3 and 5 for partitions without repeated odd parts. In this paper, we derive formulas for the generating functions for M_2-ranks modulo 6 and 10 for such partitions. With these generating functions, we find some inequalities between M_2-ranks modulo 3, 5, 6, and 10 of partitions without repeated odd parts.

Keywords M_2-rank · Rank differences · Inequalities · Partition without repeated odd parts · Generalized Lambert series · Theta identities

Mathematics Subject Classification Primary 11P81 · Secondary 05A17

1 Introduction

Let $p(n)$ denote the number of unrestricted partitions of n. Dyson [10] defined the rank of a partition to be the largest part minus the number of parts. He also conjectured that this partition statistic provided a combinatorial explanation for Ramanujan's congruences $p(5n+4) \equiv 0 \mod 5$ and $p(7n+5) \equiv 0 \mod 7$. This was proved by Atkin and Swinnerton-Dyer [4] in 1954. Recently, partitions without repeated odd parts have also been studied in many places. Berkovich and Garvan [5] introduced what they called the M_2-rank of such partitions. The M_2-rank of a partition λ without repeated odd parts is defined by

R. Mao (✉)
Division of Mathematical Sciences, School of Physical and Mathematical Sciences,
Nanyang Technological University, 21 Nanyang Link, Singapore 637371, Republic of Singapore
e-mail: maor0001@e.ntu.edu.sg

$$M_2\text{-rank}(\lambda) = \left\lceil \frac{l(\lambda)}{2} \right\rceil - \nu(\lambda),$$

where $l(\lambda)$ is the largest part of λ, $\nu(\lambda)$ is the number of parts of λ, and $\lceil . \rceil$ is the ceiling function. Let $N_2(m,n)$ denote the number of partitions of n without repeated odd parts whose M_2-rank is m. Then, we have the following generating function:

$$\sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} N_2(m,n) z^m q^n = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(zq^2, q^2/z; q^2)_n} \quad \text{([19, Eq. (1.1)]).} \tag{1.1}$$

In the equation above and for the rest of this article, we use the notations

$$\begin{aligned}
(x_1, x_2, \ldots, x_k; q)_m &:= \prod_{n=0}^{m-1} (1 - x_1 q^n)(1 - x_2 q^n) \cdots (1 - x_k q^n), \\
(x_1, x_2, \ldots, x_k; q)_\infty &:= \prod_{n=0}^{\infty} (1 - x_1 q^n)(1 - x_2 q^n) \cdots (1 - x_k q^n), \\
[x_1, x_2, \ldots, x_k; q]_\infty &:= (x_1, q/x_1, x_2, q/x_2, \ldots, x_k, q/x_k; q)_\infty, \\
J_{a,b} &:= (q^a, q^{b-a}, q^b; q^b), \\
\overline{J}_{a,b} &:= (-q^a, -q^{b-a}, q^b; q^b), \\
J_b &:= (q^b; q^b)_\infty, \\
\overline{J}_b &:= (-q^b; q^b)_\infty,
\end{aligned}$$

and we require $|q| < 1$ for absolute convergence.

The two-variable generating function for M_2-rank in (1.1) appears many times in Ramanujan's "lost" notebook [2, Ch. 12]. As an example, the identity, [2, Entry 12.4.3],

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-1)^n (q^2;q^4)_n q^{2n^2}}{(-xq^4;q^4)_n (-q^4/x;q^4)_n} + (1+x)(1+1/x) \sum_{n=0}^{\infty} \frac{(-q;q)_{2n} q^{n+1}}{(xq;q^2)_{n+1}(q/x;q^2)_{n+1}} \\
&= \frac{(-xq^2;q^4)_\infty \left(-q^2/x;q^4\right)_\infty (q^2;q^2)_\infty}{(q;q^2)_\infty^2 (-xq^2;q^4)_\infty (xq;q^2)_\infty (q/x;q^2)_\infty}
\end{aligned} \tag{1.2}$$

gives a connection between the generating function for M_2-rank of partitions without repeated odd parts and Ramanujan's ϕ function [the special case of $x = 1$ of the infinite sum in the second series on the left side of (1.2)]. It is also closely related to some mock theta functions. For positive integer t, we define

$$L_t(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(\xi_t q^2, q^2/\xi_t; q^2)_n},$$

where $\xi_t = e^{\frac{2\pi i}{t}}$. Then, $L_2(q)$ is the second-order mock theta function $\mu(-q)$ of McIntosh [17]. Also, $L_4(q)$ is the eighth order mock theta function $U_0(q)$ of Gordon and McIntosh [12].

Let $N_2(s, l, n)$ denote the number of partitions of n without repeated odd parts whose M_2-rank is congruent to s modulo l and for $0 \le s, t, d \le l$, define

$$R_{st}(d) := \sum_{n\ge 0}(N_2(s, l, ln + d) - N_2(t, l, ln + d))q^n.$$

Then, Lovejoy and Osburn [15] gave all the generating functions of $R_{st}(d)$ for $l = 3$ and 5. In our paper, after deriving the 3-dissection of $L_6(q)$ and 5-dissection of $L_{10}(q)$, we find the following results on the generating functions for M_2-ranks modulo 6 and 10 of partition without repeated odd parts.

Theorem 1.1 *We have*

$$\begin{aligned}
&\sum_{n\ge 0}(N_2(0, 6, n) + N_2(1, 6, n) - N_2(2, 6, n) - N_2(3, 6, n))q^n \\
&\quad = L_6(q) && (1.3)\\
&\quad = \frac{1}{J_{9,36}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{18n^2+9n}}{1-q^{18n+3}} \\
&\quad +q\frac{J_{6,36}^2 J_{18,36} J_{36}^3}{J_{3,36}^2 J_{9,36} J_{15,36}^2} \\
&\quad +\frac{J_{6,36} J_{18,36}^2 J_{36}^3}{2q J_{3,36}^2 J_{9,36} J_{15,36}^2} - \frac{1}{J_{9,36}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{18n^2+9n-1}}{1+q^{18n}}. && (1.4)
\end{aligned}$$

Theorem 1.2 *Let*

$$F_1(q) := \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2+n}(1+q^{2n})}{1+q^{10n}}$$

and

$$F_2(q) := \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2+n}(q^{4n}-1)}{1+q^{10n}}.$$

Then, we have

$$\begin{aligned}
&\sum_{n\ge 0}(N_2(0, 10, n) + N_2(1, 10, n) - N_2(4, 10, n) - N_2(5, 10, n))q^n \\
&\quad = F_1(q) && (1.5)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{J_{25,100}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{50n^2+25n}}{1+q^{50n+10}} \\
&\quad + 2q^5 \frac{J_{10,100} J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^2 J_{20,100}^2 J_{25,100}^3 J_{30,100} J_{35,100}^2 J_{45,100}^3} \\
&\quad + q \frac{J_{20,100} J_{30,100}^2 J_{50,100} J_{100}^{15}}{J_{5,100}^2 J_{10,100}^2 J_{15,100}^4 J_{25,100} J_{35,100}^4 J_{40,100}^3 J_{45,100}^2} \\
&\quad + q^2 \frac{J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^3 J_{20,100} J_{25,100} J_{35,100}^3 J_{40,100} J_{45,100}^3} \\
&\quad + q^3 \frac{J_{10,100}^2 J_{40,100} J_{50,100} J_{100}^{15}}{J_{5,100}^4 J_{15,100}^2 J_{20,100}^3 J_{25,100} J_{30,100}^2 J_{35,100}^2 J_{45,100}^4} \\
&\quad + \frac{1}{J_{25,100}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{50n^2+75n+24}}{1+q^{50n+30}} \\
&\quad + 2q^4 \frac{J_{30,100} J_{50,100} J_{100}^{15}}{J_{5,100}^2 J_{10,100} J_{15,100}^3 J_{25,100}^3 J_{35,100}^3 J_{40,100}^2 J_{45,100}^2}
\end{aligned}
\tag{1.6}
$$

and

$$
\sum_{n\geq 0} (N_2(1,10,n) + N_2(2,10,n) - N_2(3,10,n) - N_2(4,10,n))q^n = F_2(q)
\tag{1.7}
$$

$$
\begin{aligned}
&= \frac{J_{30,100} J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{10,100} J_{15,100}^3 J_{25,100} J_{35,100}^3 J_{40,100}^2 J_{45,100}^3} \\
&\quad - 2q^5 \frac{J_{10,100} J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^2 J_{20,100}^2 J_{25,100}^3 J_{30,100} J_{35,100}^2 J_{45,100}^3} \\
&\quad - \frac{1}{J_{25,100}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{50n^2+25n}}{1+q^{50n+10}} \\
&\quad + \frac{2q^6 J_{50,100} J_{100}^{15}}{J_{5,100}^2 J_{15,100}^3 J_{20,100} J_{25,100}^3 J_{35,100}^3 J_{40,100} J_{45,100}^2} \\
&\quad + 2q^7 \frac{J_{10,100}^2 J_{40,100} J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^2 J_{20,100}^3 J_{25,100}^3 J_{30,100}^2 J_{35,100}^2 J_{45,100}^3} \\
&\quad + \frac{J_{10,100}^2 J_{15,100}^2 J_{20,100} J_{30,100} J_{35,100}^2 J_{40,100} J_{50,100}^2}{2q^3 J_{25,100} J_{100}^9} \\
&\quad - \frac{1}{J_{25,100}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{50n^2+25n-3}}{1+q^{50n}}
\end{aligned}
$$

$$+\frac{q^3 J_{30,100}J_{50,100}J_{100}^{15}}{J_{5,100}^2 J_{10,100}J_{15,100}^4 J_{25,100}J_{35,100}^4 J_{40,100}^2 J_{45,100}^2}$$
$$+\frac{q^4 J_{10,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3 J_{15,100}^3 J_{20,100}^2 J_{25,100}J_{30,100}J_{35,100}^3 J_{45,100}^3}. \tag{1.8}$$

Although more and more inequalities between ranks of ordinary partitions have been studied over the years (see [3,11,14,16], for example), relations between M_2-ranks of partitions without repeated odd parts have not been discussed widely. In this article, with the generating functions established in Theorem 1.1, 1.2 and [15, Theorem 1.1, 1.2], we discover the following inequalities.

Corollary 1.3 *For $i = 1, 2, 3$, $j = 1, 3, 4$ and $n \geq 0$,*

$$N_2(0,3,3n+1) \geq N_2(1,3,3n+1), \tag{1.9}$$
$$N_2(0,3,3n+2) \geq N_2(1,3,3n+2), \tag{1.10}$$
$$N_2(1,5,5n+3) \geq N_2(2,5,5n+3), \tag{1.11}$$
$$N_2(0,5,5n+1) \geq N_2(1,5,5n+1), \tag{1.12}$$
$$N_2(1,5,5n+3) \geq N_2(0,5,5n+3), \tag{1.13}$$
$$N_2(0,6,3n)+N_2(1,6,3n) > N_2(2,6,3n)+N_2(3,6,3n), \tag{1.14}$$
$$N_2(0,6,3n+1)+N_2(1,6,3n+1) > N_2(2,6,3n+1)+N_2(3,6,3n+1), \tag{1.15}$$
$$N_2(0,10,5n+i)+N_2(1,10,5n+i) > N_2(4,10,5n+i)+N_2(5,10,5n+i), \tag{1.16}$$
$$N_2(1,10,5n+j)+N_2(2,10,5n+j) \geq N_2(3,10,5n+j)+N_2(4,10,5n+j), \tag{1.17}$$
$$N_2(0,6,3n+1) > N_2(2,6,3n+1), \tag{1.18}$$
$$N_2(0,10,5n+1) > N_2(4,10,5n+1), \tag{1.19}$$
$$N_2(1,10,5n+1) > N_2(3,10,5n+1), \tag{1.20}$$
$$N_2(2,10,5n+1) > N_2(4,10,5n+1), \tag{1.21}$$
$$N_1(1,10,5n+3) > N_2(3,10,5n+3), \tag{1.22}$$
$$N_1(1,10,5n+3) > N_2(5,10,5n+3). \tag{1.23}$$

All the "$\geq$" above can be replaced by "$>$", except the following cases: $n = 1$ in (1.9), $n = 3$ in (1.10), $n = 1$ and 3 in (1.12), $n = 1, 2,$ and 3 in (1.11) and (1.13), and $(n, j) = (0, 1)$ in (1.17).

Remark From the proofs of Theorem 1.1 and Theorem 1.2, we will see that

$$L_6(q) = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n(1+q^{2n})q^{2n^2+n}}{1+q^{6n}} \tag{1.24}$$

and

$$L_{10}(q) = F_1(q) + (\xi_{10}^2 - \xi_{10}^3) F_2(q). \tag{1.25}$$

Computer evidence suggests the non-negativity of coefficients of $L_6(q)$, $F_1(q)$, and $F_2(q)$. By (1.14) and (1.15), we see that $L_6(q)$ has positive coefficients of q^n for $n \equiv 0$ and $1 \mod 3$, $n \geq 0$. By (1.16), we see that $F_1(q)$ has positive coefficients of q^n for $n \equiv 1, 2,$ and $3 \mod 5$, $n \geq 0$. And by (1.17), we see that $F_2(q)$ has positive coefficients of q^n for $n \equiv 1, 3,$ and $4 \mod 5$, $n \geq 3$. We failed to prove the remaining cases and so we leave them in the following conjecture.

Conjecture 1.4

$$\begin{aligned}
N_2(0, 6, 3n+2) + N_2(1, 6, 3n+2) &> N_2(2, 6, 3n+2) \\
&\quad + N_2(3, 6, 3n+2) \textit{ for } n \geq 0, \\
N_2(0, 10, 5n) + N_2(1, 10, 5n) &> N_2(4, 10, 5n) + N_2(5, 10, 5n) \textit{ for } n \geq 0, \\
N_2(0, 10, 5n+4) + N_2(1, 10, 5n+4) &> N_2(4, 10, 5n+4) \\
&\quad + N_2(5, 10, 5n+4) \textit{ for } n \geq 0, \\
N_2(1, 10, 5n) + N_2(2, 10, 5n) &> N_2(3, 10, 5n) + N_2(4, 10, 5n) \textit{ for } n \geq 1, \\
N_2(1, 10, 5n+2) + N_2(2, 10, 5n+2) &> N_2(3, 10, 5n+2) \\
&\quad + N_2(4, 10, 5n+2) \textit{ for } n \geq 1.
\end{aligned}$$

The paper is organized as follows. We give some lemmas in Sect. 2 and prove Theorem 1.1 and 1.2 in Sects. 3 and 4, respectively. Corollary 1.3 is proved in Sect. 5. In Appendix, after applying a lot of theta identities, we provide proofs of Lemma 3.2, 4.2, and 4.3 which are used to obtain Theorem 1.1 and 1.2. Since these proofs are lengthy and complicated, we put them at the end of the paper.

2 Some lemmas

We require Lemma 3.1 in [15] for the 3-dissection and 5-dissection of $\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}$:

Lemma 2.1

$$\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} = \left(q^3, -q^6, -q^9, -q^{12}, q^{15}, q^{18}; q^{18}\right)_\infty - q\left(q^9, q^{27}, q^{36}; q^{36}\right)_\infty, \tag{2.1}$$

and

$$\begin{aligned}
\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} &= \left(-q^{10}, q^{15}, -q^{25}, q^{35}, -q^{40}, q^{50}; q^{50}\right)_\infty \\
&\quad - q\left(q^5, -q^{20}, -q^{25}, -q^{30}, q^{45}, q^{50}; q^{50}\right)_\infty
\end{aligned}$$

$$-q^3\left(q^{25},q^{75},q^{100};q^{100}\right)_\infty. \tag{2.2}$$

Lemma 2.2 *We have*

$$\begin{aligned}&\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+3n}\left[\frac{\zeta^{-4n}}{1-z^2\zeta^{-2}q^{2n}}+\frac{\zeta^{4n+6}}{1-z^2\zeta^2q^{2n}}\right]\\&=\frac{\zeta^2\left(-q,-q,\zeta^4,q^2\zeta^{-4};q^2\right)_\infty}{\left(\zeta^2,q^2\zeta^{-2},-q\zeta^{-2},-q\zeta^2;q^2\right)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{2n^2+3n}}{1-z^2q^{2n}}\\&\quad+\frac{\left(-z^2q,-qz^{-2},\zeta^4,q^2\zeta^{-4},\zeta^2,q^2\zeta^{-2};q^2\right)_\infty\left(q^2;q^2\right)_\infty^2}{\left(z^2\zeta^{-2},q^2\zeta^2z^{-2},\zeta^2z^2,q^2\zeta^{-2}z^{-2},-q\zeta^2,-q\zeta^{-2},z^2,q^2z^{-2};q^2\right)_\infty}.\end{aligned} \tag{2.3}$$

The above lemma appears in [15] as Lemma 4.1. Next, we derive an identity which is similar to (2.3).

Lemma 2.3

$$\begin{aligned}&\sum_{n=-\infty}^{\infty}(-1)^n q^{2n^2+n}\left[\frac{\zeta^{-2n}}{1-z\zeta^{-1}q^{2n}}+\frac{\zeta^{2n+1}}{1-z\zeta q^{2n}}\right]\\&=\frac{\left[-q,\zeta^2;q^2\right]_\infty}{\left[-q\zeta,\zeta;q^2\right]_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{2n^2+n}}{1-zq^{2n}}+\frac{q\left[-z/q,\zeta^2,\zeta;q^2\right]_\infty\left(q^2;q^2\right)_\infty^2}{\zeta\left[z\zeta^{-1},\zeta z,-q\zeta,z;q^2\right]_\infty}\end{aligned} \tag{2.4}$$

Proof This follows from setting $r=1$, $s=3$ and replacing q, a_1, b_1, b_2, and b_3 by q^2, $-z/q$, $z\zeta$, z/ζ, and z, respectively, in [7, Eq.(2.1)],

$$\begin{aligned}&\frac{[a_1,..,a_r]_\infty(q;q)_\infty^2}{[b_1,..,b_s]_\infty}\\&=\frac{[a_1/b_1,..,a_r/b_1]_\infty(q;q)_\infty^2}{[b_2/b_1,..,b_s/b_1]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^{s-r}q^{(s-r)k(k-1)/2}}{1-b_1q^k}\left(\frac{a_1..a_rb_1^{s-r-1}}{b_2..b_r}\right)^k\\&\quad+idem(b_1;b_2,..b_s).\end{aligned} \tag{2.5}$$

Here, we use the usual notation

$$\begin{aligned}&F(b_1,b_2,..,b_m)+idem(b_1,b_2,\dots,b_m)\\&\quad:=F(b_1,b_2,\dots,b_m)+F(b_2,b_1,b_3,\dots,b_m)+..+F(b_m,b_2,\dots,b_{m-1},b_1).\end{aligned}$$

□

Lastly, we recall the following two classical identities.

Lemma 2.4 (q-binomial theorem [1, p.19, Eq. (2.2.5)]) *For* $|z| < 1$,

$$\sum_{n=0}^{\infty} \frac{(a;q)_n z^n}{(q;q)_n} = \frac{(az;q)_\infty}{(z;q)_\infty}.$$

3 Proof of Theorem 1.1

The following lemma is used to obtain (3.8) which is a key identity to the proof of Theorem 1.1. The infinite sums on the left hand sides of equations in Lemma 3.1 come from $L_6(q)$ (see (1.24)).

Lemma 3.1 *Let*

$$V_0 := \frac{\left[q^3, q^6, q^6; q^{18}\right]_\infty \left(q^{18}; q^{18}\right)_\infty^2}{\left[-1, -q^6, -q^6, -q^3; q^{18}\right]_\infty} = \frac{J_{3,36}^2 J_{6,36}^3 J_{12,36}^2 J_{15,36}^2 J_{18,36}^2}{2J_{36}^9}$$

and

$$V_1 := \frac{\left[q^9, q^6, q^6; q^{18}\right]_\infty \left(q^{18}; q^{18}\right)_\infty^2}{q\left[-1, -q^6, -q^6, -q^3; q^{18}\right]_\infty} = \frac{J_{3,36} J_{6,36}^3 J_{9,36}^2 J_{12,36}^2 J_{15,36} J_{18,36}^2}{2qJ_{36}^9}.$$

We have

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n^2+n}}{1+q^{6n}} = V_0 - \frac{(q^2;q^2)_\infty}{J_{9,36}(-q;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{18n^2+27n+9}}{1+q^{18n+12}} \tag{3.1}$$

and

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n^2+3n}}{1+q^{6n}} = V_1 - \frac{\left(q^2;q^2\right)_\infty}{J_{9,36}\left(-q;q^2\right)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{18n^2+9n-1}}{1+q^{18n}}. \tag{3.2}$$

Proof The proofs of the above two equations are similar to each other. We give the details of the proof of (3.1) and omit the other. Splitting the series on the left side of (3.1) into three series according to the summation index n modulo 3, we find that

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n^2+n}}{1+q^{6n}} &= \sum_{\substack{n=-\infty \\ n\equiv 0 \mod 3}}^{\infty} \frac{(-1)^n q^{2n^2+n}}{1+q^{6n}} \\ &\quad + \sum_{\substack{n=-\infty \\ n\equiv 1 \mod 3}}^{\infty} \frac{(-1)^n q^{2n^2+n}}{1+q^{6n}} + \sum_{\substack{n=-\infty \\ n\equiv 2 \mod 3}}^{\infty} \frac{(-1)^n q^{2n^2+n}}{1+q^{6n}} \\ &= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{18n^2+3n}}{1+q^{18n}} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{18n^2+15n+3}}{1+q^{18n+6}}\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{18n^2+27n+10}}{1+q^{18n+12}}\\
&=: S_0 - S_1 + S_2.
\end{aligned}
$$

Applying (2.3) with q, z^2, and ζ^2 replaced by q^9, $-q^{12}$, and q^{12}, respectively, we find that

$$
\begin{aligned}
&\sum_{n=-\infty}^{\infty}(-1)^n q^{18n^2+27n}\left(\frac{q^{-24n}}{1+q^{18n}}+\frac{q^{24n+36}}{1+q^{18n+24}}\right)\\
&=\frac{q^{12}\left[-q^9,q^{24};q^{18}\right]_\infty}{\left[q^{12},-q^{21};q^{18}\right]_\infty}\sum_{n=-\infty}^{\infty}(-1)^n\frac{q^{18n^2+27n}}{1+q^{18n+12}}\\
&+\frac{\left[q^{21},q^6,q^{24};q^{18}\right]_\infty\left(q^{18};q^{18}\right)_\infty^2}{\left[-1,-q^{24},-q^6,-q^{21};q^{18}\right]_\infty}.
\end{aligned}
\tag{3.3}
$$

Replacing the summation index n by $n+1$, we find that

$$
S_1=-\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{18n^2+51n+36}}{1+q^{18n+24}}.
$$

Substituting the above equation into (3.3) and simplifying, we get

$$
S_0-S_1=-\frac{\left[-q^9;q^{18}\right]_\infty}{q\left[-q^3;q^{18}\right]_\infty}S_2+\frac{\left[q^3,q^6,q^{12};q^{18}\right]_\infty\left(q^{18};q^{18}\right)_\infty^2}{\left[-1,-q^6,-q^6,-q^3;q^{18}\right]_\infty}.
$$

Therefore,

$$
\begin{aligned}
&\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2+n}}{1+q^{6n}}=S_0-S_1+S_2\\
&=-\frac{\left[-q^9;q^{18}\right]_\infty}{q\left[-q^3;q^{18}\right]_\infty}S_2+\frac{\left[q^3,q^6,q^{12};q^{18}\right]_\infty\left(q^{18};q^{18}\right)_\infty^2}{\left[-1,-q^6,-q^6,-q^3;q^{18}\right]_\infty}+S_2\\
&=-\left\{\left(q^3,-q^6,-q^9,-q^{12},q^{15},q^{18};q^{18}\right)_\infty-q\left(q^9,q^{27},q^{36};q^{36}\right)_\infty\right\}\\
&\times\frac{\left(-q^9;q^{18}\right)_\infty S_2}{q\left(q^{18};q^{18}\right)_\infty}\\
&+\frac{\left[q^3,q^6,q^{12};q^{18}\right]_\infty\left(q^{18};q^{18}\right)_\infty^2}{\left[-1,-q^6,-q^6,-q^3;q^{18}\right]_\infty}.
\end{aligned}
$$

We complete the proof of (3.1) after invoking (2.1). □

Next, we obtain the 3-dissection of $\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \times \{V_0 + V_1\}$ through the following lemma which gives us all the products in Theorem 1.1. It will be used in the final step of the proof (1.4). We prove Lemma 3.2 in in Appendix.

Lemma 3.2

$$V_0 + V_1 = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} \times \left\{ \frac{J_{6,36}J_{18,36}^2J_{36}^3}{J_{3,36}J_{9,36}^3J_{15,36}} + q\frac{J_{6,36}^2J_{18,36}J_{36}^3}{J_{3,36}^2J_{9,36}J_{15,36}^2} + \frac{J_{6,36}J_{18,36}^2J_{36}^3}{2qJ_{3,36}^2J_{9,36}J_{15,36}^2} \right\}. \quad (3.4)$$

Now we are in a position to prove Theorem 1.1.

Proof of Theorem 1.1 First, we prove (1.3). Replacing z by $\xi_6 = e^{\frac{\pi i}{3}}$ in (1.1), we have

$$\sum_{n=0}^{\infty} q^{n^2} \frac{(-q;q^2)_n}{(\xi_6 q^2, q^2/\xi_6;q^2)_n} = \sum_{m=-\infty}^{\infty}\sum_{n=0}^{\infty} N_2(m,n)\xi_6^m q^n$$
$$= \sum_{n=0}^{\infty}\sum_{t=0}^{5}\sum_{m=-\infty}^{\infty} N_2(6m+t,n)\xi_6^t q^n.$$

By the definition of $N_2(s,l,n)$, we know that

$$\sum_{n=0}^{\infty} q^{n^2} \frac{(-q;q^2)_n}{(\xi_6 q^2, q^2/\xi_6;q^2)_n} = \sum_{t=0}^{5}\sum_{n=0}^{\infty} N_2(t,6,n)\xi_6^t q^n.$$

Expanding the last series in the above equation according to the summation index t, noting that $N(t,6,n) = N(6-t,6,n)$, $\xi_6 + \xi_6^5 = 1$, and $\xi_6^3 = -1$, we have

$$L_6(q) = \sum_{n\geq 0}(N_2(0,6,n) + N_2(1,6,n) - N_2(2,6,n) - N_2(3,6,n))q^n.$$

This completes the proof of (1.3).

Next, we prove (1.4). From a limiting case of Watson's ${}_8\phi_7$ transformation, [6, Eq. (7.2), p. 16]

$$\sum_{n=0}^{\infty} \frac{(aq/bc,d,e;q)_n(\frac{aq}{de})^n}{(q,aq/b,aq/c;q)_n} = \frac{(aq/d,aq/e;q)_\infty}{(aq,aq/de;q)_\infty} \times \sum_{n=0}^{\infty} \frac{(a,b,c,d,e;q)_n(1-aq^{2n})(-a^2)^n q^{n(n+3)/2}}{(q,aq/b,aq/c,aq/d,aq/e;q)_n(1-a)(bcde)^n}, \quad (3.5)$$

we let $e \to \infty$ and replace q, a, b, c, and d by q^2, 1, $\frac{1}{z}$, z, and $-q$, respectively. Upon simplifying, we find that

$$\sum_{n=0}^{\infty} q^{n^2} \frac{(-q;q^2)_n}{(zq^2, q^2/z; q^2)_n} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1-\frac{1}{z})(1-z)q^{2n^2+n}}{(1-zq^{2n})(1-q^{2n}/z)}. \tag{3.6}$$

Replacing z by ξ_6 in (3.6), we arrive at

$$\begin{aligned} L_6(q) &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n\left(1-\frac{1}{\xi_6}\right)(1-\xi_6)q^{2n^2+n}}{(1-\xi_6 q^{2n})(1-q^{2n}/\xi_6)} \\ &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1+\xi_6^2)(1+\xi_6^4)q^{2n^2+n}}{(1+q^{2n}\xi_6^4)(1+q^{2n}\xi_6^2)} \\ &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n(1+q^{2n})q^{2n^2+n}}{1+q^{6n}}. \end{aligned} \tag{3.7}$$

Substituting (3.1) and (3.2) into (3.7), we find that

$$\begin{aligned} L_6(q) = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \times (V_0+V_1) - \frac{1}{J_{9,36}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{18n^2+27n+9}}{1+q^{18n+12}} \\ - \frac{1}{J_{9,36}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{18n^2+9n-1}}{1+q^{18n}}. \end{aligned} \tag{3.8}$$

After invoking Lemma 3.2, we find that

$$\begin{aligned} L_6(q) = \frac{J_{6,36}J_{18,36}^2J_{36}^3}{J_{3,36}J_{9,36}^3J_{15,36}} - \frac{1}{J_{9,36}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{18n^2+27n+9}}{1+q^{18n+12}} \\ +q\frac{J_{6,36}^2J_{18,36}J_{36}^3}{J_{3,36}^2J_{9,36}J_{15,36}^2} \\ +\frac{J_{6,36}J_{18,36}^2J_{36}^3}{2qJ_{3,36}^2J_{9,36}J_{15,36}^2} - \frac{1}{J_{9,36}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{18n^2+9n-1}}{1+q^{18n}}. \end{aligned}$$

To complete the proof of Theorem 1.1, we only need to show that

$$\frac{J_{6,36}J_{18,36}^2J_{36}^3}{J_{3,36}J_{9,36}^3J_{15,36}} - \frac{1}{J_{9,36}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{18n^2+27n+9}}{1+q^{18n+12}} = \frac{1}{J_{9,36}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{18n^2+9n}}{1-q^{18n+3}},$$

which follows from setting $r=0$, $s=2$ and replacing q, b_1, and b_2 by q^{18}, $-q^{12}$, and q^3, respectively, in (2.5). ◻

4 Proof of Theorem 1.2

First, we need the following lemma on the infinite sums in $F_1(q)$ and $F_2(q)$. It will be applied to prove (4.9) and (4.10).

Lemma 4.1 *Let*

$$\begin{aligned}P_0 :&= \frac{q\left[q^{10}, q^{15}, q^{20}; q^{50}\right]_\infty (q^{50}; q^{50})_\infty^2}{\left[-q^5, -q^{10}, -q^{10}, -q^{20}; q^{50}\right]_\infty} - \frac{\left[q^{10}, q^{15}, q^{20}; q^{50}\right]_\infty (q^{50}; q^{50})_\infty^2}{\left[-1, -q^{10}, -q^{20}, -q^{35}; q^{50}\right]_\infty}\\ &= \frac{q J_{5,100} J_{10,100}^2 J_{15,100} J_{30,100}^2 J_{35,100} J_{40,100}^2 J_{45,100} J_{50,100}}{J_{100}^9}\\ &\quad - \frac{J_{10,100}^2 J_{15,100}^2 J_{20,100} J_{30,100} J_{35,100}^2 J_{40,100} J_{50,100}^2}{2J_{100}^9},\\ P_1 :&= \frac{\left[q^5, q^{10}, q^{20}; q^{50}\right]_\infty (q^{50}; q^{50})_\infty^2}{\left[-1, -q^5, -q^{10}, -q^{30}; q^{50}\right]_\infty} - \frac{q^9\left[q^5, q^{10}, q^{30}; q^{50}\right]_\infty (q^{50}; q^{50})_\infty^2}{\left[-q^{10}, -q^{15}, -q^{20}, -q^{20}; q^{50}\right]_\infty}\\ &= \frac{J_{5,100}^2 J_{10,100} J_{20,100} J_{30,100}^2 J_{40,100} J_{45,100}^2 J_{50,100}^2}{2J_{100}^9}\\ &\quad - \frac{q^9 J_{5,100} J_{10,100}^2 J_{15,100} J_{20,100}^2 J_{30,100}^2 J_{35,100} J_{45,100} J_{50,100}}{J_{100}^9}\end{aligned}$$

and

$$\begin{aligned}P_2 :&= \frac{\left[q^{10}, q^{20}, q^{25}; q^{50}\right]_\infty (q^{50}; q^{50})_\infty^2}{q^3\left[-1, -q^{10}, -q^{10}, -q^{15}; q^{50}\right]_\infty} - \frac{\left[q^{10}, q^{20}, q^{25}; q^{50}\right]_\infty (q^{50}; q^{50})_\infty^2}{q^2\left[-q^5, -q^{20}, -q^{20}, -q^{20}; q^{50}\right]_\infty}\\ &= \frac{J_{10,100}^3 J_{15,100} J_{25,100}^2 J_{35,100} J_{40,100}^3 J_{45,100}^3 J_{50,100}^2}{2q^3 J_{20,100} J_{100}^9}\\ &\quad - \frac{J_{5,100} J_{20,100}^3 J_{25,100}^2 J_{30,100}^3 J_{45,100} J_{50,100}^2}{2q^2 J_{40,100} J_{100}^9}.\end{aligned}$$

Then, we have

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n^2+n}}{1+q^{10n}} = P_0 + \frac{(q^2; q^2)_\infty}{J_{25,100}(-q; q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{50n^2+25n}}{1+q^{50n+10}}, \tag{4.1}$$

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n^2+3n}}{1+q^{10n}} = P_1 + \frac{(q^2; q^2)_\infty}{J_{25,100}(-q; q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{50n^2+75n+24}}{1+q^{50n+30}} \tag{4.2}$$

and

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{2n^2+5n}}{1+q^{10n}} = P_2 - \frac{(q^2; q^2)_\infty}{J_{25,100}(-q; q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{50n^2+25n-3}}{1+q^{50n}}. \tag{4.3}$$

Proof The proofs of the above three equations are similar to each other. We give the details of the proof of (4.1) and omit the rest. Splitting the series on the left of (4.1) into five series according to the summation index n modulo 5, we find that

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2+n}}{1+q^{10n}} &= \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+5n}}{1+q^{50n}} - \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+25n+3}}{1+q^{50n+10}}\\ &\quad+\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+45n+10}}{1+q^{50n+20}}\\ &\quad-\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+65n+21}}{1+q^{50n+30}} + \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+85n+36}}{1+q^{50n+40}}\\ &:= T_0 - T_1 + T_2 - T_3 + T_4.\end{aligned}$$

We apply (2.4) twice. First, replacing $q, z,$ and ζ by $q^{25}, -q^{10}$, and q^{10}, respectively, we find that

$$\begin{aligned}T_0 + T_2 &= \sum_{n=-\infty}^{\infty}(-1)^n q^{50n^2+25n}\left\{\frac{q^{-20n}}{1+q^{50n}}+\frac{q^{20n+10}}{1+q^{50n+20}}\right\}\\ &= \frac{\left[q^{20},-q^{25};q^{50}\right]_\infty}{\left[q^{10},-q^{15};q^{50}\right]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+25n}}{1+q^{50n+10}}\\ &\quad-\frac{\left[q^{10},q^{15},q^{20};q^{50}\right]_\infty\left(q^{50};q^{50}\right)_\infty^2}{\left[-1,-q^{10},-q^{20},-q^{35};q^{50}\right]_\infty}.\end{aligned}$$

Next, replacing $q, z,$ and ζ by $q^{25}, -q^{10}$, and q^{20}, we find that

$$\begin{aligned}\frac{1}{q}\left(T_3 - T_4\right) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{50n^2+25n}\left\{\frac{q^{-40n}}{1+q^{50n-10}}+\frac{q^{40n+20}}{1+q^{50n+30}}\right\}\\ &= \frac{\left[q^{10},-q^{25};q^{50}\right]_\infty}{\left[-q^{5},q^{20};q^{50}\right]_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+25n}}{1+q^{50n+10}}\\ &\quad-\frac{\left[q^{10},q^{15},q^{20};q^{50}\right]_\infty\left(q^{50};q^{50}\right)_\infty^2}{\left[-q^{5},-q^{10},-q^{10},-q^{20};q^{50}\right]_\infty}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2+n}}{1+q^{10n}} = T_0 - T_1 + T_2 - T_3 + T_4\\ &=\left\{\frac{\left[q^{20},-q^{25};q^{50}\right]_\infty}{\left[q^{10},-q^{15};q^{50}\right]_\infty} - q^3 - q\,\frac{\left[q^{10},-q^{25};q^{50}\right]_\infty}{\left[-q^{5},q^{20};q^{50}\right]_\infty}\right\}\times\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+25n}}{1+q^{50n+10}}\end{aligned}$$

$$
\begin{aligned}
&+\frac{q\left[q^{10},q^{15},q^{20};q^{50}\right]_\infty (q^{50};q^{50})_\infty^2}{\left[-q^5,-q^{10},-q^{10},-q^{20};q^{50}\right]_\infty}-\frac{\left[q^{10},q^{15},q^{20};q^{50}\right]_\infty (q^{50};q^{50})_\infty^2}{\left[-1,-q^{10},-q^{20},-q^{35};q^{50}\right]_\infty}\\
&=\frac{1}{J_{25,100}}\left\{\frac{J_{25,100}\left[q^{20},-q^{25};q^{50}\right]_\infty}{\left[q^{10},-q^{15};q^{50}\right]_\infty}-q^3J_{25,100}-q\frac{J_{25,100}\left[q^{10},-q^{25};q^{50}\right]_\infty}{\left[-q^5,q^{20};q^{50}\right]_\infty}\right\}\\
&\times\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+25n}}{1+q^{50n+10}}+P_0.
\end{aligned}
$$

We complete the proof of (4.1) after invoking (2.2). □

Next, we obtain the 5-dissections of $\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\times\{P_0+P_1\}$ and $\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\times\{P_2-P_0\}$ through Lemma 4.2 and Lemma 4.3, respectively. Lemma 4.2 (resp. Lemma 4.3) gives us the infinite products appearing in (1.6) (resp. (1.8)). These lemmas will be applied to complete the proof of Theorem 1.2 after we obtain (4.9) and (4.10). We prove Lemma 4.2 and Lemma 4.3 in Appendix.

Lemma 4.2 *Let*

$$
\begin{aligned}
A_0 &:= \frac{2q^5J_{10,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^2J_{20,100}^2J_{25,100}^3J_{30,100}J_{35,100}^2J_{45,100}^3},\\
A_1 &:= \frac{qJ_{20,100}J_{30,100}^2J_{50,100}J_{100}^{15}}{J_{5,100}^2J_{10,100}^2J_{15,100}^4J_{25,100}J_{35,100}^4J_{40,100}^3J_{45,100}^2},\\
A_2 &:= \frac{q^2J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^3J_{20,100}J_{25,100}J_{35,100}^3J_{40,100}J_{45,100}^3},\\
A_3 &:= \frac{q^3J_{10,100}^2J_{40,100}J_{50,100}J_{100}^{15}}{J_{5,100}^4J_{15,100}^2J_{20,100}^3J_{25,100}J_{30,100}^2J_{35,100}^2J_{45,100}^4}
\end{aligned}
$$

and

$$
A_4 := \frac{2q^4J_{30,100}J_{50,100}J_{100}^{15}}{J_{5,100}^2J_{10,100}J_{15,100}^3J_{25,100}^3J_{35,100}^3J_{40,100}^2J_{45,100}^2}. \tag{4.4}
$$

Then, we have

$$
P_0+P_1=\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty}\times\{A_0+A_1+A_2+A_3+A_4\}. \tag{4.5}
$$

Lemma 4.3 *Let*

$$
\begin{aligned}
B_0 := &\frac{J_{30,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{10,100}J_{15,100}^3J_{25,100}J_{35,100}^3J_{40,100}^2J_{45,100}^3}\\
&-2q^5\frac{J_{10,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^2J_{20,100}^2J_{25,100}^3J_{30,100}J_{35,100}^2J_{45,100}^3},
\end{aligned}
$$

$$B_1 := \frac{2q^6 J_{50,100} J_{100}^{15}}{J_{5,100}^2 J_{15,100}^3 J_{20,100} J_{25,100}^3 J_{35,100}^3 J_{40,100} J_{45,100}^2},$$

$$B_2 := 2q^7 \frac{J_{10,100}^2 J_{40,100} J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^2 J_{20,100}^3 J_{25,100}^3 J_{30,100}^2 J_{35,100}^2 J_{45,100}^3} + \frac{J_{10,100}^2 J_{15,100}^2 J_{20,100} J_{30,100} J_{35,100}^2 J_{40,100} J_{50,100}^2}{2q^3 J_{25,100} J_{100}^9},$$

$$B_3 := \frac{q^3 J_{30,100} J_{50,100} J_{100}^{15}}{J_{5,100}^2 J_{10,100} J_{15,100}^4 J_{25,100} J_{35,100}^4 J_{40,100}^2 J_{45,100}^2}$$

and

$$B_4 := \frac{q^4 J_{10,100} J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^3 J_{20,100}^2 J_{25,100} J_{30,100} J_{35,100}^3 J_{45,100}^3}.$$

Then, we have

$$P_2 - P_0 = \frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} \times \{B_0 + B_1 + B_2 + B_3 + B_4\}. \tag{4.6}$$

Now we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2 First, we prove (1.5) and (1.7). Replacing z by $\xi_{10} = e^{\frac{\pi i}{5}}$ in (1.1), we have

$$L_{10}(q) = \sum_{n=0}^{\infty} \sum_{m=-\infty}^{\infty} N_2(m,n)\xi_{10}^m q^n = \sum_{n=0}^{\infty} \sum_{t=0}^{9} \sum_{m=-\infty}^{\infty} N_2(10m+t,n)\xi_{10}^t q^n.$$

By the definition of $N_2(s,l,n)$, we know that

$$L_{10}(q) = \sum_{n=0}^{\infty} \sum_{t=0}^{9} N_2(t,10,n)\xi_{10}^t q^n = \sum_{t=0}^{9} \sum_{n=0}^{\infty} N_2(t,10,n)\xi_{10}^t q^n.$$

Expanding the last series in the above equation according to the summation index t, noting that $N_2(t,10,n) = N_2(10-t,10,n)$, $\xi_{10}^5 = -1$ and $\xi_{10} + \xi_{10}^3 + \xi_{10}^7 + \xi_{10}^9 = 1$, we have

$$\begin{aligned} L_{10}(q) = &\sum_{n\geq 0} (N_2(0,10,n) + N_2(1,10,n) - N_2(4,10,n) - N_2(5,10,n))\, q^n \\ &+(\xi_{10}^2 - \xi_{10}^3) \sum_{n\geq 0} (N_2(1,10,n) + N_2(2,10,n) \\ &-N_2(3,10,n) - N_2(4,10,n))\, q^n. \end{aligned} \tag{4.7}$$

Next, replacing z by ξ_{10} in (3.6) and simplifying, we find that

$$\begin{aligned}
L_{10}(q) &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n(1-\frac{1}{\xi_{10}})(1-\xi_{10})q^{2n^2+n}}{(1-\xi_{10}q^{2n})(1-q^{2n}/\xi_{10})}\\
&= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n(1+\xi_{10}^4)(1+\xi_{10}^6)q^{2n^2+n}}{(1+\xi_{10}^6q^{2n})(1+\xi_{10}^4q^{2n})}\\
&= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n(1+\xi_{10}^4)(1+\xi_{10}^6)q^{2n^2+n}(1+q^{2n})(1+\xi_{10}^2q^{2n})(1+\xi_{10}^8q^{2n})}{(1+\xi_{10}^6q^{2n})(1+\xi_{10}^4q^{2n})(1+q^{2n})(1+\xi_{10}^2q^{2n})(1+\xi_{10}^8q^{2n})}\\
&= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{2n^2+n}(1+q^{2n})}{1+q^{10n}}\\
&\quad+\frac{(\xi_{10}^2-\xi_{10}^3)(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{2n^2+n}(q^{4n}-1)}{1+q^{10n}}\\
&= F_1(q)+(\xi_{10}^2-\xi_{10}^3)F_2(q).
\end{aligned}$$

By the above Eq. (4.7), we have

$$\begin{aligned}
&\sum_{n\geq 0}(N_2(0,10,n)+N_2(1,10,n)-N_2(4,10,n)-N_2(5,10,n))\,q^n+(\xi_{10}^2-\xi_{10}^3)\\
&\quad\times\sum_{n\geq 0}(N_2(1,10,n)+N_2(2,10,n)-N_2(3,10,n)-N_2(4,10,n))\,q^n\\
&=F_1(q)+(\xi_{10}^2-\xi_{10}^3)F_2(q).
\end{aligned}\tag{4.8}$$

Since the coefficients of $F_1(q)$ and $F_2(q)$ are all integers and $[\mathbb{Q}(\xi_{10}):\mathbb{Q}]=4$, we equate the coefficient of ξ_{10}^k on both sides of (4.8) and find that

$$\sum_{n=0}^{\infty}(N_2(0,10,n)+N_2(1,10,n)-N_2(4,10,n)-N_2(5,10,n))q^n=F_1(q)$$

and

$$\sum_{n=0}^{\infty}(N_2(1,10,n)+N_2(2,10,n)-N_2(3,10,n)-N_2(4,10,n))q^n=F_2(q).$$

This completes the proofs of (1.5) and (1.7).

Next, we prove (1.6) and (1.8). Since we have

$$F_1(q)=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{2n^2+n}(1+q^{2n})}{1+q^{10n}},$$

substituting (4.1) and (4.2) into the series on the right side of the above equation, we find

$$\begin{aligned}F_1(q) &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left\{P_0 + \frac{(q^2;q^2)_\infty}{J_{25,100}(-q;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+25n}}{1+q^{50n+10}}\right\} \\ &\quad + \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left\{P_1 + \frac{(q^2;q^2)_\infty}{J_{25,100}(-q;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+75n+24}}{1+q^{50n+30}}\right\} \\ &= \frac{1}{J_{25,100}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+25n}}{1+q^{50n+10}} + \frac{1}{J_{25,100}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+75n+24}}{1+q^{50n+30}} \\ &\quad + \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}(P_0+P_1). \end{aligned} \tag{4.9}$$

We complete our proof of (1.6) after invoking Lemma 4.2. Similarly, since we have

$$F_2(q) = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{2n^2+n}(q^{4n}-1)}{1+q^{10n}},$$

substituting (4.2) and (4.3) into the series on the right side of the above equation, we find

$$\begin{aligned}F_2(q) &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left\{P_2 - \frac{(q^2;q^2)_\infty}{J_{25,100}(-q;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+25n-3}}{1+q^{50n}}\right\} \\ &\quad - \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left\{P_0 + \frac{(q^2;q^2)_\infty}{J_{25,100}(-q;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+25n}}{1+q^{50n+10}}\right\} \\ &= -\frac{1}{J_{25,100}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+25n-3}}{1+q^{50n}} - \frac{1}{J_{25,100}}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{50n^2+25n}}{1+q^{50n+10}} \\ &\quad + \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}(P_2-P_0). \end{aligned} \tag{4.10}$$

We complete our proof after invoking Lemma 4.3. □

5 Proof of Corollary 1.3

Sketch of the proof First, we consider (1.9)–(1.17) which can be proved by the generating functions in [15, Theorem 1.1, 1.2], Theorem 1.1, and 1.2. The proofs are all quite similar to each other. We illustrate our ideas by the proof of (1.9) and omit the proofs of the rest inequalities. By [15, Theorem 1.1], we have

$$\begin{aligned}\sum_{n\geq 0} &(N_2(0,3,3n+1)-N_2(1,3,3n+1))q^n\\ &=\frac{(-q^3,q^6;q^6)_\infty}{(q^2,q^4;q^6)_\infty}\\ &=\frac{(-q^3;q^6)_\infty}{(q^2;q^6)_\infty}\times\sum_{n\geq 0}\frac{q^{4n}(q^2;q^6)_n}{(q^6;q^6)_n}\quad\text{(by the q-binomial theorem)}\\ &=(-q^3;q^6)_\infty\times\sum_{n\geq 0}\frac{q^{4n}}{(q^{6n+2};q^6)_\infty\left(q^6;q^6\right)_n}.\end{aligned}$$

It is clear that a product of terms of the type $\frac{1}{1-q^m}$ has non-negative coefficients, and the term $\frac{1+q^3}{1-q^2}$ appearing in the last expression has positive coefficients of q^n for all $n\geq 2$. Inequality (1.9) follows.

Inequalities (1.18)–(1.23) are obtained from (1.9)–(1.17). Note that, for $0\leq t<l$, we have

$$N_2(t,l,n)=N_2(t,2l,n)+N_2(l-t,2l,n).$$

Substituting the above equation with the suitable t and l into the relations between M_2-ranks mod 3 and 5, by comparing the results with (1.14), (1.15), (1.16), and (1.17), we can prove rest of the inequalities. For example, since we have $N_2(0,3,3n+1)=N_2(0,6,3n+1)+N_2(3,6,3n+1)$ and $N_2(1,3,3n+1)=N_2(1,6,3n+1)+N_2(2,6,3n+1)$, substitute these two equalities into (1.9), we find

$$N_2(0,6,3n+1)+N_2(3,6,3n+1)\geq N_2(1,6,3n+1)+N_2(2,6,3n+1).$$

Adding this to (1.14), we get (1.18). We omit the proofs of the other inequalities since they are all similar to the above.

Appendix: Proofs of Lemmas 3.2, 4.2 and 4.3

First, we derive some theta identities. Let

$$H(a,b,c,q):=\frac{(ab,q/(ab),bc,q/(bc),ca,q/(ca);q)_\infty\,(q;q)_\infty^2}{(a,q/a,b,q/b,c,q/c,abc,q/(abc);q)_\infty}.$$

Then, we have the following equivalent version of Corollary 4.4 in [8]:

Lemma 6.1

$$H(a,b,c,q)-H(a,b,d,q)=H(c,1/d,abd,q), \tag{6.1}$$
$$H(a,a,q/a,q^2)+H(b,b,q/b,q^2)=2H(a,q/a,b,q^2), \tag{6.2}$$
$$H(a,a,q/a,q^2)-H(b,b,q/b,q^2)=2H(a,q/a,b/q,q^2). \tag{6.3}$$

We derive the following special cases. Replacing q by q^{50} and setting $(a, b, c, d) = (q^{10}, q^{25}, q^{10}, -q^{-5})$ in (6.1), we get

$$\frac{J_{15,50}^2 J_{20,50}}{J_{5,50} J_{10,50}^2 J_{25,50}} - \frac{q^5 J_{15,50}}{J_{10,50} J_{25,50}} = \frac{\overline{J}_{10,50} \overline{J}_{15,50} J_{15,50}}{J_{5,50} \overline{J}_{5,50} J_{10,50} \overline{J}_{20,50}}.$$

Multiplying by $\frac{J_{5,50} J_{10,50}^2 J_{25,50}}{J_{15,50}}$ throughout, the above equation becomes

$$J_{15,50} J_{20,50} - q^5 J_{5,50} J_{10,50} = \frac{J_{5,50} J_{20,50}^2 J_{25,50}}{J_{10,50} J_{15,50}}. \tag{6.4}$$

Replacing q by q^{50} and setting $(a, b) = (q^5, q^{15})$ in (6.3), we find

$$\frac{J_{10,100}}{J_{5,100}^2 J_{45,100}^2} - \frac{J_{30,100}}{J_{15,100}^2 J_{35,100}^2} = \frac{2q^5 J_{10,100} J_{30,100}}{J_{5,100} J_{15,100} J_{35,100} J_{45,100} J_{50,100}}. \tag{6.5}$$

Replacing q by q^{50} and setting $(a, b) = (q^{10}, q^{20})$ in (6.3), we find that

$$\frac{J_{20,100}}{J_{10,100}^2 J_{40,100}^2} - \frac{J_{40,100}}{J_{20,100}^2 J_{30,100}^2} = \frac{2q^{10}}{J_{30,100} J_{40,100} J_{50,100}}. \tag{6.6}$$

Next, we recall [9, Theorem 1.1]:

Lemma 6.2 *For five complex parameters A, b, c, d, e satisfying $A^2 = bced$, there holds the theta function identity*

$$[A/b, A/c, A/d, A/e; q]_\infty - [b, c, d, e; q]_\infty = b[A, A/bc, A/bd, A/be; q]_\infty. \tag{6.7}$$

We need the following special cases. Replacing q by q^{50} and setting $(A, b, c, d, e) = (-q^{25}, q^5, q^{10}, q^{15}, q^{20})$ in (6.7), we have

$$\overline{J}_{5,50} \overline{J}_{10,50} \overline{J}_{15,50} \overline{J}_{20,50} - J_{5,50} J_{10,50} J_{15,50} J_{20,50} = q^5 \overline{J}_{0,50} \overline{J}_{5,50} \overline{J}_{10,50} \overline{J}_{25,50}. \tag{6.8}$$

Now, we are ready to Lemma 3.2, 4.2, and 4.3, respectively.

Proof of Lemma 3.2 By (2.1), we know that

$$\frac{(q^2; q^2)_\infty}{(-q; q^2)_\infty} = \frac{J_{3,36} J_{15,36} J_{18,36}}{J_{6,36} J_{9,36}} - q J_{9,36}.$$

Expanding the right side and comparing both sides according to the powers of q modulo 3, we find that it suffices to prove the following three identities:

$$\frac{J_{3,36}^2 J_{6,36}^3 J_{12,36}^2 J_{15,36}^2 J_{18,36}^2}{2J_{36}^9} = \frac{J_{18,36}^3 J_{36}^3}{J_{9,36}^4} - \frac{J_{6,36} J_{18,36}^2 J_{36}^3}{2J_{3,36}^2 J_{15,36}^2}, \tag{6.9}$$

$$0 = q\frac{J_{6,36} J_{18,36}^2 J_{36}^3}{J_{3,36} J_{9,36}^2 J_{15,36}} - q\frac{J_{6,36} J_{18,36}^2 J_{36}^3}{J_{3,36} J_{9,36}^2 J_{15,36}}, \tag{6.10}$$

$$\frac{J_{3,36} J_{6,36}^3 J_{9,36}^2 J_{12,36}^2 J_{15,36} J_{18,36}^2}{2qJ_{36}^9} = \frac{J_{18,36}^3 J_{36}^3}{2qJ_{3,36} J_{9,36}^2 J_{15,36}} - q^2\frac{J_{6,36}^2 J_{18,36} J_{36}^3}{J_{3,36}^2 J_{15,36}^2}. \tag{6.11}$$

Simplifying, each of these three identities, we see that to prove (6.9), it suffices to show that

$$2\frac{J_{18,36}}{J_{9,36}^4} - \frac{J_{6,36}}{J_{3,36}^2 J_{15,36}^2} = \frac{J_{3,36}^2 J_{6,36}^3 J_{12,36}^2 J_{15,36}^2}{J_{36}^{12}}. \tag{6.12}$$

Noting that

$$\frac{J_{18,36} J_{36}^3}{J_{9,36}^4} = \frac{\overline{J}_{9,18}}{J_{9,18}},$$
$$\frac{J_{6,36} J_{36}^3}{J_{3,36}^2 J_{15,36}^2} = \frac{\overline{J}_{3,18}}{J_{3,18}}$$

and

$$\frac{J_{3,36}^2 J_{6,36}^3 J_{12,36}^2 J_{15,36}^2}{J_{36}^9} = \frac{J_{3,18} J_{6,18}^2}{\overline{J}_{3,18} \overline{J}_{6,18}^2},$$

we find that (6.12) is equivalent to

$$\frac{2\overline{J}_{9,18}}{J_{9,18}} - \frac{\overline{J}_{3,18}}{J_{3,18}} = \frac{J_{3,18} J_{6,18}^2}{\overline{J}_{3,18} \overline{J}_{6,18}^2}.$$

Multiplying by $\frac{J_{9,18}^2}{\overline{J}_{3,18} J_{3,18} J_{6,18}}$ throughout and rearranging, the above equation becomes

$$\frac{J_{9,18}^2 J_{6,18}}{J_{3,18}^2 J_{6,18}^2} + \frac{J_{9,18}^2 J_{6,18}}{\overline{J}_{3,18}^2 \overline{J}_{6,18}^2} = \frac{2J_{9,18} \overline{J}_{6,18} \overline{J}_{9,18}}{J_{3,18} J_{6,18} \overline{J}_{3,18} \overline{J}_{6,18}}.$$

This follows from replacing q, a, and b by q^9, q^3, and $-q^3$, respectively, in (6.2).

Equation (6.10) is trivial.

To prove (6.11), it suffices to show that

$$\frac{J_{18,36}^2}{J_{9,36}^2} - \frac{2q^3 J_{6,36}^2}{J_{3,36}J_{15,36}} = \frac{J_{3,36}^2 J_{6,36}^3 J_{9,36}^2 J_{12,36}^2 J_{15,36}^2 J_{18,36}}{J_{36}^{12}},$$

which is equivalent to

$$\frac{J_{6,12}^2}{J_{3,12}^2} - 2q\frac{J_{2,12}^2}{J_{1,12}J_{5,12}} = \frac{J_1^2 J_{2,12}}{J_{12}^3}. \tag{6.13}$$

Equation (6.13) follows from two identities:

$$\frac{J_{6,12}^2}{J_{3,12}^2} + q\frac{J_{2,12}^2}{J_{1,12}J_{5,12}} = \frac{J_{4,12}^2}{J_{1,12}J_{5,12}}, \tag{6.14}$$

$$\frac{J_{4,12}^2}{J_{1,12}J_{5,12}} - 3q\frac{J_{2,12}^2}{J_{1,12}J_{5,12}} = \frac{J_1^2 J_{2,12}}{J_{12}^3}. \tag{6.15}$$

Equation (6.14) follows from replacing q by q^{12} and setting $(A, b, c, d, e) = (q^9, q, q^5, q^6, q^6)$ in (6.7).

Next, we prove (6.15). Multiplying throughout by $\frac{J_{1,12}J_{4,12}J_{5,12}}{J_{12}}$, Eq. (6.15) becomes

$$\frac{J_{4,12}^3}{J_{12}} - 3q\frac{J_{2,12}^2 J_{4,12}}{J_{12}} = \frac{J_1^2 J_{1,12}J_{2,12}J_{4,12}J_{5,12}}{J_{12}^4},$$

which upon simplifying and rearranging gives,

$$\frac{J_1^3}{J_3} = \frac{J_4^3}{J_{12}} - 3q\frac{J_{12}^3 J_2^2}{J_4 J_6^2}.$$

Setting $b(q) = \frac{J_1^3}{J_3}$, we see that it suffices to show that

$$b(q) = b(q^4) - 3q\frac{J_{12}^3 J_2^2}{J_4 J_6^2},$$

which is proved in [13, Eq. (1.35)]. This completes the proof of Lemma 3.2. □

Sketch of the proof of Lemma 4.2 By (2.2), we know that

$$\frac{(q^2;q^2)_\infty}{(-q;q^2)_\infty} = \frac{J_{15,100}J_{20,100}J_{35,100}J_{50,100}}{J_{10,100}J_{25,100}J_{40,100}} - q\frac{J_{5,100}J_{40,100}J_{45,100}J_{50,100}}{J_{20,100}J_{25,100}J_{30,100}} - q^3 J_{25,100}.$$

Expanding the right side of (4.5) and comparing both sides according to the powers of q modulo 5, we find that it suffices to prove the following five identities:

$$\begin{aligned}
&\frac{J_{5,100}^2 J_{10,100} J_{20,100} J_{30,100}^2 J_{40,100} J_{45,100}^2 J_{50,100}^2}{2J_{100}^9} \\
&-\frac{J_{10,100}^2 J_{15,100}^2 J_{20,100} J_{30,100} J_{35,100}^2 J_{40,100} J_{50,100}^2}{2J_{100}^9} \\
&=\frac{J_{15,100} J_{20,100} J_{35,100} J_{50,100}}{J_{10,100} J_{25,100} J_{40,100}} \times \frac{2q^5 J_{10,100} J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^2 J_{20,100}^2 J_{25,100}^3 J_{30,100} J_{35,100}^2 J_{45,100}^3} \\
&-\frac{q J_{5,100} J_{40,100} J_{45,100} J_{50,100}}{J_{20,100} J_{25,100} J_{30,100}} \times \frac{2q^4 J_{30,100} J_{50,100} J_{100}^{15}}{J_{5,100}^2 J_{10,100} J_{15,100}^3 J_{25,100}^3 J_{35,100}^3 J_{40,100}^2 J_{45,100}^2} \\
&-q^3 J_{25,100} \times \frac{q^2 J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^3 J_{20,100} J_{25,100} J_{35,100}^3 J_{40,100} J_{45,100}^3},
\end{aligned} \tag{6.16}$$

$$\begin{aligned}
&\frac{q J_{5,100} J_{10,100}^2 J_{15,100} J_{30,100}^2 J_{35,100} J_{40,100}^2 J_{45,100} J_{50,100}}{J_{100}^9} \\
&=\frac{J_{15,100} J_{20,100} J_{35,100} J_{50,100}}{J_{10,100} J_{25,100} J_{40,100}} \times \frac{q J_{20,100} J_{30,100}^2 J_{50,100} J_{100}^{15}}{J_{5,100}^2 J_{10,100}^2 J_{15,100}^4 J_{25,100} J_{35,100}^4 J_{40,100}^3 J_{45,100}^2} \\
&-q\frac{J_{5,100} J_{40,100} J_{45,100} J_{50,100}}{J_{20,100} J_{25,100} J_{30,100}} \times \frac{2q^5 J_{10,100} J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^2 J_{20,100}^2 J_{25,100}^3 J_{30,100} J_{35,100}^2 J_{45,100}^3} \\
&-q^3 J_{25,100} \times \frac{q^3 J_{10,100}^2 J_{40,100} J_{50,100} J_{100}^{15}}{J_{5,100}^4 J_{15,100}^2 J_{20,100}^3 J_{25,100} J_{30,100}^2 J_{35,100}^2 J_{45,100}^4},
\end{aligned} \tag{6.17}$$

$$\begin{aligned}
0 &=\frac{J_{15,100} J_{20,100} J_{35,100} J_{50,100}}{J_{10,100} J_{25,100} J_{40,100}} \times \frac{q^2 J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^3 J_{20,100} J_{25,100} J_{35,100}^3 J_{40,100} J_{45,100}^3} \\
&-\frac{q J_{5,100} J_{40,100} J_{45,100} J_{50,100}}{J_{20,100} J_{25,100} J_{30,100}} \times \frac{q J_{20,100} J_{30,100}^2 J_{50,100} J_{100}^{15}}{J_{5,100}^2 J_{10,100}^2 J_{15,100}^4 J_{25,100} J_{35,100}^4 J_{40,100}^3 J_{45,100}^2} \\
&-q^3 J_{25,100} \times \frac{2q^4 J_{30,100} J_{50,100} J_{100}^{15}}{J_{5,100}^2 J_{10,100} J_{15,100}^3 J_{25,100}^3 J_{35,100}^3 J_{40,100}^2 J_{45,100}^2},
\end{aligned} \tag{6.18}$$

$$\begin{aligned}
0 &=\frac{J_{15,100} J_{20,100} J_{35,100} J_{50,100}}{J_{10,100} J_{25,100} J_{40,100}} \times \frac{q^3 J_{10,100}^2 J_{40,100} J_{50,100} J_{100}^{15}}{J_{5,100}^4 J_{15,100}^2 J_{20,100}^3 J_{25,100} J_{30,100}^2 J_{35,100}^2 J_{45,100}^4} \\
&-\frac{q J_{5,100} J_{40,100} J_{45,100} J_{50,100}}{J_{20,100} J_{25,100} J_{30,100}} \times \frac{q^2 J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^3 J_{20,100} J_{25,100} J_{35,100}^3 J_{40,100} J_{45,100}^3} \\
&-q^3 J_{25,100} \times \frac{2q^5 J_{10,100} J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^2 J_{20,100}^2 J_{25,100}^3 J_{30,100} J_{35,100}^2 J_{45,100}^3},
\end{aligned} \tag{6.19}$$

$$-\frac{q^9 J_{5,100} J_{10,100}^2 J_{15,100} J_{20,100}^2 J_{30,100}^2 J_{35,100} J_{45,100} J_{50,100}}{J_{100}^9}$$

$$
\begin{aligned}
&= \frac{J_{15,100}J_{20,100}J_{35,100}J_{50,100}}{J_{10,100}J_{25,100}J_{40,100}} \times \frac{2q^4 J_{30,100}J_{50,100}J_{100}^{15}}{J_{5,100}^2 J_{10,100}J_{15,100}^3 J_{25,100}^3 J_{35,100}^3 J_{40,100}^2 J_{45,100}^2} \\
&\quad - \frac{q J_{5,100}J_{40,100}J_{45,100}J_{50,100}}{J_{20,100}J_{25,100}J_{30,100}} \times \frac{q^3 J_{10,100}^2 J_{40,100}J_{50,100}J_{100}^{15}}{J_{5,100}^4 J_{15,100}^2 J_{20,100}^3 J_{25,100}J_{30,100}^2 J_{35,100}^2 J_{45,100}^4} \\
&\quad - q^3 J_{25,100} \times \frac{q J_{20,100}J_{30,100}^2 J_{50,100}J_{100}^{15}}{J_{5,100}^2 J_{10,100}^2 J_{15,100}^4 J_{25,100}J_{35,100}^4 J_{40,100}^3 J_{45,100}^2}. \qquad (6.20)
\end{aligned}
$$

The proof of the above five identities are quite similar to each other. We give the details of the proof of the first identity and omit the rest. After simplifying, we see that to prove (6.16), it suffices to show that

$$
\begin{aligned}
&\frac{J_{10,100}J_{20,100}J_{30,100}J_{40,100}J_{50,100}}{2J_{100}^{24}} \left\{J_{5,100}^2 J_{30,100}J_{45,100}^2 - J_{10,100}J_{15,100}^2 J_{35,100}^2\right\} \\
&\quad = \frac{2q^5 J_{50,100}}{J_{5,100}J_{15,100}J_{20,100}J_{25,100}^4 J_{35,100}J_{40,100}J_{45,100}} \\
&\qquad \times \left\{\frac{1}{J_{5,100}^2 J_{30,100}J_{45,100}^2} - \frac{1}{J_{10,100}J_{15,100}^2 J_{35,100}^2}\right\} \\
&\quad - \frac{q^5}{J_{5,100}^3 J_{15,100}^3 J_{20,100}J_{35,100}^3 J_{40,100}J_{45,100}^3}. \qquad (6.21)
\end{aligned}
$$

Multiplying by $J_{5,100}^2 J_{15,100}^2 J_{35,100}^2 J_{45,100}^2$ throughout, (6.5) becomes

$$
\begin{aligned}
&J_{5,100}^2 J_{30,100}J_{45,100}^2 - J_{10,100}J_{15,100}^2 J_{35,100}^2 \\
&\qquad = -\frac{2q^5 J_{5,100}J_{10,100}J_{15,100}J_{30,100}J_{35,100}J_{45,100}}{J_{50,100}}. \qquad (6.22)
\end{aligned}
$$

Next, we divide by $J_{10,100}J_{30,100}$ on both sides of (6.5) and get

$$
\frac{1}{J_{5,100}^2 J_{30,100}J_{45,100}^2} - \frac{1}{J_{10,100}J_{15,100}^2 J_{35,100}^2} = \frac{2q^5}{J_{5,100}J_{15,100}J_{35,100}J_{45,100}J_{50,100}}. \qquad (6.23)
$$

Substituting (6.22) and (6.23) into (6.21), we find that (6.21) is equivalent to

$$
\begin{aligned}
&-q^5 \frac{J_{5,100}J_{10,100}^2 J_{15,100}J_{20,100}J_{30,100}^2 J_{35,100}J_{40,100}J_{45,100}}{J_{100}^{24}} \\
&\quad = \frac{q^5}{J_{5,100}^2 J_{15,100}^2 J_{20,100}J_{35,100}^2 J_{40,100}J_{45,100}^2} \\
&\qquad \times \left\{\frac{4q^5}{J_{25,100}^4} - \frac{1}{J_{5,100}J_{15,100}J_{35,100}J_{45,100}}\right\}. \qquad (6.24)
\end{aligned}
$$

Multiplying by $\frac{1}{q^5} J_{5,100} J_{15,100} J_{20,100} J_{35,100} J_{40,100} J_{45,100} J_{100}^8$ throughout and noting that,

$$-\frac{J_{5,100}^2 J_{10,100}^2 J_{15,100}^2 J_{20,100}^2 J_{30,100}^2 J_{35,100}^2 J_{40,100}^2 J_{45,100}^2}{J_{100}^{16}} = -\frac{J_{5,25}^2 J_{10,25}^2}{J_{25}^4},$$

$$\frac{4q^5 J_{100}^8}{J_{5,100} J_{15,100} J_{25,100}^4 J_{35,100} J_{45,100}} = \frac{q^5 \overline{J}_{0,25}^2 \overline{J}_{5,25} \overline{J}_{10,25}}{J_{25}^4}$$

and

$$-\frac{J_{100}^8}{J_{5,100}^2 J_{15,100}^2 J_{35,100}^2 J_{45,100}^2} = -\frac{\overline{J}_{5,25}^2 \overline{J}_{10,25}^2}{J_{25}^4},$$

we find that (6.24) is equivalent to

$$-J_{5,25}^2 J_{10,25}^2 = q^5 \overline{J}_{0,25}^2 \overline{J}_{5,25} \overline{J}_{10,25} - \overline{J}_{5,25}^2 \overline{J}_{10,25}^2.$$

This follows from (6.8). So, we complete the proof of (6.16). □

Sketch of the proof of Lemma 4.3 Expanding the right side of (4.6) and comparing both sides according to the powers of q modulo 5, we find that it suffices to prove the five identities,

$$\begin{aligned}
&\frac{J_{10,100}^2 J_{15,100}^2 J_{20,100} J_{30,100} J_{35,100}^2 J_{40,100} J_{50,100}^2}{2J_{100}^9} \\
&= \left\{ \frac{J_{30,100} J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{10,100} J_{15,100}^3 J_{25,100} J_{35,100}^3 J_{40,100}^2 J_{45,100}^3} \right. \\
&\quad \left. -2q^5 \frac{J_{10,100} J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^2 J_{20,100}^2 J_{25,100}^3 J_{30,100} J_{35,100}^2 J_{45,100}^3} \right\} \\
&\quad \times \frac{J_{15,100} J_{20,100} J_{35,100} J_{50,100}}{J_{10,100} J_{25,100} J_{40,100}} - q^3 J_{25,100} \\
&\quad \times \left\{ \frac{2q^7 J_{10,100}^2 J_{40,100} J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^2 J_{20,100}^3 J_{25,100}^3 J_{30,100}^2 J_{35,100}^2 J_{45,100}^3} \right. \\
&\quad \left. + \frac{J_{10,100}^2 J_{15,100}^2 J_{20,100} J_{30,100} J_{35,100}^2 J_{40,100} J_{50,100}^2}{2q^3 J_{25,100} J_{100}^9} \right\} \\
&\quad - \frac{q J_{5,100} J_{40,100} J_{45,100} J_{50,100}}{J_{20,100} J_{25,100} J_{30,100}} \\
&\quad \times \frac{q^4 J_{10,100} J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^3 J_{20,100}^2 J_{25,100} J_{30,100} J_{35,100}^3 J_{45,100}^3},
\end{aligned} \tag{6.25}$$

$$
\begin{aligned}
&-\frac{q J_{5,100} J_{10,100}^2 J_{15,100} J_{30,100}^2 J_{35,100} J_{40,100}^2 J_{45,100} J_{50,100}}{J_{100}^9}\\
&\quad=\frac{J_{15,100} J_{20,100} J_{35,100} J_{50,100}}{J_{10,100} J_{25,100} J_{40,100}} \times \frac{2q^6 J_{50,100} J_{100}^{15}}{J_{5,100}^2 J_{15,100}^3 J_{20,100} J_{25,100}^3 J_{35,100}^3 J_{40,100} J_{45,100}^2}\\
&\qquad-\left\{\frac{J_{30,100} J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{10,100} J_{15,100}^3 J_{25,100} J_{35,100}^3 J_{40,100}^2 J_{45,100}^3}\right.\\
&\qquad\left.-\frac{2q^5 J_{10,100} J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^2 J_{20,100}^2 J_{25,100}^3 J_{30,100} J_{35,100}^2 J_{45,100}^3}\right\}\\
&\qquad\times\frac{q J_{5,100} J_{40,100} J_{45,100} J_{50,100}}{J_{20,100} J_{25,100} J_{30,100}} - q^3 J_{25,100}\\
&\qquad\times\frac{q^3 J_{30,100} J_{50,100} J_{100}^{15}}{J_{5,100}^2 J_{10,100} J_{15,100}^4 J_{25,100} J_{35,100}^4 J_{40,100}^2 J_{45,100}^2},
\end{aligned}
\tag{6.26}
$$

$$
\begin{aligned}
&\frac{J_{10,100}^3 J_{15,100} J_{25,100}^2 J_{35,100} J_{40,100}^3 J_{45,100}^3 J_{50,100}^2}{2q^3 J_{20,100} J_{100}^9}\\
&\quad=\left\{\frac{2q^7 J_{10,100}^2 J_{40,100} J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^2 J_{20,100}^3 J_{25,100}^3 J_{30,100}^2 J_{35,100}^2 J_{45,100}^3}\right.\\
&\qquad\left.+\frac{J_{10,100}^2 J_{15,100}^2 J_{20,100} J_{30,100} J_{35,100}^2 J_{40,100} J_{50,100}^2}{2q^3 J_{25,100} J_{100}^9}\right\}\\
&\qquad\times\frac{J_{15,100} J_{20,100} J_{35,100} J_{50,100}}{J_{10,100} J_{25,100} J_{40,100}} - q^3 J_{25,100}\\
&\qquad\times\frac{q^4 J_{10,100} J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^3 J_{20,100}^2 J_{25,100} J_{30,100} J_{35,100}^3 J_{45,100}^3}\\
&\qquad-\frac{q J_{5,100} J_{40,100} J_{45,100} J_{50,100}}{J_{20,100} J_{25,100} J_{30,100}} \times \frac{2q^6 J_{50,100} J_{100}^{15}}{J_{5,100}^2 J_{15,100}^3 J_{20,100} J_{25,100}^3 J_{35,100}^3 J_{40,100} J_{45,100}^2},
\end{aligned}
\tag{6.27}
$$

$$
\begin{aligned}
&-\frac{J_{5,100} J_{20,100}^3 J_{25,100}^2 J_{30,100}^3 J_{45,100} J_{50,100}^2}{2q^2 J_{40,100} J_{100}^9}\\
&\quad=\frac{J_{15,100} J_{20,100} J_{35,100} J_{50,100}}{J_{10,100} J_{25,100} J_{40,100}} \times \frac{q^3 J_{30,100} J_{50,100} J_{100}^{15}}{J_{5,100}^2 J_{10,100} J_{15,100}^4 J_{25,100} J_{35,100}^4 J_{40,100}^2 J_{45,100}^2}\\
&\qquad-\left\{\frac{2q^7 J_{10,100}^2 J_{40,100} J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^2 J_{20,100}^3 J_{25,100}^3 J_{30,100}^2 J_{35,100}^2 J_{45,100}^3}\right.\\
&\qquad\left.+\frac{J_{10,100}^2 J_{15,100}^2 J_{20,100} J_{30,100} J_{35,100}^2 J_{40,100} J_{50,100}^2}{2q^3 J_{25,100} J_{100}^9}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&\times q \frac{J_{5,100}J_{40,100}J_{45,100}J_{50,100}}{J_{20,100}J_{25,100}J_{30,100}} - q^3 J_{25,100} \\
&\times \left\{ \frac{J_{30,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3 J_{10,100} J_{15,100}^3 J_{25,100} J_{35,100}^3 J_{40,100}^2 J_{45,100}^3} \right. \\
&\left. - \frac{2q^5 J_{10,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3 J_{15,100}^2 J_{20,100}^2 J_{25,100}^3 J_{30,100} J_{35,100}^2 J_{45,100}^3} \right\}, \qquad (6.28)
\end{aligned}
$$

$$
\begin{aligned}
0 = {} & \frac{J_{15,100}J_{20,100}J_{35,100}J_{50,100}}{J_{10,100}J_{25,100}J_{40,100}} \times \frac{q^4 J_{10,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3 J_{15,100}^3 J_{20,100}^2 J_{25,100} J_{30,100} J_{35,100}^3 J_{45,100}^3} \\
& - q \frac{J_{5,100}J_{40,100}J_{45,100}J_{50,100}}{J_{20,100}J_{25,100}J_{30,100}} \times \frac{q^3 J_{30,100}J_{50,100}J_{100}^{15}}{J_{5,100}^2 J_{10,100} J_{15,100}^4 J_{25,100} J_{35,100}^4 J_{40,100}^2 J_{45,100}^2} \\
& - q^3 J_{25,100} \times \frac{2q^6 J_{50,100}J_{100}^{15}}{J_{5,100}^2 J_{15,100}^3 J_{20,100} J_{25,100}^3 J_{35,100}^3 J_{40,100} J_{45,100}^2}. \qquad (6.29)
\end{aligned}
$$

The proof of the above five identities are quite similar to each other. We give the details of the proof of the first identity and omit the rest. After simplifying, we find that, to prove (6.25), it suffices to show that

$$
\begin{aligned}
&\frac{J_{10,100}^2 J_{15,100}^2 J_{20,100} J_{30,100} J_{35,100}^2 J_{40,100} J_{50,100}^2}{J_{100}^9} \\
&= \left\{ \frac{J_{20,100}J_{30,100}J_{50,100}^2 J_{100}^{15}}{J_{5,100}^3 J_{10,100}^2 J_{15,100}^2 J_{25,100}^2 J_{35,100}^2 J_{40,100}^3 J_{45,100}^3} \right. \\
&\quad \left. - \frac{2q^5 J_{50,100}^2 J_{100}^{15}}{J_{5,100}^3 J_{15,100} J_{20,100} J_{25,100}^4 J_{30,100} J_{35,100} J_{40,100} J_{45,100}^3} \right\} \\
&\quad - \left\{ \frac{q^5 J_{10,100}J_{40,100}J_{50,100}^2 J_{100}^{15}}{J_{5,100}^2 J_{15,100}^3 J_{20,100}^3 J_{25,100}^2 J_{30,100}^2 J_{35,100}^3 J_{45,100}^2} \right. \\
&\quad \left. + \frac{2q^{10} J_{10,100}^2 J_{40,100} J_{50,100} J_{100}^{15}}{J_{5,100}^3 J_{15,100}^2 J_{20,100}^3 J_{25,100}^2 J_{30,100}^2 J_{35,100}^2 J_{45,100}^3} \right\}. \qquad (6.30)
\end{aligned}
$$

Replacing q by $q^{1/2}$ in (6.6), we have

$$
\frac{J_{10,50}}{J_{5,50}^2 J_{20,50}^2} - \frac{J_{20,50}}{J_{10,50}^2 J_{15,50}^2} = \frac{2q^5}{J_{15,50}J_{20,50}J_{25,50}}.
$$

Noting that $\frac{J_{k,50}}{J_{50}} = \frac{J_{k,100}J_{50-k,100}}{J_{100}^2}$ and rearranging, the above equation gives

$$
\frac{J_{10,100}J_{40,100}}{J_{5,100}^2 J_{20,100}^2 J_{30,100}^2 J_{45,100}^2} = \frac{J_{20,100}J_{30,100}}{J_{10,100}^2 J_{15,100}^2 J_{35,100}^2 J_{40,100}^2}
$$

$$+\frac{2q^5}{J_{15,100}J_{20,100}J_{25,100}^2J_{30,100}J_{35,100}}. \tag{6.31}$$

Multiplying by $\frac{J_{50,100}^2J_{100}^{15}}{J_{5,100}^3J_{25,100}^2J_{40,100}J_{45,100}^3}$ throughout and subtracting

$$\frac{4q^5J_{50,100}^2J_{100}^{15}}{J_{5,100}^3J_{15,100}J_{20,100}J_{25,100}^4J_{30,100}J_{35,100}J_{40,100}J_{45,100}^3}$$

on both sides, (6.31) becomes

$$\begin{aligned}&\frac{J_{20,100}J_{30,100}J_{50,100}^2J_{100}^{15}}{J_{5,100}^3J_{10,100}^2J_{15,100}^2J_{25,100}^2J_{35,100}^2J_{40,100}^3J_{45,100}^3}\\&\quad-\frac{2q^5J_{50,100}^2J_{100}^{15}}{J_{5,100}^3J_{15,100}J_{20,100}J_{25,100}^4J_{30,100}J_{35,100}J_{40,100}J_{45,100}^3}\\&=\frac{J_{10,100}J_{50,100}^2J_{100}^{15}}{J_{5,100}^5J_{20,100}^2J_{25,100}^2J_{30,100}^2J_{45,100}^5}\\&\quad-\frac{4q^5J_{50,100}^2J_{100}^{15}}{J_{5,100}^3J_{15,100}J_{20,100}J_{25,100}^4J_{30,100}J_{35,100}J_{40,100}J_{45,100}^3}.\end{aligned} \tag{6.32}$$

Next, multiplying by $\frac{q^5J_{10,100}J_{40,100}J_{50,100}^2J_{100}^{15}}{J_{5,100}^2J_{15,100}J_{20,100}^3J_{25,100}^2J_{30,100}^3J_{35,100}J_{45,100}^2}$ on the both sides of (6.5) and rearranging, we get

$$\begin{aligned}&\frac{q^5J_{10,100}J_{40,100}J_{50,100}^2J_{100}^{15}}{J_{5,100}^2J_{15,100}^3J_{20,100}^3J_{25,100}^2J_{30,100}^2J_{35,100}^3J_{45,100}^2}\\&\quad+\frac{2q^{10}J_{10,100}^2J_{40,100}J_{50,100}J_{100}^{15}}{J_{5,100}^3J_{15,100}^2J_{20,100}^3J_{25,100}^2J_{30,100}^2J_{35,100}^2J_{45,100}^3}\\&=\frac{q^5J_{10,100}^2J_{40,100}J_{50,100}^2J_{100}^{15}}{J_{5,100}^4J_{15,100}J_{20,100}^3J_{25,100}^2J_{30,100}^3J_{35,100}J_{45,100}^4}.\end{aligned} \tag{6.33}$$

Substituting (6.32) and (6.33) into (6.30), we find that (6.30) is equivalent to

$$\begin{aligned}&\frac{J_{10,100}^2J_{15,100}^2J_{20,100}J_{30,100}J_{35,100}^2J_{40,100}J_{50,100}^2}{J_{100}^9}\\&=\frac{J_{10,100}J_{50,100}^2J_{100}^{15}}{J_{5,100}^5J_{20,100}^2J_{25,100}^2J_{30,100}^2J_{45,100}^5}\\&\quad-\frac{4q^5J_{50,100}^2J_{100}^{15}}{J_{5,100}^3J_{15,100}J_{20,100}J_{25,100}^4J_{30,100}J_{35,100}J_{40,100}J_{45,100}^3}\end{aligned}$$

$$
-\frac{q^5 J_{10,100}^2 J_{40,100} J_{50,100}^2 J_{100}^{15}}{J_{5,100}^4 J_{15,100} J_{20,100}^3 J_{25,100}^2 J_{30,100}^3 J_{35,100} J_{45,100}^4}. \tag{6.34}
$$

This follows from the two identities,

$$
\begin{aligned}
&\frac{J_{50,100}^2 J_{100}^{15}}{J_{5,100}^4 J_{15,100}^2 J_{20,100} J_{30,100} J_{35,100}^2 J_{40,100} J_{45,100}^4} \\
&\quad -\frac{4q^5 J_{50,100}^2 J_{100}^{15}}{J_{5,100}^3 J_{15,100} J_{20,100} J_{25,100}^4 J_{30,100} J_{35,100} J_{40,100} J_{45,100}^3} \\
&= \frac{J_{10,100}^2 J_{15,100}^2 J_{20,100} J_{30,100} J_{35,100}^2 J_{40,100} J_{50,100}^2}{J_{100}^9}
\end{aligned} \tag{6.35}
$$

and

$$
\begin{aligned}
&\frac{J_{10,100} J_{50,100}^2 J_{100}^{15}}{J_{5,100}^5 J_{20,100}^2 J_{25,100}^2 J_{30,100}^2 J_{45,100}^5} - \frac{q^5 J_{10,100}^2 J_{40,100} J_{50,100}^2 J_{100}^{15}}{J_{5,100}^4 J_{15,100} J_{20,100}^3 J_{25,100}^2 J_{30,100}^3 J_{35,100} J_{45,100}^4} \\
&= \frac{J_{50,100}^2 J_{100}^{15}}{J_{5,100}^4 J_{15,100}^2 J_{20,100} J_{30,100} J_{35,100}^2 J_{40,100} J_{45,100}^4}.
\end{aligned} \tag{6.36}
$$

We obtain (6.35) after multiplying by $\frac{J_{15,100} J_{35,100} J_{50,100}^2 J_{100}^{15}}{q^5 J_{5,100} J_{30,100} J_{45,100}}$ on both sides of (6.24).

Next, we prove (6.36). Noting that $\frac{J_{k,50}}{J_{50}} = \frac{J_{k,100} J_{50-k,100}}{J_{100}^2}$, we find that (6.4) is equivalent to

$$
\begin{aligned}
&J_{15,100} J_{20,100} J_{30,100} J_{35,100} - q^5 J_{5,100} J_{10,100} J_{40,100} J_{45,100} \\
&= \frac{J_{5,100} J_{20,100}^2 J_{25,100}^2 J_{30,100}^2 J_{45,100}}{J_{10,100} J_{15,100} J_{35,100} J_{40,100}}.
\end{aligned} \tag{6.37}
$$

Multiplying by $\frac{J_{10,100} J_{50,100}^2 J_{100}^{15}}{J_{5,100}^5 J_{15,100} J_{20,100}^3 J_{25,100}^2 J_{30,100}^3 J_{35,100} J_{45,100}^5}$ on both sides of the above equation, we recover (6.36). This completes our proof of (6.25). □

References

1. Andrews, G.E.: The Theory of Partitions, the Encyclopedia of Mathematics and Its Applications Series, p. 300. Addison-Wesley, New York (1976)
2. Andrews, G.E., Berndt, B.C.: Ramanujan's Lost Notebook Part I. Springer, New York (2005)
3. Andrews, G.E., Lewis, R.: The ranks and cranks of partitions moduli 2, 3, and 4. J. Number Theory **85**(1), 74–84 (2000)
4. Atkin, A.O.L., Swinnerton-Dyer, H.P.F.: Some properties of partitions. Proc. Lond. Math. Soc. **4**, 84–106 (1954)
5. Berkovich, A., Garvan, F.G.: Some observations on Dyson's new symmetries of partitions. J. Combin. Theory Ser. A **100**, 61–93 (2002)

6. Berndt, B.C.: Ramanujan's Notebooks. Part III. Springer, New York (1991)
7. Chan, S.H.: Generalized Lambert series identities. Proc. Lond. Math. Soc. **91**(3), 598–622 (2005)
8. Chan, S.H.: Congruences for Ramanujan's ϕ function. Acta Arith. **153**(2), 161–189 (2012)
9. Chu, W.: Theta function identities and Ramanujan's congruences on the partition function. Q. J. Math. **56**(4), 491–506 (2005)
10. Dyson, F.J.: Some guess in the theory of partitions. Eureka (Cambridge) **8**, 10–15 (1944)
11. Garvan, F.G.: New combinatorial interpretations of Ramanujan's partition congruences mod 5, 7 and 11. Trans. Am. Math. Soc. **305**(1), 47–77 (1988)
12. Gordon, B., McIntosh, R.: Some eighth order mock theta functions. J. Lond. Math. Soc. **62**, 321–335 (2000)
13. Hirschhorn, M.D., Garvan, F., Borwein, J.: Cubic analogues of the Jacobian theta function $\theta(z, q)$. Can. J. Math. **45**(4), 673–694 (1993)
14. Bringmann, K., Kane, B.: Inequalities for differences of Dyson's rank for all odd moduli. Math. Res. Lett. **17**(5), 927–942 (2010)
15. Lovejoy, J., Osburn, R.: M_2-rank differences for partitions without repeated odd parts. J. Theor. Nombres Bordeaux **21**(2), 313–334 (2009)
16. Mao, R.: Rank of partitions mod 10, submitted
17. McIntosh, R.J.: Second order mock theta functions. Can. Math. Bull. **50**(2), 284–290 (2007)