Outline

Given a resource semiring $\mathbf{R}$ with elements $s, t, u, \dots \in R$, define the following equivalence relation on the set $R^T := \{0_{\mathbf{R}}\} \cup \mu X.(R \setminus \{0_{\mathbf{R}}\}) + X \times X$:

$$\tau \sim \rho \iff \begin{cases} \tau = s,\ \rho = t \text{ and } s =_R t \\ \tau = s,\ \rho = (\rho, \rho'') \text{ and } \tau \sim \rho' \text{ and } \tau \sim \rho'' \\ \tau = (\tau', \tau''),\ \rho = t \text{ and } \tau' \sim \rho \text{ and } \tau'' \sim \rho \\ \tau = (\tau', \tau''),\ \rho = (\rho', \rho'') \text{ and } \tau' \sim \rho' \text{ and } \tau'' \sim \rho'' \end{cases}$$

Claim. *$R^T/\sim$ can be endowed with usage semiring structure.*

Adjust the $R{-}LCA$ to account for the tree structure by augmenting the operation $!_\tau : \mathscr{A} \to \mathscr{A}$ and the elements $W_{\pi\rho}$, F_ρ, $\delta_{\pi\rho}$:

$$!^T_\tau x := \begin{cases} [\ulcorner true \urcorner, !_s x] & \text{if } s = contract(\tau) \\ [\ulcorner false \urcorner, E(!^T_{\tau'} x, !^T_{\tau''} x)], & \text{if } \langle \tau', \tau'' \rangle = contract(\tau) \end{cases}$$

The outermost pairing with Bool allows us to inspect/ the shape of the resource annotation internally within the BCI algebra.
Futhermore, it is also possible to define $W_{\pi\rho}$, F_ρ, $\delta_{\pi\rho}$, but that involves quite a few subtleties.

As an example usage of tree resource annotation, let's define the semantics of $\oplus$-types and check the rules are sound wrt. to it.
$[\![(A \oplus B)(\gamma)]\!] := (|A(y)| \sqcup |B(y)|, \vDash_{A \oplus B(\gamma)})$, where

$$a \vDash_{A \oplus B(\gamma)} (i, x) \text{ iff } \begin{array}{l} (\exists\, a_x \in \mathscr{A}. a = [\ulcorner true \urcorner, a_x] \wedge a_x \vDash_{A(\gamma)} x \wedge \ulcorner true \urcorner \vDash_{Bool(\gamma)} i) \vee \\ (\exists\, b_x \in \mathscr{A}. a = [\ulcorner false \urcorner, b_x] \wedge b_x \vDash_{B(\gamma)} x \wedge \ulcorner false \urcorner \vDash_{Bool(\gamma)} i) \end{array}$$

$$\frac{0\Gamma \vdash A \quad 0\Gamma \vdash B}{0\Gamma \vdash A \oplus B}\ \oplus\text{-type} \qquad \frac{\Gamma \vdash M \overset{\sigma}{:} A}{\Gamma \vdash \mathbf{inl}\, M \overset{\sigma}{:} A \oplus B}\ \text{inl} \qquad \frac{\Gamma \vdash N \overset{\sigma}{:} B}{\Gamma \vdash \mathbf{inr}\, N \overset{\sigma}{:} A \oplus B}\ \text{inr}$$

$$\frac{\begin{array}{c} 0\Gamma, x \overset{0}{:} A \oplus B \vdash C \\ \Gamma \vdash M \overset{\sigma}{:} A \oplus B \qquad \Gamma', a \overset{\rho}{:} A \vdash T_a \overset{\sigma}{:} C[\mathbf{inl}\, a/x] \qquad \Gamma', b \overset{\pi}{:} B \vdash T_b \overset{\sigma}{:} C[\mathbf{inr}\, b/x] \qquad 0\Gamma = 0\Gamma' \end{array}}{\Gamma' + \langle \rho, \pi \rangle \Gamma \vdash \mathbf{case}(M, T_a, T_b) \overset{\sigma}{:} C[M/x]}\ \oplus\text{-elim}$$

Figure 1: Rules for $\oplus$-types

WLOG, let's verify the soundness of **inl** and **case**.
Given a realiser a_M for $\forall \gamma \in |\Gamma|. M(\gamma)$, let $a_{\mathbf{inl}} := \lambda^* a_\gamma. [\ulcorner true \urcorner, a_M \cdot a_\gamma]$

Given realisers a_M, a_T and b_T, s.t:
$\forall \gamma \in |\Gamma|, a_\gamma \in \mathscr{A}.a_\gamma \vDash_\Gamma y \implies a_M \cdot a_y \vDash_{A \oplus B(\gamma)} M(\gamma)$,
$\forall \gamma' \in |\Gamma'|, s \in |A(\gamma')|, a'_\gamma \in \mathscr{A}, a_s \in \mathscr{A}.a'_\gamma \vDash_\Gamma \gamma' \wedge a_s \vDash_{A(\gamma)} s \implies a_T \cdot [a'_\gamma, !_\rho a_s] \vDash_{C(\gamma', inl(s))} T_a(\gamma', s)$,
$\forall \gamma' \in |\Gamma'|, t \in |B(\gamma')|, a'_\gamma \in \mathscr{A}, a_t \in \mathscr{A}.a'_\gamma \vDash_\Gamma \gamma' \wedge a_t \vDash_{B(\gamma)} t \implies b_T \cdot [a'_\gamma, !_\pi a_t] \vDash_{C(\gamma', inr(t))} T_b(\gamma', t)$,

let $a_{case} :=$
$$W_{1\langle\rho,\pi\rangle} \cdot (\lambda^* a'_{!\gamma}\ a_{!\gamma}. \qquad -\ \textit{unwrap}\ (1 + \langle\rho,\pi\rangle)\Gamma$$
$$\ldots F_{\langle\pi,\rho\rangle} \cdot !_{\langle\pi,\rho\rangle}(\lambda^* r.a_M \cdot r).a_{!\gamma} \ldots$$

Problem : We end up with something of the shape $!_{\langle\pi,\rho\rangle}([\ulcorner boolean \urcorner, x])$ and want to select the projection depending on the inner boolean value.