

# Relational realizability model for QTT

Our aim is to build a concrete realizability model for QTT which allows for metareasoning with results derived from parametricity. To that end, a construction of a quantified category with families (QCwF[**fill**]) is presented. We follow the relational approach to types introduced by Reynolds for typed lambda calculus[**fill**] and later refined for dependent types theories.[**fill**] Once and for all fix a usage semiring  $R$  and an  $R$ -linear combinatory algebra  $\mathcal{A}$ <sup>1</sup>.

## Taking stocks

**Definition 1** (Assembly). An assembly  $\Gamma$  is a pair  $(|\Gamma|, e)$  where  $|\Gamma|$  is a carrier set and  $e$  is a realizability function  $|\Gamma| \rightarrow \mathcal{P}(\mathcal{A})$ , s.t  $e(\gamma)$  is nonempty for every  $\gamma \in |\Gamma|$ .

We interpret  $E(\gamma)$  as a set of witnesses for the existence of  $\gamma$  and write  $a \Vdash_{\Gamma} \gamma$  to denote  $a \in e(\gamma)$ .

A morphism between two assemblies  $(|\Gamma|, E_{\Gamma})$  and  $(|\Delta|, E_{\Delta})$  is a function  $f : |\Gamma| \rightarrow |\Delta|$  that is realizable - there exists  $a_f \in \mathcal{A}$ , s.t the following holds for every  $\gamma \in |\Gamma|$  and  $a_{\gamma} \in \mathcal{A}$ :

$$a_{\gamma} \Vdash_{\Gamma} \gamma \implies a_f.a_{\gamma} \Vdash_{\Delta} f(\gamma)$$

We say that  $a_f$  tracks  $f$ . Note that only an existence of a realizer for  $f$  is stipulated in the definition - multiple realizers do not induce multiple morphisms.

Using these notions we can construct a category  $\mathcal{A}sm(\mathcal{A})$ .

**Definition 2** (Reflexive graph). A reflexive graph (r.g.)  $G$  is a tuple  $(G_O, G_R, G_{refl}, G_{src}, G_{tgt})$ , where  $G_O$  and  $G_R$  are sets,  $G_{src}$  and  $G_{tgt}$  are functions  $G_R \rightarrow G_O$  and  $G_{refl}$  is a function  $G_O \rightarrow G_R$ , s.t  $G_{src} \circ G_{refl} = id_{G_O} = G_{tgt} \circ G_{refl}$ .

$G_O$  and  $G_R$  stand for a set of objects and a set of relations, respectively. In general, reflexive graphs are less structured than categories as relations need not compose.

A morphism between reflexive graphs  $G$  and  $H$  is a pair of functions  $(f_o : G_O \rightarrow H_O, f_r : G_R \rightarrow H_R)$ , s.t all of the depicted squares commute:

$$\begin{array}{ccc} G_O & \xrightarrow{f_o} & H_O \\ \begin{array}{c} \uparrow | \downarrow \\ G_{src} \left( G_{refl} \right) G_{tgt} \end{array} & & \begin{array}{c} \uparrow | \downarrow \\ H_{src} \left( H_{refl} \right) H_{tgt} \end{array} \\ G_R & \xrightarrow{f_r} & H_R \end{array}$$

Reflexive graphs equipped with r.g. morphisms form a category  $\mathcal{RGph}(\mathcal{Set})$ .

We use reflexive graphs to give a dyadic interpretation of types in the spirit of [**fill**].

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<sup>1</sup>In case some non-trivial properties of  $\mathcal{A}$  are required, we will tacitly assume that  $\mathcal{A}$  is a graph model(see [**fill**]) - also to fix

## Assembling a reflexive graph

One could easily generalize reflexive graphs by considering object and relation components from an arbitrary category  $\mathcal{C}$  instead of  $\mathcal{Set}$ . As our purpose is to build a realizability model, we pick  $\mathcal{Asm}(\mathcal{A})$  as the category of interest and identify two appropriate notions of a family of reflexive graphs of assemblies and internal r.g. in  $\mathcal{Asm}(\mathcal{A})$ .

**Definition 3** (Internal reflexive graph in  $\mathcal{Asm}(\mathcal{A})$ ). An internal reflexive graph  $G$  in  $\mathcal{Asm}(\mathcal{A})$  is a pair of assemblies  $(G_O, G_R)$  and a triple of morphisms  $(G_{refl} : G_O \rightarrow G_R, G_{src} : G_R \rightarrow G_O, G_{tgt} : G_R \rightarrow G_O)$ , such that the identities in Definition 2 are satisfied.

In essence, sets and functions are directly replaced by the objects and morphisms from  $\mathcal{Asm}(\mathcal{A})$ . The same transformation is carried out when defining morphisms between internal reflexive graphs as well. With these components, we obtain a category  $\mathcal{RGph}(\mathcal{Asm}(\mathcal{A}))$ . By considering internal r.g. of shape  $(X, X, id_X, id_X, id_X)$ , we identify isomorphic copy of  $\mathcal{Asm}(\mathcal{A})$  inside  $\mathcal{RGph}(\mathcal{Asm}(\mathcal{A}))$ .

A terminal object in  $\mathcal{RGph}(\mathcal{Asm}(\mathcal{A}))$  is a tuple  $(1, 1, id, id, id)$ , where 1 is the terminal assembly  $(\{\star\}, f)$ , with  $f$  defined as  $\star \mapsto \{I\}$ .

**Definition 4** (Family of reflexive graphs of assemblies). Let  $\mathcal{C}$  be a category with a terminal object. Given a reflexive graph  $\Gamma \in Ob(\mathcal{C})$ , a family of internal r.g. over  $\Gamma$  is a tuple  $S := (S_O, S_R, S_{refl}, S_{src}, S_{tgt})$ , where:

- $S_O : \Gamma_O \rightarrow \mathcal{Asm}(\mathcal{A})$
- $S_R : \Gamma_R \rightarrow \mathcal{Asm}(\mathcal{A})$
- a  $\Gamma$ -indexed collection of morphisms  $S_{refl} := \{f : S_O(\gamma) \rightarrow S_R(\Gamma_{refl}(\gamma))\}_{\gamma \in \Gamma_O}$
- $S_{src} := \{f : S_R(\gamma) \rightarrow S_O(\Gamma_{src}(\gamma))\}_{\gamma \in \Gamma_R}$
- $S_{tgt} := \{f : S_R(\gamma) \rightarrow S_O(\Gamma_{tgt}(\gamma))\}_{\gamma \in \Gamma_R}$

such that

- each identity in the following collection is satisfied:

$$S_\sigma(\Gamma_{refl}(\gamma)) \circ S_{refl}(\gamma) = id \text{ for every } \gamma \in \Gamma_O, \sigma \in \{src, tgt\}$$

- there exists an element  $a_\sigma \in \mathcal{A}$  which tracks each morphism in the collection  $S_\sigma$  for  $\sigma \in \{src, tgt, refl\}$

We are only interested in cases when  $\mathcal{C} = \mathcal{Set}$  or  $\mathcal{C} = \mathcal{Asm}(\mathcal{A})$ .

A morphism  $M$  between two families  $S$  and  $T$  of internal r.g. over  $\Gamma$  is a pair of  $\Gamma$ -indexed collection of functions:

- $M_O := \{f : |S_O(\gamma)| \rightarrow |T_O(\gamma)|\}_{\gamma \in \Gamma_O}$
- $M_R := \{f : |S_R(\gamma)| \rightarrow |T_R(\gamma)|\}_{\gamma \in \Gamma_O}$

such that:

- $\check{T}_{refl}(M_O(\gamma)(x)) = M_R(\Gamma_{refl}(\gamma))(|S_{refl}(\gamma)|(x))$  for every  $\gamma \in \Gamma_O$ ,  $x \in S_O(\gamma)$
- $\check{T}_{src}(M_R(\gamma)(r)) = M_O(\Gamma_{src}(\gamma))(|S_{src}(\gamma)|(r))$  for every  $\gamma \in \Gamma_R$ ,  $r \in S_R(\gamma)$
- $\check{T}_{tgt}(M_R(\gamma)(r)) = M_O(\Gamma_{tgt}(\gamma))(|S_{tgt}(\gamma)|(r))$  for every  $\gamma \in \Gamma_R$ ,  $r \in S_R(\gamma)$

where given a morphism  $f : A \rightarrow B$  between assemblies,  $\check{f}$  is the underlying set-theoretic function.

A terminal family of r.g. over  $\Gamma$ ,  $1_\Gamma$ , consists of two constant functions, mapping  $\gamma \in \Gamma$  to a terminal assembly 1, and three  $\Gamma$ -indexed collections with a sole element  $id_1$ .

There is an apparent discrepancy in definition of family of r.g of assemblies and morphisms between them - morphisms ignore any realizability information. This is due to the role they play in interpretation of QTT - semantic terms in the 0-fragment of the theory are interpreted as morphisms from the terminal family and hence do not require any realizers.

It is then tempting to continue discarding realizability information in Definition 4. However, as families of r.g of assemblies will model the semantic types in QCwF, the approach seems infeasible.

- types provide realizability blueprint for resourced semantic terms. Thus we cannot substitute  $Set$  for  $Asm(\mathcal{A})$ .
- given a context  $\Gamma \in Ob(\mathcal{RGph})$ ,  $\rho \in R$  and a type  $S \in Ty(\Gamma)$ , the resourced context extension  $\Gamma.\rho S \in Ob(\mathcal{RGph}(Asm(\mathcal{A})))$  makes use of the trackers for  $src, tgt$  and  $refl$  morphisms. (Note that these are not needed when building an ordinary CwF).

## A CwF from families of reflexive graphs of assemblies

As a first step toward obtaining a relationally parametric realizability model of QTT, we construct a concrete CwF using families of internal graphs of assemblies.

Consider the category  $\mathcal{RGph}$  with terminal object  $1 := (\{\star\}, \{\star\}, id, id, id)$ .

Let  $\Gamma, \Delta \in Ob(\mathcal{RGph})$ , define:

- the collection of semantic types  $Ty(\Gamma)$  as the collection of families of internal r. g. over  $\Gamma$ .
- given a type  $S \in Ty(\Gamma)$ , let  $Tm(\Gamma, S) := Hom(1_\Gamma, S)$ .  
Spelling this out and ignoring the contribution of the terminal family, we get:  
An element  $M \in Tm(\Gamma, S)$  is a pair of functions  $(M_O : \forall \gamma \in \Gamma_O. S_O(\gamma), M_R : \forall \gamma \in \Gamma_R. S_R(\gamma))$ , such that

$$\begin{aligned} \forall \gamma \in \Gamma_O. \check{S}_{refl}(M_O(\gamma)) &= M_R(\Gamma_{refl}(\gamma)) \\ \forall \gamma \in \Gamma_R. \check{S}_{src}(M_R(\gamma)) &= M_O(\Gamma_{src}(\gamma)) \\ \forall \gamma \in \Gamma_R. \check{S}_{tgt}(M_R(\gamma)) &= M_O(\Gamma_{tgt}(\gamma)) \end{aligned}$$

- given  $f : \Gamma \rightarrow \Delta$ , substitutions in types and terms is a precomposition with  $f$  on the object and relation components of types and terms respectively. Clearly, these operations are compatible with identity and composition in  $\mathcal{RGph}$
- context extension : Suppose  $S \in Ty(\Gamma)$ , construct a r.g.  $\Gamma.S$  as :

$$\begin{aligned}
(\Gamma.S)_O &= \{(\gamma, x) : \gamma \in \Gamma_O, x \in S_O(\gamma)\} \\
(\Gamma.S)_R &= \{(\gamma, r) : \gamma \in \Gamma_R, r \in S_R(\gamma)\} \\
(\Gamma.S)_{refl}(\gamma, x) &= (\Gamma_{refl}(\gamma), S_{refl}(\gamma)(x)) \\
(\Gamma.S)\sigma(\gamma, r) &= (\Gamma\sigma(\gamma), S\sigma(\gamma)(r)), \quad \sigma \in \{src, tgt\}
\end{aligned}$$

**Claim.**  $Hom_{\mathcal{RGph}}(\Delta, \Gamma.S) \cong \{(f, M) : f : \Delta \rightarrow \Gamma, M \in TM(\Delta, S\{f\})\}$ , natural in  $\Delta$ .

## Upgrading to a QCwF

We simultanously recall the definition of a QCwF and add structure to our CwF.

1. A CwF  $(\mathcal{C}, 1, Ty, Tm, \dashv, \langle -, - \rangle)$ .  
Consider the CwF built in the previous section
2. Category  $\mathcal{L}$  for context with resource annotation, equipped with a faithful functor  $U : \mathcal{L} \rightarrow \mathcal{C}$ .  
Take  $c\mathcal{L} := \mathcal{RGph}(\mathcal{Asm}(\mathcal{A}))$  and let  $U$  be the forgetful functor  $\mathcal{RGph}(\mathcal{Asm}(\mathcal{A})) \rightarrow \mathcal{RGph}$ .
3. Addition structure - let  $\mathcal{L} \times_{\mathcal{C}} \mathcal{L}$  denote the pullback  $\mathcal{L} \xrightarrow{U} \mathcal{C} \xleftarrow{U} \mathcal{L}$ , we stipulate the existence of a functor  $(+) : \mathcal{L} \times_{\mathcal{C}} \mathcal{L} \rightarrow \mathcal{L}$ , s.t  $U(\Gamma' + \Gamma'') = U(\Gamma') = U(\Gamma'')$ . Moreover, there is a distinguished object  $\diamond \in \mathcal{L}$ , s.t  $U(\diamond) = 1$ .  
Suppose  $\Gamma', \Gamma''$  are internal r.g. in  $\mathcal{Asm}(\mathcal{A})$ , such that  $|\Gamma'_O| = |\Gamma''_O|$  and  $|\Gamma'_R| = |\Gamma''_R|$ . Construct the internal r.g.  $\Gamma := \Gamma' + \Gamma''$ , where:
  - $\Gamma_O := (|\Gamma'_O|, \models_{\Gamma})$  with  $a \models_{\Gamma} \gamma$  iff there exist  $x, y \in \mathcal{A}$ , s.t.  $a = [x, y]$  and  $x \models_{\Gamma'} \gamma$  and  $y \models_{\Gamma''} \gamma$ .
  - define  $\Gamma_R$  similarly as  $\Gamma_O$ .
  - $\Gamma_{refl}, \Gamma_{src}, \Gamma_{tgt}$  are all inherited from  $\Gamma'$  (or  $\Gamma''$ ). (*should check this is indeed well-defined*)

To wit that addition preserves realizability of functions, wlog take some  $\gamma \in \Gamma_R$ , s.t  $a \models_{\Gamma_R} \gamma$ . Then by definition there exists  $x, y \in \mathcal{A}$ , s.t  $x \models_{\Gamma'_R} \gamma$  and  $y \models_{\Gamma''_R} \gamma$ . Moreover, there exist  $b, c \in \mathcal{A}$ , s.t  $b.x \models_{\Gamma'_O} \Gamma'_{src}(\gamma)$  and  $c.y \models_{\Gamma''_O} \Gamma''_{src}(\gamma)$ . Let  $a^* = \lambda x \lambda y. [b.x, c.y]$ , then  $a^*$  tracks  $\Gamma_{src}$ .  
Pick a terminal internal r.g. as an interpretation for  $\diamond$ .
4. Scaling with  $\rho \in R$  - there is a functor  $\rho(-) : \mathcal{L} \rightarrow \mathcal{L}$ , s.t.  $U(\rho(-)) = U(-)$ .  
Again we define an internal r.g.  $\Gamma := \rho(\Gamma')$  as :
  - $\Gamma_{\sigma} = (|\Gamma'_{\sigma}|, \models_{\Gamma_{\sigma}})$  with  $a \models_{\Gamma_{\sigma}} \gamma$  iff there is  $x \in \mathcal{A}$ , s.t  $a = !_\rho x$  and  $x \models_{\Gamma'_{\sigma}} \gamma$  for  $\sigma \in \{O, R\}$

- $\Gamma_\sigma$  is just  $\Gamma'_\sigma$  for  $\sigma \in \{src, tgt, rfl\}$

To see that, for instance,  $\Gamma_{refl}$  retain realizability, let  $a \Vdash_{\Gamma_O} \gamma$ . Therefore, there is some  $x, b \in \mathcal{A}$ , s.t  $x \Vdash_{\Gamma'_O} \gamma$  and  $b.x \Vdash_{\Gamma'_R} \Gamma'_{refl}(\gamma)$ . Now consider  $a^* := F_\rho(!_\rho b)$ , we have  $a^*.x \Vdash_{\Gamma_R} \Gamma_{refl}(\gamma)$ .

As the functor  $\rho$  modifies only realizability information, it is clear that  $U(\rho(-)) = U(-)$  holds.

## Type formers

### Universe of small types

**Dependent function types in 1-fragment** Given an internal reflexive graph  $\Gamma$ ,  $S \in Ty(\Gamma)$ ,  $T \in Ty(\Gamma.S)$ ,  $\rho \in R$ , define  $\Pi\rho ST \in Ty(\Gamma)$  as the tuple of:

- Object assembly. Let  $\gamma \in |\Gamma_O|$

$$|(\Pi\rho ST)_O|(\gamma) := \{(f_O, f_R) \mid$$

$$f_O : \forall s \in S_O(\gamma). T_O(\gamma, s),$$

$$\exists a \in \mathcal{A} \forall s \in |S_O(\gamma)|, b \in \mathcal{A}.$$

$$b \Vdash_{S_O(\gamma)} s \implies a.!_\rho b \Vdash_{T_O(\gamma, s)} f_O(s)$$

$$f_R : \forall r \in S_R(\Gamma_{refl}(\gamma)). T_R(\Gamma_{refl}(\gamma), r),$$

$$\exists a \in \mathcal{A} \forall r \in |S_R(\Gamma_{refl}(\gamma))|, b \in \mathcal{A}.$$

$$b \Vdash_{S_R(\Gamma_{refl}(\gamma))} r \implies a.!_\rho b \Vdash_{T_R(\Gamma_{refl}(\gamma), r)} f_R(r),$$

$$\forall s \in S_O(\gamma). T_{refl}(\gamma, s)(f_O(s)) = f_R(S_{refl}(\gamma)(s)),$$

$$\forall r \in S_R(\Gamma_{refl}(\gamma)). T_{src}(\Gamma_{refl}(\gamma), r)(f_R(r)) = f_O(S_{src}(\Gamma_{refl}(\gamma))(r)),$$

$$\forall r \in S_R(\Gamma_{refl}(\gamma)). T_{tgt}(\Gamma_{refl}(\gamma), r)(f_R(r)) = f_O(S_{tgt}(\Gamma_{refl}(\gamma))(r))\}$$

$a \Vdash_{(\Pi\rho ST)_O(\gamma)} (f_O, f_R)$  iff  $a = [a_o, a_r]$  where the realizers  $a_o, a_r$  are given by the existential statements above.

- Relation assembly. Let  $\gamma \in |\Gamma_R|$

$$|(\Pi\rho ST)_R|(\gamma) := \{((f_O^{src}, f_R^{src}), (f_O^{tgt}, f_R^{tgt}), r) \mid$$

$$(f_O^{src}, f_R^{src}) \in (\Pi\rho ST)_O(\Gamma_{src}(\gamma)),$$

$$(f_O^{tgt}, f_R^{tgt}) \in (\Pi\rho ST)_O(\Gamma_{tgt}(\gamma)),$$

$$r : \forall s \in S_R(\gamma). T_R(\gamma, s),$$

$$\exists a \in \mathcal{A} \forall s \in |S_R(\gamma)|, b \in \mathcal{A}. b \Vdash_{S_R(\gamma)} s \implies a.!_\rho b \Vdash_{T_R(\gamma, s)} r(s),$$

$$\forall s \in S_R(\gamma). T_{src}(\gamma, s)(r(s)) = f_O^{src}(S_{src}(\gamma)(s)),$$

$$\forall s \in S_R(\gamma). T_{tgt}(\gamma, s)(r(s)) = f_O^{tgt}(S_{tgt}(\gamma)(s))\}$$

- collections of morphisms:

$$(\Pi\rho ST)_{refl}(\gamma)(f_O, f_R) := ((f_O, f_R), (f_O, f_R), f_R) \text{ for } \gamma \in \Gamma_O$$

$$(\Pi\rho ST)_{src}(\gamma)(f^{src}, f^{tgt}, r) := f^{src} \text{ for } \gamma \in \Gamma_R$$

$$(\Pi\rho ST)_{src}(\gamma)(f^{src}, f^{tgt}, r) := f^{tgt} \text{ for } \gamma \in \Gamma_R$$

These collections satisfy the identity conditions in Definition 4.

Suppose we specialize  $\mathcal{A}$  to the graph model,  $\mathcal{D}$ . Then we can find uniform realizers for the morphisms in each collection. Assuming that syntactic tupling in the BCI-algebra is represented using right-nested tuples in  $\mathcal{D}$ , then:

$$a_{refl} := \{ \langle \langle x_o, \langle x_r, 0 \rangle \rangle, \langle \langle x_o, \langle x_r, 0 \rangle \rangle, \langle \langle x_o, \langle x_r, 0 \rangle \rangle, x_r \rangle \rangle \rangle | x_o, x_r \in D \}$$

$$a_{src} := \{ \langle \langle x_{src}, \langle x_{tgt}, \langle x_r, 0 \rangle \rangle \rangle, \langle x_{src}, 0 \rangle \rangle | x_{src}, x_{tgt}, x_r \in D \}$$

$$a_{tgt} := \{ \langle \langle x_{src}, \langle x_{tgt}, \langle x_r, 0 \rangle \rangle \rangle, \langle x_{tgt}, 0 \rangle \rangle | x_{src}, x_{tgt}, x_r \in D \}$$

## Dependent pairs