

Outline

1 Quantitative polynomial functors

1.1 Category of closed types and linear functions

Let \mathcal{C} be the category of closed types and linear functions $f : (x \vdash^1 X) \rightarrow Y$ for which derivations in QTT exist. Composition of morphisms $\Gamma \vdash f \vdash^\sigma (x \vdash^1 X) \rightarrow Y$ and $\Gamma \vdash g \vdash^\sigma (y \vdash^1 Y) \rightarrow Z$ is given by ordinary function composition $\Gamma \vdash \lambda x \vdash^1 X. g(f(x)) \vdash^\sigma (x \vdash^1 X) \rightarrow Z$.

The linearity restriction on the morphisms does not lead to loss of expressiveness - a function with arbitrary resource annotations can be represented as linear one via the exponential type $!_\rho A := (a \vdash^\rho A) \otimes I$.

For an arbitrary $\Gamma \vdash f \vdash^1 X \xrightarrow{\rho} Y$, the embedding is given by:

$$f \mapsto \lambda z \vdash^1 (a \vdash^\rho X) \otimes I. \text{ let } (x, i) = z \text{ in} \\ \text{ let } * = i \text{ in } f(x) \vdash^1 ((a \vdash^\rho X) \otimes I) \xrightarrow{1} Y$$

Suppose \mathcal{D} stands for the derivation of $\Gamma \vdash f \vdash^1 X \xrightarrow{\rho} Y$ and \mathcal{D}_x , \mathcal{D}_z and \mathcal{D}_i for the derivations of $0\Gamma, x \vdash^1 X \vdash x \vdash^1 X$, $0\Gamma, z \vdash^1 (a \vdash^\rho X) \otimes I \vdash z \vdash^1 (a \vdash^\rho X) \otimes I$ and $0\Gamma, i \vdash^1 I \vdash i \vdash^1 I$ obtained by the applications of the **Var** rule.

$$\frac{\frac{\frac{\mathcal{D}_x \quad \mathcal{D}}{\mathcal{D}_i \quad \Gamma, x \vdash^\rho X \vdash f(x) \vdash^1 Y}}{\mathcal{D}_z \quad \Gamma, x \vdash^\rho X, i \vdash^1 I \vdash \text{let } * = i \text{ in } f(x) \vdash^1 Y}}{\Gamma, z \vdash^1 (a \vdash^\rho X) \otimes I \vdash \text{let } (x, z) = z \text{ in let } * = i \text{ in } f(x) \vdash^1 Y}}{\Gamma \vdash \lambda z \vdash^1 (a \vdash^\rho X) \otimes I. \text{let } (x, z) = z \text{ in let } * = i \text{ in } f(x) \vdash^1 (a \vdash^\rho X) \otimes I \xrightarrow{1} Y}$$

1.2 Category of closed types and linear functions in a nonempty context

Let Δ be an “underlying context”, i.e. a context of the form $\Delta = 0\Gamma_0$ for some Γ_0 . There is a category \mathcal{C}_Δ where the objects are types X such that $\Delta \vdash X$ type, and a morphism X to Y consists of pair (Γ, f) , where Γ is a context such that $0\Gamma = \Delta$, and $\Gamma \vdash f : X \xrightarrow{1} Y$.

- The identity morphism is given by $(\Delta, \lambda x. x)$;
- Composition of (Γ_2, g) and (Γ_1, f) is given by $(\Gamma_1 + \Gamma_2, \lambda x. f(g(x)))$.

This is a category since $0\Delta = \Delta$, and context addition is associative, and with $\Gamma + \Delta = \Delta + \Gamma = \Gamma$. Note that the category of \mathcal{C} closed types from Section 1.1 is a special case $\mathcal{C} = \mathcal{C}_\diamond$ where $\Delta = \diamond$, because the only Γ with $0\Gamma = \diamond$ is $\Gamma = \diamond$.

Lemma 1. *Fix Δ as above, and let $\Delta \vdash A$ type and $\Delta, x : A \vdash B[x]$ type. The operation $F(X) = (a : A) \otimes (Ba \xrightarrow{1} X)$ is a functor $\mathcal{C}_\Delta \rightarrow \mathcal{C}_\Delta$.*

Proof. If $\Delta \vdash X$ type then $\Delta \vdash F(X)$ type. On morphisms, we can define

$$F(\Gamma, f) = (\Gamma, \lambda z. \text{let } (a, h) = z \text{ in } (a, \lambda b. f(h(b))))$$

as the following derivation shows:

$$\frac{\frac{\frac{}{0\Gamma, z : F(X), a : A, h : B[a] \xrightarrow{1} X \vdash a : A} \quad \dot{\mathcal{D}}}{\Gamma, z : F(X), a : A, h : B[a] \xrightarrow{1} X \vdash (a, \lambda b. f(h(b))) : F(Y)} \quad \frac{}{0\Gamma, z : F(X) \vdash z : F(X)}}{\Gamma, z : F(X) \vdash \text{let } (a, h) = z \text{ in } (a, \lambda b. f(h(b))) : F(Y)} \quad \frac{}{\Gamma \vdash \lambda z. \text{let } (a, h) = z \text{ in } (a, \lambda b. f(h(b))) : F(X) \xrightarrow{1} F(Y)}$$

where \mathcal{D} is the derivation

$$\frac{\frac{\frac{}{\Gamma, a : A, h : B[a] \xrightarrow{1} X, b : B[a] \vdash f : X \xrightarrow{1} Y} \quad \dot{\mathcal{D}}'}{\Gamma, a : A, h : B[a] \xrightarrow{1} X, b : B[a] \vdash f(h(b)) : Y}}{\Gamma, a : A, h : B[a] \xrightarrow{1} X \vdash \lambda b. f(h(b)) : B[a] \xrightarrow{1} Y}$$

weakened by $z : F(X)$, where again \mathcal{D}' is the derivation

$$\frac{\frac{}{0\Gamma, a : A, h : B[a] \xrightarrow{1} X, b : B[a] \vdash h : B[a] \xrightarrow{1} X} \quad \frac{}{0\Gamma, a : A, h : B[a] \xrightarrow{0} X, b : B[a] \vdash b : B[a]}}{0\Gamma, a : A, h : B[a] \xrightarrow{1} X, b : B[a] \vdash h(b) : X}$$

similarly weakened. This is functorial by the η -rules for functions and pairs. \square

1.3 Internal representation

Let $F_0 : \text{Obj}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{C})$ be the function mapping a type X to the type $(a : A) \otimes (Ba \xrightarrow{1} X)$.

Claim. F_0 can be extended to an endofunctor $F : \mathcal{C} \rightarrow \mathcal{C}$.

Suppose $\Gamma \vdash f : X \xrightarrow{1} Y$ is some morphism with a derivation \mathcal{D}_f , then $Ff : (a : A) \otimes (Ba \xrightarrow{1} X) \xrightarrow{1} (a : A) \otimes (Ba \xrightarrow{1} Y)$ is defined by precomposition with f on the second component.

By applying **Var** rule, we get the following:

- a derivation \mathcal{D}_a of $0\Gamma, a : A, g : Ba \xrightarrow{1} X \vdash a : A$
- a derivation \mathcal{D}_g of $0\Gamma, a : A, g : Ba \rightarrow X \vdash g : Ba \xrightarrow{1} X$
- a derivation \mathcal{D}_b of $0\Gamma, a : A, g : Ba \rightarrow X, b : Ba \vdash b : Ba$
- a derivation \mathcal{D}_z of $0\Gamma, z : (a : A) \otimes (Ba \xrightarrow{1} X) \vdash z : (a : A) \otimes (Ba \xrightarrow{1} X)$

Form an intermediate derivation \mathcal{D} by :

$$\frac{\frac{\frac{\frac{D_g \quad D_b}{0\Gamma, a : A, g : Ba \rightarrow X, b : Ba \vdash g(b) : X} \text{App}}{0\Gamma, a : A, g : Ba \rightarrow X, b : Ba, f : X \xrightarrow{1} Y \vdash f(g(b)) : Y} \text{App}}{0\Gamma, a : A, g : Ba \rightarrow X, f : X \xrightarrow{1} Y \vdash \lambda b.f(g(b)) : Ba \xrightarrow{1} Y} \text{Abs}}{\Gamma, a : A, g : Ba \xrightarrow{1} X \vdash (a, \lambda b.f(g(b))) : F(Y)} \otimes - \text{intro}$$

and use it to get a final one for Ff :

$$\frac{\frac{\frac{D_z \quad \mathcal{D}}{\Gamma, z : F(X)) \vdash \text{let } (x, u) = a \text{ in } (x, \lambda b.f(u(b))) : F(Y)} \oplus - E}{\Gamma \vdash \lambda z.\text{let } (x, u) = a \text{ in } (x, \lambda b.f(u(b))) : F(X) \xrightarrow{1} F(Y)} \text{Lam}$$

Definition 2. We will call any functor isomorphic to F a quantitative polynomial functor.

1.4 External representation (using adjoints)

1.5 Generalising to non-empty contexts

1.6 Properties of quantitative polynomial functors

2 Algebras for QPFs

Recall that an algebra for an endofunctor $F : \mathcal{C} \rightarrow \mathcal{C}$ is a pair (X, a) , where X is an object of \mathcal{C} and a - a morphism $F(X) \rightarrow X$. A morphism between F -algebras is a map $f : X \rightarrow Y$,

making the square commute:

$$\begin{array}{ccc}
 F(X) & \xrightarrow{F(f)} & F(Y) \\
 \downarrow a & & \downarrow b \\
 X & \xrightarrow{f} & Y
 \end{array}$$

2.1 \mathbb{N}

Fix $A := \mathbf{Bool}$ and B , such that $B(false) := \emptyset$ and $B(true) := I$ where $x : A \vdash B$ type.

Sketch:

Observe that a function $\Gamma \vdash f : F_{A,B}(X) \xrightarrow{1} X$ is equivalent to functions $\Gamma \vdash f_l : (\emptyset \xrightarrow{1} X) \rightarrow X$ and $\Gamma \vdash f_r : (I \xrightarrow{1} X) \rightarrow X$.

Assuming that the type $\emptyset \xrightarrow{1} X$ is a (sub?)singleton (or $\emptyset \xrightarrow{1} X \cong I$, but maybe too strong), f_l just encodes a choice of an element of X . Similarly, $I \xrightarrow{1} X$ also encodes the same data, that is $I \xrightarrow{1} X \cong X$ (provable in QTT?).

Essentially, an algebra for $F_{A,B}$ is a diagram $I \rightarrow X \rightarrow X$.

Now, we prove the following propositions:

- (i) Suppose $\Gamma \vdash \mathbf{Nat}$ type, then there exists an initial algebra for $F_{A,B}(\mathbf{Nat})$
- (ii) Suppose $\Gamma \vdash X$ type, such that there exists an initial algebra for $F_{A,B}(X)$, then $F_{A,B}(X) \cong \mathbf{Nat}$.

Before embarking on proving that, let's examine a simpler construction:

Interlude : Pseudobooleans as a constant QPF Let $A := \mathbf{Bool}$, $\Gamma, a : A \vdash B$ type, $B(a) := \emptyset$. An algebra for this $F_{A,B}(X)$ is a diagram $X \leftarrow \{*\} \rightarrow X$.

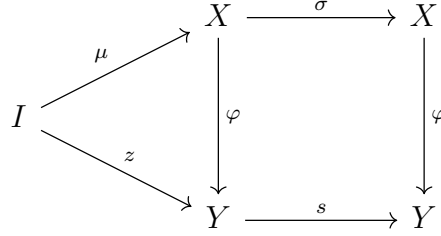
Assume $\Gamma \vdash \mathbf{Bool}$ type. Then define $t(i) := true$ and $f(i) := false$ using the unit elimination. Let $(F_{A,B}(Y), m, n)$ be another algebra. Define $\varphi(x) := Elim(m(*), n(*), k)$, where $k : F_{A,B}(\mathbf{Bool}) \xrightarrow{1} \mathbf{Bool}$ is the structure map. The squares commute by the computation rules for \mathbf{Bool} .

$$\begin{array}{ccccc}
 \mathbf{Bool} & \xleftarrow{t} & \{*\} & \xrightarrow{f} & \mathbf{Bool} \\
 \downarrow \varphi & & \parallel & & \downarrow \varphi \\
 Y & \xleftarrow{m} & \{*\} & \xrightarrow{n} & Y
 \end{array}$$

Conversely, assume that there is an initial algebra $(X, k : F_{A,B}X \rightarrow X)$. Designate some elements $t, f \in X$, such that $t \neq f$. For any other algebra represented diagrammatically $(Y, m : I \xrightarrow{1} Y, n : I \xrightarrow{1} Y)$, we have that $\varphi(t) = m$ and $\varphi(f) = n$ by initiality. Define $Elim(m, n, x) := \varphi(x)$.

Back to \mathbb{N} , let **Nat** be defined as:

$$\begin{array}{ccc} \frac{0\Gamma \vdash A}{0\Gamma \vdash \mathbf{Nat}} \text{Nat} & \frac{0\Gamma \vdash}{0\Gamma \vdash 0 : \mathbf{Nat}} & \frac{\Gamma \vdash N : \mathbf{Nat}}{\Gamma \vdash suc(N) : \mathbf{Nat}} \text{suc} \\ \dots & & \end{array}$$



2.2 Lists

2.3 Trees

2.4 Induction principle

3 Rules for W-types in QTT

4 Parametricity and W-types

5 Appendix

5.1 (Stand-alone) Sum types

$$\begin{array}{c}
\frac{0\Gamma \vdash A \quad 0\Gamma \vdash B}{0\Gamma \vdash \rho A \oplus \pi B} \oplus\text{-type} \quad \frac{\Gamma \vdash S_1 \overset{\sigma}{:} A}{\rho\Gamma \vdash \mathbf{inl} S_1 \overset{\sigma}{:} \rho A \oplus \pi B} \text{inl} \quad \frac{\Gamma \vdash S_2 \overset{\sigma}{:} B}{\pi\Gamma \vdash \mathbf{inr} S_2 \overset{\sigma}{:} \rho A \oplus \pi B} \text{inr} \\
\\
\frac{\Gamma \vdash M \overset{\sigma}{:} \rho A \oplus \pi B \quad \Gamma', a \overset{\rho}{:} A \vdash T_1 \overset{\sigma}{:} C[\mathbf{inl} a/x] \quad \Gamma', b \overset{\pi}{:} B \vdash T_2 \overset{\sigma}{:} C[\mathbf{inr} b/x] \quad 0\Gamma = 0\Gamma'}{\Gamma' + \Gamma \vdash \mathbf{case}(M, T_1, T_2) \overset{\sigma}{:} C[M/x]} \oplus\text{-elim} \\
\\
\frac{\Gamma \vdash S_1 \overset{\sigma}{:} A \quad \Gamma \vdash M \overset{\sigma}{:} \rho A \oplus \pi B \quad \Gamma', a \overset{\rho}{:} A \vdash T_1 \overset{\sigma}{:} C[\mathbf{inl} a/x] \quad 0\Gamma = 0\Gamma'}{\Gamma' + \rho\Gamma \vdash \mathbf{case}(\mathbf{inl}(S_1), T_1, T_2) \equiv T_1[S_1/a]} \oplus\text{-comp}
\end{array}$$

Figure 1: Rules for \oplus -type

We give the following semantics for the \oplus -type:

$$\begin{aligned}
|\rho A \oplus \pi B(\gamma)| &:= |A(\gamma)| \sqcup |B(\gamma)| \\
a \models_{\rho A \oplus \pi B(\gamma)} (i, x) &\text{ iff } (\exists b. a = [!_{\rho} b, \ulcorner \text{true} \urcorner] \wedge b \models_{A(\gamma)} x \wedge i = 0) \vee \\
&\quad (\exists c. a = [!_{\pi} c, \ulcorner \text{false} \urcorner] \wedge c \models_{B(\gamma)} x \wedge i = 1)
\end{aligned}$$

Claim. *The rules are sound when interpreted wrt to the given semantics for \oplus -types and realisability model.*

Proof. The underlying set-theoretic functions are immediate. For the realisers, let

- $a_{\mathbf{inl}} := \lambda^* x. [F_{\rho} \cdot !_\rho a_{s_1} \cdot x, \ulcorner \text{true} \urcorner]$

$$\text{if } a_{s_1} \cdot a_{\gamma} \models s_1, \text{ then } a_{\mathbf{inl}} \cdot !_\rho a_{\gamma} \models \mathbf{inl} s_1; a_{\mathbf{inl}} \cdot !_\rho a_{\gamma} \rightsquigarrow [!_{\rho}(a_{s_1} \cdot a_{\gamma}), \ulcorner \text{true} \urcorner]$$

$$\lambda^* x. \text{ let } [a'_{\gamma}, a_{\gamma}] = x,$$

- $a_{\mathbf{case}} := \begin{array}{l} [a, b] = a_m \cdot a_{\gamma} \text{ in} \\ E(b, a_{T_1}, a_{T_2}) \cdot [a'_{\gamma}, a] \end{array}$

assuming $a_m \cdot a_{\gamma} \models M$, $a_{T_1} \cdot [a'_{\gamma}, !_\rho a_a] \models T_1$, $a_{T_2} \cdot [a'_{\gamma}, !_\pi a_b] \models T_2$, then we want to find $a_{\mathbf{case}}$, s.t. $a_{\mathbf{case}} \cdot [a'_{\gamma}, a_{\gamma}] \models \mathbf{case}(M, T_1, T_2)$.

if $a_m \cdot a_\gamma = [!_\rho a_a, \ulcorner true \urcorner]$, then $a_{\text{case}} \cdot [a_\gamma, a'_\gamma] \rightsquigarrow E(\ulcorner true \urcorner, a_{T_1}, a_{T_2}) \cdot [a'_\gamma, !_\rho a_a] \rightsquigarrow$
 $a_{T_1} \cdot [a'_\gamma, !_\rho a_a]$
 if $a_m \cdot a_\gamma = [!_\pi a_b, \ulcorner false \urcorner]$, then ...

□

Claim. *There is a bijection:*

$$RTm(\Gamma, \Pi(x \overset{\tau}{:} \rho A \oplus \pi B) C) \cong RTm(\Gamma, \Pi(y \overset{\tau\rho}{:} A) C[\mathbf{inl}y/x]) \times RTm(\Gamma, \Pi(z \overset{\tau\pi}{:} B) C[\mathbf{inl}z/x])$$

(natural in Γ).

Proof. Given a term $\Gamma \vdash f \overset{1}{:} (x \overset{\tau}{:} \rho A \oplus \pi B) \rightarrow C$, we can derive another term $\Gamma \vdash f^l \overset{1}{:} (y \overset{\tau\rho}{:} A) \rightarrow C[\mathbf{inl}y/x]$:

$$\frac{\frac{\Gamma \vdash f : (x \overset{\tau}{:} \rho A \oplus \pi B) \rightarrow C}{\Gamma, y \overset{0}{:} A \vdash f \overset{1}{:} (x \overset{\tau}{:} \rho A \oplus \pi B) \rightarrow C} \text{Weak} \quad \frac{\frac{\vdash 0\Gamma, y \overset{1}{:} A}{0\Gamma, y \overset{1}{:} \vdash y \overset{1}{:} A} \text{var} \quad \frac{}{0\Gamma, y \overset{\rho}{:} A \vdash \mathbf{inl}y \overset{1}{:} \rho A \oplus \pi B} \text{inl}}{\frac{}{0\Gamma, y \overset{\rho}{:} A \vdash \mathbf{inl}y \overset{1}{:} \rho A \oplus \pi B} \text{App}} \text{App} \quad \frac{\Gamma, y \overset{\tau\rho}{:} A \vdash f(\mathbf{inl}y) \overset{1}{:} C[\mathbf{inl}y/x]}{\Gamma \vdash \lambda y \overset{\tau\rho}{:} A. f(\mathbf{inl}y) : (y \overset{\tau\rho}{:} A) \rightarrow C[\mathbf{inl}y/x]} \text{Lam}$$

Analogously, we can obtain $\Gamma \vdash f^r \overset{1}{:} (z \overset{\tau\pi}{:} B) \rightarrow C[\mathbf{inr}y/x]$.

Now suppose we have terms $\Gamma \vdash l \overset{1}{:} (y \overset{\tau\rho}{:} A) \rightarrow C[\mathbf{inl}y/x]$ and $\Gamma \vdash r \overset{1}{:} (z \overset{\tau\pi}{:} B) \rightarrow C[\mathbf{inr}z/x]$.

Using the isomorphism $\Lambda^{\mathcal{L}}$, we get judgements $\Gamma, y \overset{\tau\rho}{:} A \vdash l^* : C[\mathbf{inl}y/x]$ and

$\Gamma, z \overset{\tau\pi}{:} B \vdash r^* : C[\mathbf{inr}z/x]$.

□