

Relational realizability model for QTT

Our aim is to build a concrete realizability model for QTT which allows for metareasoning with results derived from parametricity. To that end, a construction of a quantified category with families (QCwF[**fill**]) is presented. We follow the relational approach to types introduced by Reynolds for typed lambda calculus[**fill**] and later refined for dependent types theories.[**fill**] Once and for all fix a usage semiring R and an R -linear combinatory algebra \mathcal{A} ¹.

Taking stocks

Definition 1 (Assembly). An assembly Γ is a pair $(|\Gamma|, e)$ where $|\Gamma|$ is a carrier set and e is a realizability function $|\Gamma| \rightarrow \mathcal{P}(\mathcal{A})$, s.t $e(\gamma)$ is nonempty for every $\gamma \in |\Gamma|$.

We interpret $E(\gamma)$ as a set of witnesses for the existence of γ and write $a \Vdash_{\Gamma} \gamma$ to denote $a \in e(\gamma)$.

A morphism between two assemblies $(|\Gamma|, E_{\Gamma})$ and $(|\Delta|, E_{\Delta})$ is a function $f : |\Gamma| \rightarrow |\Delta|$ that is realizable - there exists $a_f \in \mathcal{A}$, s.t the following holds for every $\gamma \in |\Gamma|$ and $a_{\gamma} \in \mathcal{A}$:

$$a_{\gamma} \Vdash_{\Gamma} \gamma \implies a_f.a_{\gamma} \Vdash_{\Delta} f(\gamma)$$

We say that a_f tracks f . Note that only an existence of a realizer for f is stipulated in the definition - multiple realizers do not induce multiple morphisms.

Using these notions we can construct a category $\mathcal{A}sm(\mathcal{A})$.

Definition 2 (Reflexive graph). A reflexive graph (r.g.) G is a tuple $(G_O, G_R, G_{refl}, G_{src}, G_{tgt})$, where G_O and G_R are sets, G_{src} and G_{tgt} are functions $G_R \rightarrow G_O$ and G_{refl} is a function $G_O \rightarrow G_R$, s.t $G_{src} \circ G_{refl} = id_{G_O} = G_{tgt} \circ G_{refl}$.

G_O and G_R stand for a set of objects and a set of relations, respectively. In general, reflexive graphs are less structured than categories as relations need not compose.

A morphism between reflexive graphs G and H is a pair of functions $(f_o : G_O \rightarrow H_O, f_r : G_R \rightarrow H_R)$, s.t all of the depicted squares commute:

$$\begin{array}{ccc} G_O & \xrightarrow{f_o} & H_O \\ \begin{array}{c} \uparrow | \uparrow \\ G_{src} \left(G_{refl} \right) G_{tgt} \\ \downarrow | \downarrow \end{array} & & \begin{array}{c} \uparrow | \uparrow \\ H_{src} \left(H_{refl} \right) H_{tgt} \\ \downarrow | \downarrow \end{array} \\ G_R & \xrightarrow{f_r} & H_R \end{array}$$

Reflexive graphs equipped with r.g. morphisms form a category $\mathcal{RGph}(\mathcal{Set})$.

We use reflexive graphs to give a dyadic interpretation of types in the spirit of [**fill**].

¹In case some non-trivial properties of \mathcal{A} are required, we will tacitly assume that \mathcal{A} is a graph model(see [**fill**]) - also to fix

Assembling a reflexive graph

One could easily generalize reflexive graphs by considering object and relation components from an arbitrary category \mathcal{C} instead of \mathcal{Set} . As our purpose is to build a realizability model, we pick $\mathcal{Asm}(\mathcal{A})$ as the category of interest and identify two appropriate notions of reflexive graph of assemblies and a family of reflexive graphs of assemblies.

Definition 3 (Reflexive graph of assemblies). A reflexive graph of assemblies G is a pair of assemblies (G_O, G_R) and a triple of set-theoretic $(G_{refl} : |G_O| \rightarrow |G_R|, G_{src} : |G_R| \rightarrow |G_O|, G_{tgt} : |G_R| \rightarrow |G_O|)$, such that the identities in Definition 2 are satisfied.

In essence, sets are directly replaced by the objects from $\mathcal{Asm}(\mathcal{A})$, while the morphism are kept intact - plain set-theoretic functions.

However, we require realizable functions only in the definition morphism between reflexive graph of assemblies. With these components, we obtain a category $\mathcal{RGph}(\mathcal{A})$. By considering r.g. of assemblies of shape (X, X, id_X, id_X, id_X) , we identify isomorphic copy of $\mathcal{Asm}(\mathcal{A})$ inside $\mathcal{RGph}(\mathcal{A})$.

A terminal object in $\mathcal{RGph}(\mathcal{A})$ is a tuple $(1, 1, id, id, id)$, where 1 is the terminal assembly $(\{\star\}, f)$, with f defined as $\star \mapsto \{I\}$.

Definition 4 (Family of reflexive graphs of assemblies). Let \mathcal{C} be a category with a terminal object. Given a reflexive graph $\Gamma \in Ob(\mathcal{C})$, a family of internal r.g. over Γ is a tuple $S := (S_O, S_R, S_{refl}, S_{src}, S_{tgt})$, where:

- $S_O : \Gamma_O \rightarrow \mathcal{Asm}(\mathcal{A})$
- $S_R : \Gamma_R \rightarrow \mathcal{Asm}(\mathcal{A})$
- a Γ -indexed collection of functions $S_{refl} := \{f : |S_O(\gamma)| \rightarrow |S_R(\Gamma_{refl}(\gamma))|\}_{\gamma \in \Gamma_O}$
- $S_{src} := \{f : |S_R(\gamma)| \rightarrow |S_O(\Gamma_{src}(\gamma))|\}_{\gamma \in \Gamma_R}$
- $S_{tgt} := \{f : |S_R(\gamma)| \rightarrow |S_O(\Gamma_{tgt}(\gamma))|\}_{\gamma \in \Gamma_R}$

such that

- each identity in the following collection is satisfied:

$$S_\sigma(\Gamma_{refl}(\gamma)) \circ S_{refl}(\gamma) = id \text{ for every } \gamma \in \Gamma_O, \sigma \in \{src, tgt\}$$

We are only interested in cases when $\mathcal{C} = \mathcal{Set}$ or $\mathcal{C} = \mathcal{Asm}(\mathcal{A})$.

A morphism M between two families S and T of internal r.g. over Γ is a pair of Γ -indexed collection of functions:

- $M_O := \{f : |S_O(\gamma)| \rightarrow |T_O(\gamma)|\}_{\gamma \in \Gamma_O}$
- $M_R := \{f : |S_R(\gamma)| \rightarrow |T_R(\gamma)|\}_{\gamma \in \Gamma_O}$

such that:

- $T_{refl}(M_O(\gamma)(x)) = M_R(\Gamma_{refl}(\gamma))(S_{refl}(\gamma)(x))$ for every $\gamma \in \Gamma_O$, $x \in S_O(\gamma)$
- $T_{src}(M_R(\gamma)(r)) = M_O(\Gamma_{src}(\gamma))(S_{src}(\gamma)(r))$ for every $\gamma \in \Gamma_R$, $r \in S_R(\gamma)$
- $T_{tgt}(M_R(\gamma)(r)) = M_O(\Gamma_{tgt}(\gamma))(S_{tgt}(\gamma)(r))$ for every $\gamma \in \Gamma_R$, $r \in S_R(\gamma)$

A terminal family of r.g. over Γ , 1_Γ , consists of two constant functions, mapping $\gamma \in \Gamma$ to a terminal assembly 1, and three Γ -indexed collections with a sole element id_1 .

A CwF from families of reflexive graphs of assemblies

As a first step toward obtaining a relationally parametric realizability model of QTT, we construct a concrete CwF using families of internal graphs of assemblies.

Consider the category \mathcal{RGph} with terminal object $1 := (\{\star\}, \{\star\}, id, id, id)$.

Let $\Gamma, \Delta \in Ob(\mathcal{RGph})$, define:

- the collection of semantic types $Ty(\Gamma)$ as the collection of families of reflexive graphs of assemblies Γ .
- given a type $S \in Ty(\Gamma)$, let $Tm(\Gamma, S) := Hom(1_\Gamma, S)$.
Spelling this out and ignoring the contribution of the terminal family, we get:
An element $M \in Tm(\Gamma, S)$ is a pair of functions $(M_O : \forall \gamma \in \Gamma_O. S_O(\gamma), M_R : \forall \gamma \in \Gamma_R. S_R(\gamma))$, such that

$$\begin{aligned} \forall \gamma \in \Gamma_O. S_{refl}(M_O(\gamma)) &= M_R(\Gamma_{refl}(\gamma)) \\ \forall \gamma \in \Gamma_R. S_{src}(M_R(\gamma)) &= M_O(\Gamma_{src}(\gamma)) \\ \forall \gamma \in \Gamma_R. S_{tgt}(M_R(\gamma)) &= M_O(\Gamma_{tgt}(\gamma)) \end{aligned}$$

- given $f : \Gamma \rightarrow \Delta$, substitutions in types and terms is a precomposition with f on the object and relation components of types and terms respectively. Clearly, these operations are compatible with identity and composition in \mathcal{RGph}
- context extension : Suppose $S \in Ty(\Gamma)$, construct a r.g. $\Gamma.S$ as :

$$\begin{aligned} (\Gamma.S)_O &= \{(\gamma, x) : \gamma \in \Gamma_O, x \in S_O(\gamma)\} \\ (\Gamma.S)_R &= \{(\gamma, r) : \gamma \in \Gamma_R, r \in S_R(\gamma)\} \\ (\Gamma.S)_{refl}(\gamma, x) &= (\Gamma_{refl}(\gamma), S_{refl}(\gamma)(x)) \\ (\Gamma.S)\sigma(\gamma, r) &= (\Gamma\sigma(\gamma), S\sigma(\gamma)(r)), \quad \sigma \in \{src, tgt\} \end{aligned}$$

Claim. $Hom_{\mathcal{RGph}}(\Delta, \Gamma.S) \cong \{(f, M) : f : \Delta \rightarrow \Gamma, M \in Tm(\Delta, S\{f\})\}$, natural in Δ .

Upgrading to a QCwF

Recall the definition of a QCwF from [fill]. Given a usage semiring R , a R -QCwF consists of:

1. A CwF $(\mathcal{C}, 1, Ty, Tm, -, -, \langle -, - \rangle)$
2. A category \mathcal{L} with a faithful functor $U : \mathcal{L} \rightarrow \mathcal{C}$
3. A functor $(+) : \mathcal{L} \times_{\mathcal{C}} \mathcal{L} \rightarrow \mathcal{L}$, s.t. $U(\Gamma_1 + \Gamma_2) = U(\Gamma_1) = U(\Gamma_2)$ ². $\mathcal{L} \times_{\mathcal{C}} \mathcal{L}$ denotes the pullback $\mathcal{L} \xrightarrow{U} \mathcal{C} \xleftarrow{U} \mathcal{L}$.
Additionally, there exists an object $\diamond \in \mathcal{L}$, s.t. $U\diamond = 1$.
4. A functor $\rho(-) : \mathcal{L} \rightarrow \mathcal{L}$ for each $\rho \in R$, s.t. $U(\rho(-)) = U(-)$.
5. A collection $RTm(\Gamma, S)$ for each $\Gamma \in \mathcal{L}$ and $S \in Ty(U\Gamma)$, equipped with an injective function $U_{\Gamma.S} : RTm(\Gamma, S) \rightarrow Tm(U\Gamma, S)$.
For \mathcal{L} morphisms $f : \Gamma \rightarrow \Delta$ and types $S \in Ty(U\Gamma)$, a function $- \{f\} : RTm(\Delta, S) \rightarrow RTm(\Gamma, S \{f\})$, s. t. $U(- \{f\}) = (U(-)) \{Uf\}$.
6. Given $\Gamma \in \mathcal{L}$, $\rho \in R$ and $S \in Ty(U\Gamma)$, an object $\Gamma.\rho S$, s.t. $U(\Gamma.\rho S) = U\Gamma.S$.
Additionally, there exist the following natural transformations:
 - $emp_{\pi} : \diamond \rightarrow \pi\diamond$, s.t. $U(emp_{\pi}) = id_1$
 - $emp_+ : \diamond \rightarrow \diamond + \diamond$, s.t. $U(emp_+) = id_1$
 - $ext_{\pi} : \pi\Gamma.(\pi\rho S) \rightarrow \pi(\Gamma.\rho S)$, s.t. $U(ext_{\pi}) = id$
 - $ext_+ : (\Gamma_1 + \Gamma_2).(\rho_1 + \rho_2)S \rightarrow \Gamma_1.\rho_1 S + \Gamma_2.\rho_2 S$, s.t. $U(ext_+) = id$
7. Given $\Gamma \in \mathcal{L}$, $S \in Ty(U\Gamma)$, there exists:
 - a morphism $p_{\Gamma.S} : \Gamma.0S \rightarrow \Gamma$, s.t. $U(p_{\Gamma.S}) = p_{U\Gamma.S}$
 - an element $v_{\Gamma.S} \in RTm(0\Gamma.1S, S \{p_{U\Gamma.S}\})$, s.t. $U(v_{\Gamma.S}) = v_{U\Gamma.S}$
 - a morphism $wk(f, \rho S) : \Gamma.\rho S \{Uf\} \rightarrow \Delta.\rho S$ for each $f : \Gamma \rightarrow \Delta$, s.t. $U(wk(f, \rho S)) = wk(Uf, S)$
 - let $\Gamma_1, \Gamma_2 \in \mathcal{L}$, s.t. $U\Gamma_1 = U\Gamma_2$ and $M \in RTm(\Gamma_2, S)$. There is a morphism $\overline{\rho M} : \Gamma_1 + \rho\Gamma_2 \rightarrow \Gamma_1.\rho S$, s.t. $U(\overline{\rho M}) = \overline{UM}$
 - a morphism $\overline{M} : \Gamma \rightarrow \Gamma.0S$ for $M \in Tm(U\Gamma, S)$, s.t. $U(\overline{M}) = \overline{M}$.

Now to construct the concrete QCwF, consider the CwF from previous section as the underlying category.

Take $\mathcal{L} := \mathcal{RGph}(\mathcal{A})$ and let U be the functor $\mathcal{RGph}(\mathcal{A}) \rightarrow \mathcal{RGph}$, sending an assembly to its underlying set, forgetting the realizability function.

Type formers

Dependent function types in 1-fragment Given an internal reflexive graph Γ , $S \in Ty(\Gamma)$, $T \in Ty(\Gamma.S)$, $\rho \in R$, define $\Pi\rho ST \in Ty(\Gamma)$ as the tuple of:

²the second equality being trivially satisfied

- Object assembly. Let $\gamma \in |\Gamma_O|$

$$|(\Pi\rho ST)_O|(\gamma) := \{(f_O, f_R)|$$

$$f_O : \forall s \in S_O(\gamma). T_O(\gamma, s),$$

$$\exists a \in \mathcal{A} \forall s \in |S_O(\gamma)|, b \in \mathcal{A}.$$

$$b \models_{S_O(\gamma)} s \implies a.!\rho b \models_{T_O(\gamma, s)} f_O(s)$$

$$f_R : \forall r \in S_R(\Gamma_{refl}(\gamma)). T_R(\Gamma_{refl}(\gamma), r),$$

$$\exists a \in \mathcal{A} \forall r \in |S_R(\Gamma_{refl}(\gamma))|, b \in \mathcal{A}.$$

$$b \models_{S_R(\Gamma_{refl}(\gamma))} r \implies a.!\rho b \models_{T_R(\Gamma_{refl}(\gamma), r)} f_R(r),$$

$$\forall s \in S_O(\gamma). T_{refl}(\gamma, s)(f_O(s)) = f_R(S_{refl}(\gamma)(s)),$$

$$\forall r \in S_R(\Gamma_{refl}(\gamma)). T_{src}(\Gamma_{refl}(\gamma), r)(f_R(r)) = f_O(S_{src}(\Gamma_{refl}(\gamma))(r)),$$

$$\forall r \in S_R(\Gamma_{refl}(\gamma)). T_{tgt}(\Gamma_{refl}(\gamma), r)(f_R(r)) = f_O(S_{tgt}(\Gamma_{refl}(\gamma))(r))\}$$

$a \models_{(\Pi\rho ST)_O(\gamma)} (f_O, f_R)$ iff $a = [a_o, a_r]$ where the realizers a_o, a_r are given by the existential statements above.

- Relation assembly. Let $\gamma \in |\Gamma_R|$

$$|(\Pi\rho ST)_R|(\gamma) := \{((f_O^{src}, f_R^{src}), (f_O^{tgt}, f_R^{tgt}), r)|$$

$$(f_O^{src}, f_R^{src}) \in (\Pi\rho ST)_O(\Gamma_{src}(\gamma)),$$

$$(f_O^{tgt}, f_R^{tgt}) \in (\Pi\rho ST)_O(\Gamma_{tgt}(\gamma)),$$

$$r : \forall s \in S_R(\gamma). T_R(\gamma, s),$$

$$\exists a \in \mathcal{A} \forall s \in |S_R(\gamma)|, b \in \mathcal{A}. b \models_{S_R(\gamma)} s \implies a.!\rho b \models_{T_R(\gamma, s)} r(s),$$

$$\forall s \in S_R(\gamma). T_{src}(\gamma, s)(r(s)) = f_O^{src}(S_{src}(\gamma)(s)),$$

$$\forall s \in S_R(\gamma). T_{tgt}(\gamma, s)(r(s)) = f_O^{tgt}(S_{tgt}(\gamma)(s))\}$$

- collections of morphisms:

$$(\Pi\rho ST)_{refl}(\gamma)(f_O, f_R) := ((f_O, f_R), (f_O, f_R), f_R) \text{ for } \gamma \in \Gamma_O$$

$$(\Pi\rho ST)_{src}(\gamma)(f^{src}, f^{tgt}, r) := f^{src} \text{ for } \gamma \in \Gamma_R$$

$$(\Pi\rho ST)_{tgt}(\gamma)(f^{src}, f^{tgt}, r) := f^{tgt} \text{ for } \gamma \in \Gamma_R$$

These collections satisfy the identity conditions in Definition 4.