Relational realisability model for QTT

Our aim is to build a concrete realisability model for QTT which allows for meta-reasoning with results derived from parametricity. To that end, a construction of a quantitative category with families (QCwF, see [1]) is presented. We follow the relational approach to types introduced by Reynolds for typed lambda calculus [4] and later refined for dependent types theories [2].

Once and for all fix a usage semiring R and an R-linear combinatory algebra $\mathscr{A}$[1].

0 Preliminaries

We recall some basic structures used in constructing the model.

Definition 1 (Assemblies†)**.** An assembly† Γ is a pair $(|\Gamma|, e)$ where $|\Gamma|$ is a carrier set and e is a realisability function $|\Gamma| \to \mathcal{P}(\mathscr{A})$.

Given some $\gamma \in |\Gamma|$, $e(\gamma)$ is interpreted as the set of witnesses for the existence of γ. To emphasize on that aspect, we write $a \vDash_\Gamma \gamma$ to denote $a \in e(\gamma)$. Moreover, let $\lfloor\Gamma\rfloor$ stand for the set of realisable elements - $\{\gamma \in |\Gamma| : e_\Gamma(\gamma) \neq \emptyset\} \subseteq |\Gamma|$.

A morphism between two assemblies† Γ **and** Δ is a function $f : |\Gamma| \to |\Delta|$ that is realisable - there exists a realiser $a_f \in \mathscr{A}$ that tracks the function f in the following sense:

$$\text{for every } \gamma \text{ in } |\Gamma| \text{ and } a_\gamma \text{ in } \mathscr{A},\ a_\gamma \vDash_\Gamma \gamma \implies a_f.a_\gamma \vDash_\Delta f(\gamma) \text{ holds.}$$

Note that multiple realisers for the same function f do not induce multiple morphisms.
Using these notions we can construct a category $\mathcal{A}sm^\dagger(\mathscr{A})$.

Definition 2 (Reflexive graph)**.** A reflexive graph (r.g.) G is a tuple $(G_O, G_R, G_{refl}, G_{src}, G_{tgt})$, where G_O and G_R are sets, $G_{src} : G_R \to G_O$, $G_{tgt} : G_R \to G_O$ and $G_{refl} : G_O \to G_R$ are functions, s.t. the identities hold:

$$G_{src} \circ G_{refl} = id_{G_O} = G_{tgt} \circ G_{refl}$$

G_O and G_R stand for a set of objects and a set of relations, respectively. In general, reflexive graphs are less structured than categories as relations need not compose.

A morphism between reflexive graphs G **and** H is a pair of functions ($f_o : G_O \to H_O$, $f_r : G_R \to H_R$), s.t. all of the depicted squares commute:

[1] In case some non-trivial properties of $\mathscr{A}$ are required, we will assume that $\mathscr{A}$ is a graph model (see [3])

$$\begin{tikzcd} G_O \arrow[r, "f_o"] \arrow[d, "G_{refl}"] & H_O \arrow[d, "H_{refl}"] \\ G_R \arrow[u, bend left, "G_{src}"] \arrow[u, bend right, "G_{tgt}"'] \arrow[r, "f_r"] & H_R \arrow[u, bend left, "H_{src}"] \arrow[u, bend right, "H_{tgt}"'] \end{tikzcd}$$

Reflexive graphs equipped with r.g. morphisms form **a category** $\mathcal{R}Gph$. The terminal object $\mathbf{1}_{\mathcal{R}Gph}$ is $(\{\star\}, \{\star\}, id, id, id)$.

Reflexive graphs provide enough structure for a dyadic interpretation of types in the spirit of [4].

1 Reflexive graphs with realisability information

One could easily generalize reflexive graphs by considering object and relation components from arbitrary categories instead of $\mathcal{S}et$. As our purpose is to build a relational model incorporating realisability information, we replace the set of objects with an assembly† and retain the $\mathcal{S}et$-based representation of relations.

Definition 3 (Reflexive graph with realisable objects)**.** A reflexive graph with realisable objects G is a tuple $(G_O, G_R, G_{refl}, G_{src}, G_{tgt})$, where $G_O \in Ob(\mathcal{A}sm^\dagger(\mathscr{A}))$, G_R is a set and the functions $G_{refl} : |G_O| \to G_R$, $G_{src} : G_R \to |G_O|$, $G_{tgt} : G_R \to |G_O|$ are such that the identities in Definition 2 are satisfied.

Accordingly, **a morphism between reflexive graphs with realisable objects,** G **and** H, is a pair $(f_O : G_O \to H_O, f_R : G_R \to H_R)$, s.t. (f_O, f_R) is an $\mathcal{R}Gph$ morphism between the reflexive graphs $(|G_O|, G_R, G_{refl}, G_{src}, G_{tgt})$ and $(|H_O|, H_R, H_{refl}, H_{src}, H_{tgt})$.

With these components, we obtain a category $\mathcal{O}RGph(\mathcal{A}sm^\dagger(\mathscr{A}))$. By considering reflexive graphs with realisable objects of shape $(X, |X|, id_X, id_X, id_X)$, we identify an isomorphic copy of $\mathcal{A}sm^\dagger(\mathscr{A})$ inside $\mathcal{O}RGph(\mathcal{A}sm^\dagger(\mathscr{A}))$.

A terminal object $\mathbf{1}_{\mathcal{O}RGph(\mathcal{A}sm^\dagger(\mathscr{A}))}$ in $\mathcal{O}RGph(\mathcal{A}sm^\dagger(\mathscr{A}))$ is a tuple $(\mathbf{1}_{\mathcal{A}sm^\dagger(\mathscr{A})}, \{*\}, id, id, id)$, where $\mathbf{1}_{\mathcal{A}sm^\dagger(\mathscr{A})}$ is the terminal assembly† $(\{\star\}, f)$, with f defined as $\star \mapsto \{I\}$.

Definition 4 (Family of reflexive graphs with realisable objects)**.** Let $\mathcal{C}$ be a category with a terminal object. Given a reflexive graph $\Gamma \in Ob(\mathcal{C})$, a family of reflexive graphs with realisable objects over Γ is a tuple $S := (S_O, S_R, S_{refl}, S_{src}, S_{tgt})$, where:

- $S_O : \Gamma_O \to \mathcal{A}sm^\dagger(\mathscr{A})$
- $S_R : \Gamma_R \to \mathcal{S}et$

- a Γ-indexed collection of functions $S_{refl} := \{f_\gamma : |S_O(\gamma)| \to S_R(\Gamma_{refl}(\gamma))\}_{\gamma \in \Gamma_O}$
- $S_{src} := \{f_\gamma : S_R(\gamma) \to |S_O(\Gamma_{src}(\gamma))|\}_{\gamma \in \Gamma_R}$
- $S_{tgt} := \{f_\gamma : S_R(\gamma) \to |S_O(\Gamma_{tgt}(\gamma))|\}_{\gamma \in \Gamma_R}$

such that

- each identity in the following collection is satisfied:

$$S_\sigma(\Gamma_{refl}(\gamma)) \circ S_{refl}(\gamma) = id \text{ for every } \gamma \in \Gamma_O, \sigma \in \{src, tgt\}$$

A morphism M between two families S and T of reflexive graphs with realisable objects over Γ is a pair (M_O, M_R) of Γ-indexed collections of morphisms:

- $M_O := \{f_\gamma : S_O(\gamma) \to T_O(\gamma)\}_{\gamma \in \Gamma_O}$
- $M_R := \{f_\gamma : S_R(\gamma) \to T_R(\gamma)\}_{\gamma \in \Gamma_R}$

s.t. the following identities are satisfied:

- $T_{refl}(M_O(\gamma)(s_o)) = M_R(\Gamma_{refl}(\gamma))(S_{refl}(\gamma)(s_o))$ for every $\gamma \in \Gamma_O$, $s_o \in |S_O(\gamma)|$
- $T_{src}(M_R(\gamma)(s_r)) = M_O(\Gamma_{src}(\gamma))(S_{src}(\gamma)(s_r))$ for every $\gamma \in \Gamma_R$, $s_r \in S_R(\gamma)$
- $T_{tgt}(M_R(\gamma)(s_r)) = M_O(\Gamma_{tgt}(\gamma))(S_{tgt}(\gamma)(s_r))$ for every $\gamma \in \Gamma_R$, $s_r \in S_R(\gamma)$

Families of reflexive graphs with realisable objects over Γ and their morphisms forms a category $\mathcal{F}am{-}\mathcal{ORG}(\mathcal{A}sm^\dagger(\mathscr{A}), \Gamma)$.

The terminal object $\mathbf{1}_{\mathcal{F}am-\mathcal{ORG}(\mathcal{A}sm^\dagger(\mathscr{A}),\boldsymbol{\Gamma})}$ is given by $(\lambda\gamma_r.\mathbf{1}_{\mathcal{A}sm^\dagger(\mathscr{A})}, \lambda\gamma_r.\{*\}, \mathbf{1_{refl}}, \mathbf{1_{tgt}}, \mathbf{1_{src}})$, where $\mathbf{1}_\sigma$ are the appropriate functions.

2 A CwF from families of reflexive graphs with realisable objects

As a first step toward obtaining a relationally parametric realisability model of QTT, we construct a concrete CwF using families of reflexive graphs with realisable objects.

Consider the category $\mathcal{R}Gph$ with terminal object $\mathbf{1}_{\mathcal{R}Gph}$.
Let $\Gamma, \Delta \in Ob(\mathcal{R}Gph)$, define:

- the collection of semantic types $Ty(\Gamma)$ as the collection of families of reflexive graph with realisable objects over Γ.
- given a type $S \in Ty(\Gamma)$, an element $M \in Tm(\Gamma, S)$ is a pair of functions
  $(M_O : \forall\gamma \in \Gamma_O.|S_O(\gamma)|,\ M_R : \forall\gamma \in \Gamma_R.S_R(\gamma))$, s.t.

$$\forall\gamma \in \Gamma_O.S_{refl}(M_O(\gamma)) = M_R(\Gamma_{refl}(\gamma))$$
$$\forall\gamma \in \Gamma_R.S_{src}(M_R(\gamma)) = M_O(\Gamma_{src}(\gamma))$$
$$\forall\gamma \in \Gamma_R.S_{tgt}(M_R(\gamma)) = M_O(\Gamma_{tgt}(\gamma))$$

- given $f : \Gamma \to \Delta$, substitutions in types and terms is a precomposition with f on the object and relation components of types and terms respectively. Clearly, these operations are compatible with identity and composition in $\mathcal{R}Gph$.
- context extension: Suppose $S \in Ty(\Gamma)$, construct a r.g. $\Gamma.S$ as :

$$\begin{aligned}
(\Gamma.S)_O &= \{(\gamma, x) : \gamma \in \Gamma_O, x \in |S_O(\gamma)|\} \\
(\Gamma.S)_R &= \{(\gamma, r) : \gamma \in \Gamma_R, r \in S_R(\gamma)\} \\
(\Gamma.S)_{refl}(\gamma, x) &= (\Gamma_{refl}(\gamma), S_{refl}(\gamma)(x)) \\
(\Gamma.S)_\sigma(\gamma, r) &= (\Gamma_\sigma(\gamma), S_\sigma(\gamma)(r)), \quad \sigma \in \{src, tgt\}
\end{aligned}$$

Claim. *$Hom_{\mathcal{R}Gph}(\Delta, \Gamma.S) \cong \{(f, M) : f : \Delta \to \Gamma, M \in Tm(\Delta, S\{f\}\}$, natural in Δ.*

Upgrading to a QCwF

Recall the definition of a QCwF from [1]. Given a usage semiring R, a R-QCwF consists of:

1. A CwF $(\mathcal{C}, 1, Ty, Tm, -.-, \langle -.- \rangle)$
2. A category $\mathcal{L}$ with a faithful functor $U : \mathcal{L} \to \mathcal{C}$
3. A functor $(+) : \mathcal{L} \times_\mathcal{C} \mathcal{L} \to \mathcal{L}$, s.t $U(\Gamma_1 + \Gamma_2) = U(\Gamma_1) = U(\Gamma_2)$[2]. $\mathcal{L} \times_\mathcal{C} \mathcal{L}$ denotes the pullback $\mathcal{L} \xrightarrow{U} \mathcal{C} \xleftarrow{U} \mathcal{L}$.
   Additionally, there exists an object $\Diamond \in \mathcal{L}$, s.t. $U\Diamond = 1$.
4. A functor $\rho(-) : \mathcal{L} \to \mathcal{L}$ for each $\rho \in R$, s.t $U(\rho(-)) = U(-)$.
5. A collection $RTm(\Gamma, S)$ for each $\Gamma \in \mathcal{L}$ and $S \in Ty(U\Gamma)$, equipped with an injective function $U_{\Gamma,S} : RTm(\Gamma, S) \to Tm(U\Gamma, S)$.
   For an $\mathcal{L}$ morphisms $f : \Gamma \to \Delta$ and types $S \in Ty(U\Gamma)$, a function $-\{f\} : RTm(\Delta, S) \to RTm(\Gamma, S\{f\})$, s. t. $U(-\{f\}) = (U(-))\{Uf\}$.
6. Given $\Gamma \in \mathcal{L}$, $\rho \in R$ and $S \in Ty(U\Gamma)$, an object $\Gamma.\rho S$, s.t $U(\Gamma.\rho S) = U\Gamma.S$.
   Additionally, the following natural transformations exist:
   $emp_\pi : \Diamond \to \pi\Diamond$
   $emp_+ : \Diamond \to \Diamond + \Diamond$
   $ext_\pi : \pi\Gamma.(\pi\rho S) \to \pi(\Gamma.\rho S)$, s.t. $U(ext_\pi) = id$
   $ext_+ : (\Gamma_1 + \Gamma_2).(\rho_1 + \rho_2)S \to \Gamma_1.\rho_1 S + \Gamma_2.\rho_2 S$, s.t. $U(ext_+) = id$
7. Given $\Gamma \in \mathcal{L}$, $S \in Ty(U\Gamma)$, there exists :
   a morphism $p_{\Gamma.S} : \Gamma.0S \to \Gamma$, s.t. $U(p_{\Gamma.S}) = p_{U\Gamma.S}$
   an element $v_{\Gamma.S} \in RTm(0\Gamma.1S, S\{p_{U\Gamma.S}\})$, s.t. $U(v_{\Gamma.S}) = v_{U\Gamma.S}$
   a morphism $wk(f, \rho S') : \Gamma.\rho S'\{Uf\} \to \Delta.\rho S'$ for each $f : \Gamma \to \Delta$, $S' \in Ty(U\Gamma, \Delta)$ s.t. $U(wk(f, \rho S')) = wk(Uf, S')$
   let $\Gamma_1, \Gamma_2 \in \mathcal{L}$, s.t $U\Gamma_1 = U\Gamma_2$ and $M \in RTm(\Gamma_2, S)$. There is a morphism $\overline{\rho M} : \Gamma_1 + \rho\Gamma_2 \to \Gamma_1.\rho S$, s.t $U(\overline{\rho M}) = \overline{UM}$
   a morphism $\overline{M} : \Gamma \to \Gamma.0S$ for $M \in Tm(U\Gamma, S)$, s.t. $U(\overline{M}) = \overline{M}$.

[2]the second equality being trivially satisfied

Now to construct the concrete QCwF, consider the CwF from previous section as the underlying category.

Let $\mathcal{L} := \mathcal{ORGph}(\mathcal{A}sm^{\dagger}(\mathscr{A}))$, the category of reflexive graphs with realisable objects
For the addition structure, let Γ', Γ'' be reflexive graphs with realisable objects, s.t $|\Gamma'_O| = |\Gamma''_O|$ and $\Gamma'_R = \Gamma''_R$

- $\Gamma_O := (|\Gamma'_O|, \vDash_\Gamma)$ with $a \vDash_\Gamma \gamma$ iff there exist $x, y \in \mathscr{A}$, s.t. $a = [x, y]$ and $x \vDash_{\Gamma'} \gamma$ and $y \vDash_{\Gamma''} \gamma$.
- $\Gamma_R := \Gamma'_R (= \Gamma''_R)$.
- $\Gamma_\sigma := \Gamma'_\sigma$, where $\sigma \in \{src, \, tgt, \, refl\}$.

Define $\Diamond$ as the terminal object $\mathbf{1}_{\mathcal{ORGph}(\mathcal{A}sm^{\dagger}(\mathscr{A}))}$.

Consider the scaling structure and let $\Gamma := \rho(\Gamma')$:

- $\Gamma_\sigma = (|\Gamma'_\sigma|, \vDash_{\Gamma_\sigma})$ with $a \vDash_{\Gamma_\sigma} \gamma$ iff there is $x \in \mathscr{A}$, s.t $a = !_\rho x$ and $x \vDash_{\Gamma'_\sigma} \gamma$ for $\sigma \in \{O, R\}$
- again, scaling leaves unmodified Γ_σ for $\sigma \in \{src, \, tgt, \, refl\}$.

Let $RTm(\Gamma, S)$ be the collection of morphisms from the terminal family $\mathbf{1}_{\mathcal{F}am-\mathcal{ORG}(\mathcal{A}sm^{\dagger}(\mathscr{A}),\mathbf{\Gamma})}$ to S. Spelling this out and simplifying it, an element of $RTm(\Gamma, S)$ is a tuple of functions $(M_O : \forall \gamma_o \in \Gamma_O.|S_O(\gamma_o)|, M_R : \forall \gamma_r \in \Gamma_R.S_R(\gamma_r))$, s.t. the conditions from Definition 4 are satisfied, namely

$$\begin{aligned}
\forall \gamma_o \in \Gamma_O.S_{refl}(M_O(\gamma_o)) &= M_R(\Gamma_{refl}(\gamma_o)) \\
\forall \gamma_r \in \Gamma_R.S_{src}(M_R(\gamma_r)) &= M_O(\Gamma_{src}(\gamma_r)) \\
\forall \gamma_r \in \Gamma_R.S_{tgt}(M_R(\gamma_r)) &= M_O(\Gamma_{tgt}(\gamma_r))
\end{aligned}$$

such that M_O is tracked - $\exists a_M \in \mathscr{A}. \forall a_\gamma \in \mathscr{A}, \gamma \in \Gamma_O. \, a_\gamma \vDash_{\Gamma_O} \gamma \implies a_m \cdot a_\gamma \vDash_{S(\gamma)} M_O(\gamma)$.
$U_{\Gamma,S}$ is the just identity function.
Substitution in terms is given by precomposition with $f : \Gamma \to \Delta$, let $-\{f_O\} := \lambda M_O. \forall \gamma \in \Gamma.M_O(f(\gamma))$ and similarly, $-\{f_R\} := \lambda M_R. \forall \gamma \in \Gamma.M_R(f(\gamma))$. The functor U interacts nicely with the so-defined $-\{f\}$ as essentially the substitution in terms in the underlying CwF is defined in the same way.

Resourced context extension is given by $\Gamma.\rho S := (\Gamma'_O, \Gamma'_R, \Gamma'_\sigma)$, where

- $|\Gamma'_O| := \{(\gamma, x) : \gamma \in |\Gamma_O|, x \in S_O(\gamma)\}$
  $a \vDash_{\Gamma.\rho S} (\gamma, x)$ iff there exists $b, c \in \mathscr{A}$, s.t $a = [b, !_\rho c]$, $b \vDash_\Gamma \gamma$ and $c \vDash_{S(\gamma)} \pi_1(\widecheck{(\gamma, x)})$, where $\widecheck{(-)} : \Gamma.S \to U(\Gamma.S)$, $\widecheck{(-)} := id$ as the set-theoretic part of the extensions in the CwF and $\mathcal{L}$ is the same by definition.

- $\Gamma'_R := \{(\gamma, r) : \gamma \in \Gamma_R, r \in S_R(\gamma)\}$
- Each Γ'_σ is defined pointwise.

The natural transformation $emp_\pi : \Diamond \to \pi\Diamond$ is given by the identity functions on both the object and relational part. It is realised by $K.!_\rho I$.
We list the realisers for the remaining transformations:

- emp_+ - $K.[I, I]$,
- ext_π - $\lambda^* q.let\ [x, y] = q\ in\ F_\pi.(F_\pi.(!_\pi \lambda^* stu.ust).x).\delta_{\pi\rho} y$
- ext_+ - $\lambda^* q.let\ [[x, y], z] = q\ in\ W_{\pi\rho}.(\lambda^* ab.[[x, a], [y, b]]).z$

The underlying function part of the above-defined natural transformations is given by identity, hence naturality follows trivially.

Finally, we construct the morphisms, listed in Item 7:

- $p_{\Gamma.S} : \Gamma.0S \to \Gamma$ is the first projection of $(\Gamma.0S)_\sigma = \{(\gamma, s) : \gamma \in \Gamma_\sigma.s \in S(\gamma)\}$, $(\sigma \in \{_{O,R}\})$ and is realized by $\lambda^* t.(t.K)$.
  The equality $U(p_{\Gamma.S}) = p_{U\Gamma.S}$ holds trivially due to the identical structure of context extension in the underlying CwF and $\mathcal{L}$.

- define $v_{\Gamma.S} \in RTm(0\Gamma.1S, S\{p_{U\Gamma.S}\})$ as the second projection. $v_{\Gamma.S}$ is realized by $\lambda^* t.\ B.t.K.D$.

- Let a^σ_f realize f_σ, then $wk(f, \rho S') := \lambda(\gamma, s).(f(\gamma), s)$ is realized by $\lambda^* q.let\ [x, y] = q\ in\ [a^\sigma_f.x, y]$

- given a $M_\sigma \in RTm(\Gamma, S)$, $M_\sigma : \forall\gamma \in U(\Gamma_\sigma).S_\sigma(\gamma)$ with realizers a^σ_m, let $\overline{\rho M}_\sigma := \lambda\gamma.(\gamma, M_\sigma(\gamma))$ realized by $\lambda^* q.let\ [x, y] = q\ in\ [x, F_\rho.(!_\rho a^\sigma_m).y]$

- given a $M_\sigma \in Tm(U\Gamma, S) = M_\sigma : \forall\gamma \in U\Gamma_\sigma.S_\sigma(\gamma)$, let $\overline{M}_\sigma := \lambda\gamma.(\gamma, M_\sigma(\check{\gamma}))$ realized by the K combinator.

From now on, we refer to the constructed model as $\mathbb{M}$.

3 Type formers

Definition 5 (Dependent function types a QCwF)**.** A QCwF supports dependent function types with usage information, if

- the underlying CwF $\mathcal{C}$ supports them, namely, if for all $\Gamma \in Ob(\mathcal{C}), S \in Ty(\Gamma), T \in Ty(\Gamma.S)$, $\pi \in R$, there exist type $\Pi\pi ST \in Ty(\Gamma)$ and a bijection

$$\Lambda : Tm(\Gamma.S, T) \cong Tm(\Gamma, \Pi\pi ST),$$

  natural in Γ.

- for $\Gamma \in Ob(\mathcal{L}), S \in Ty(U\Gamma), T \in Ty(U\Gamma.S), \pi \in R$, there exists a bijection

$$\Lambda_{\mathcal{L}} : RTm(\Gamma.\pi S, T) \cong RTm(\Gamma, \Pi\pi ST),$$

  natural in Γ such that $U \circ \Lambda_{\mathcal{L}} = \Lambda \circ U$ and $U \circ \Lambda_{\mathcal{L}}^{-1} = \Lambda^{-1} \circ U$.

To show that our model supports Π types, fix some $\pi \in R$, suppose Γ is a r.g in $Ob(\mathcal{C})$, $S \in Ty(\Gamma)$, $T \in Ty(\Gamma.S)$. Define the semantic type $\Pi\pi ST$ as the family of assemblies over Γ, consisting of:

- $(\Pi\pi ST)_O(\gamma) := (X, \vDash_X)$ for $\gamma \in \Gamma_O$, where

$$\begin{aligned} X := \{(f_O, f_R) \quad | \quad & \\ & f_O : \forall s \in |S_O(\gamma)|.T_O(\gamma, s), \\ & f_R : \forall r \in S_R(\Gamma_{refl}(\gamma)).T_R(\Gamma_{refl}(\gamma), r), \\ & \forall s \in S_O(\gamma).T_{refl}(\gamma, s)(f_O(s)) = f_R(S_{refl}(\gamma)(s)), \\ & \forall r \in S_R(\Gamma_{refl}(\gamma)).T_{src}(\Gamma_{refl}(\gamma), r)(f_R(r)) = f_O(S_{src}(\Gamma_{refl}(\gamma))(r)), \\ & \forall r \in S_R(\Gamma_{refl}(\gamma)).T_{tgt}(\Gamma_{refl}(\gamma), r)(f_R(r)) = f_O(S_{tgt}(\Gamma_{refl}(\gamma))(r))\} \end{aligned}$$

$$a \vDash_X (f_O, f_R) \text{ iff } \forall s \in |S_O(\gamma)|, a_s \in \mathscr{A}.a_s \vDash_{S_O(\gamma)} s \implies a \cdot !_\pi a_s \vDash_{T_O(\gamma,s)} f_O(s).$$

  Note that f_R does not contribute any realisability information to $\vDash_X$.

- $(\Pi\pi ST)_R(\gamma) :=$

$$\begin{aligned} \{((f_O^{src}, f_R^{src}), (f_O^{tgt}, f_R^{tgt}), r) \quad | \quad & \\ & (f_O^{src}, f_R^{src}) \in (\Pi\pi ST)_O(\Gamma_{src}(\gamma)), \\ & (f_O^{tgt}, f_R^{tgt}) \in (\Pi\pi ST)_O(\Gamma_{tgt}(\gamma)), \\ & r : \forall s \in S_R(\gamma).T_R(\gamma, s), \\ & \forall s \in S_R(\gamma).T_{src}(\gamma, s)(r(s)) = f_O^{src}(S_{src}(\gamma)(s)), \\ & \forall s \in S_R(\gamma).T_{tgt}(\gamma, s)(r(s)) = f_O^{tgt}(S_{tgt}(\gamma)(s))\} \end{aligned}$$

- $(\Pi\pi ST)_{refl}(\gamma) := \lambda(f_O, f_R).((f_O, f_R), (f_O, f_R), f_R)$ for $\gamma \in \Gamma_O$.
- $(\Pi\pi ST)_{src}(\gamma) := \lambda(f^{src}, f^{tgt}, r).f^{src}$ for $\gamma \in \Gamma_R$.
- $(\Pi\pi ST)_{tgt}(\gamma) := \lambda(f^{src}, f^{tgt}, r), f^{tgt}$ for $\gamma \in \Gamma_R$.

Unwinding Definition 5, we get

$$\{(M_O, M_R) : \forall(\gamma, s) \in (\Gamma.\pi S).T(\gamma, s)\} \cong \{(N_O, N_R) : \forall\gamma \in \Gamma.(\Pi\pi ST)(\gamma)\}$$

where the terms have the following elaborated types:

$$
\begin{aligned}
&M_O : \forall(\gamma, s) \in (\Gamma.\pi S)_O.T_O(\gamma, s)\\
&M_R : \forall(\gamma, r) \in (\Gamma.\pi S)_R.T_R(\gamma, s)\\
&N_O : \forall\gamma \in \Gamma_O.\\
&\quad \{(f_O, f_R) \quad | \quad f_O : \Pi S(\gamma)_O.T(\gamma)_O\\
&\qquad\qquad\qquad\quad\; f_R : \Pi S_R(\Gamma_{refl}(\gamma).T(\Gamma_{refl}(\gamma)))\}\\
&N_R : \forall\gamma \in \Gamma_R.\\
&\quad \{(f^{src}, f^{tgt}, r) \quad | \quad f^{src} \in (\Pi\pi ST)_O(\Gamma_{src}(\gamma))\\
&\qquad\qquad\qquad\qquad\quad\; f^{tgt} \in (\Pi\pi ST)_O(\Gamma_{tgt}(\gamma))\\
&\qquad\qquad\qquad\qquad\quad\; r : \Pi S_R(\gamma).T_R(\gamma)\}
\end{aligned}
$$

Thus, we can define Λ as $\Lambda(M_O, M_R) = (N_O, N_R)$, where

$$N_O := \lambda\gamma_o.(\lambda s.M_O(\gamma_o, s), \lambda s_r.M_R(\Gamma_{refl}(\gamma_o), s_r))$$

$$N_R := \lambda\gamma_r.(N_O(\Gamma_{src}(\gamma)), N_O(\Gamma_{tgt}(\gamma)), \lambda s_r.M_R(\gamma, s_r)))$$

For $\Lambda_{\mathcal{L}}$, a realizer a_m of M
(*that is* $\forall(\gamma, s), \forall(a_\gamma, a_s), [a_\gamma, a_s] \vDash_{\Gamma.\pi S} (\gamma, s) \implies a_m.[a_\gamma, a_s] \vDash_{T(\gamma,s)} M(\gamma, s)$)
can be transformed to a realizer a_n of N by:

$$a_n := \lambda^* y.(\lambda^* s.(a_m.[y, s]))$$

The conditions $U \circ \Lambda_{\mathcal{L}} = \Lambda \circ U$ and $U \circ \Lambda_{\mathcal{L}}^{-1} = \Lambda^{-1} \circ U$ follow trivially.

Universe of small types A plausible candidate for the universe U is given by the general construction:
$U_O :=$ the set of small reflexive graphs
$U_R := \{(A, B, R, R_{src}, R_{tgt}) : R \in \mathcal{U}, R_{src} : R \to A_O, R_{tgt} : R \to A_O, A, B \text{ are small r.g.}\}^{\cdot}$

However, this universe turns out to be "too big' - types do not carry enough structure to model parametricity accurately. To remedy the definition, we impose restrictions laid out by the following characterization:

A family of reflexive graphs of assemblies S over Γ is

- small - if for all $\gamma_\sigma \in \Gamma_\sigma$, $S_R(\gamma_R) \in \mathcal{U}$ and $|S_O(\gamma_O)| \in \mathcal{U}$.
- discrete - if for every $\gamma \in \Gamma_O$, there exists $X \in \mathcal{A}sm^\dagger(\mathscr{A})$, s.t.

$$\begin{tikzcd} S_O(\gamma) \arrow[d, "S_{refl}(\gamma)" description] \\ S_R(\Gamma_{refl}(\gamma)) \arrow[u, bend left=50, "S_{src}(\Gamma_{refl}(\gamma))"] \arrow[u, bend right=50, "S_{src}(\Gamma_{refl}(\gamma))"'] \end{tikzcd} \quad \cong \quad \begin{tikzcd} X \arrow[d, "id" description] \\ |X| \arrow[u, bend left=40, "id"] \arrow[u, bend right=40, "id"'] \end{tikzcd}$$

- proof-irrelevant - if for all $\gamma \in \Gamma_R$, the function $\langle S_{src}(\gamma), S_{tgt}(\gamma)\rangle : S_R(\gamma) \to |S_O(\Gamma_{src}(\gamma))| \times |S_O(\Gamma_{tgt}(\gamma))|$ is injective.

For any reflexive graph Γ , define the small, discrete, proof-irrelevant universe $U \in Ty(\Gamma)$ and the type decoder $T \in Ty(\Gamma.U)$ as:

- $|U_O(\gamma)| :=$ the set of small, discrete r.g. of assemblies
  $a \vDash_{U_O(\gamma)} S$ for $a \in \mathscr{A}$, $a = I$ and $S \in |U_O(\gamma)|$.

- $U_R(\gamma_R) :=$
  $$\begin{aligned} \{(S, T, R, R_{src}, R_{tgt}) \quad | \quad & R \in \mathcal{U} \\ & S, T \text{ are small discrete r.g. of assemblies} \\ & \langle R_{src}, R_{tgt}\rangle : R \to |S_O| \times |T_O| \ \textit{is injective}\} \end{aligned}$$

- $U_{refl}(\gamma_R)(S) := (S, |S|, S_{refl}, S_{src}, S_{tgt})$

- $U_{src}(\gamma_R)(S, T, R, R_{src}, R_{tgt}) = S$

- $U_{src}(\gamma_R)(S, T, R, R_{src}, R_{tgt}) = T$

and $T \in Ty(\Gamma.U)$ as:

- $T_O(\gamma_O, S) := S_O$

- $T_R(\gamma_R, (S, T, R, R_{src}, R_{tgt})) := R$

- $T_{refl}(\gamma_O, S) := S_{refl}$

- $T_{src}(\gamma_R, (S, T, R, R_{src}, R_{tgt})) := R_{src}$

- $T_{tgt}(\gamma_R, (S, T, R, R_{src}, R_{tgt})) := R_{tgt}$.

Claim. *U is closed under Π types.*

Given some r.g Γ and $S \in Ty(\Gamma)$, $T \in Ty(\Gamma.S)$, it suffices to show that $\Pi\pi TS \in Ty(\Gamma)$ is a small, discrete and proof-irrelevant family of r.g. of assemblies† [3] For brevity, let $V := \Pi\pi ST$.

Smallness follows by the closure under Π-types in the ambient set-theoretical universe $\mathcal{U}$.
For proof-irrelevance, take some $\gamma_R \in \Gamma$ and $f, g \in V_R(\gamma_R)$, s.t $\langle V_{src}(\gamma_R), V_{tgt}(\gamma_R)\rangle(f) = \langle V_{src}(\gamma_R), V_{tgt}(\gamma_R)\rangle(g)$. WTP $f = g$, by def. we get immediately that $f^{src} = g^{src}$ and $f^{tgt} = g^{tgt}$. Given that $(f^{src}, f^{tgt}, r_f), (f^{src}, f^{tgt}, r_g) \in V_R(\gamma_R)$, note that ,

$$\forall s \in S_R(\gamma_R).\langle T_{src}(\gamma_R, s), T_{src}(\gamma_R, s)\rangle(r_f(s)) = (f_O^{src}(S_{src}(\gamma_R)(s)), f_O^{tgt}(S_{tgt}(\gamma_R)(s)))$$
$$\forall s \in S_R(\gamma_R).\langle T_{src}(\gamma_R, s), T_{src}(\gamma_R, s)\rangle(r_g(s)) = (f_O^{src}(S_{src}(\gamma_R)(s)), f_O^{tgt}(S_{tgt}(\gamma_R)(s)))$$

Since T is proof-irrelevant, it follows directly that $r_f = r_g$ and thus $f = g$.

Reduced Π types - $\Pi^R\pi ST$ Our proposed definition of Π types give rise to few subtle issues with realisability. Ideally, the elements of an assembly† and their realisers are in 1:1 correspondence. But clearly this is not the case with Π types:

- $|(\Pi\pi ST)_O(-)|$ contains all set-theoretic functions $f : (s : S) \to T$, so some elements do not have realisers
- two distinct functions $f, f' : (s : S) \to T$ might have identical behaviour on $\lfloor S \rfloor$ but differ on some element of $|S| \setminus \lfloor S \rfloor$. Thus f and f' will share the same realiser a.

To remedy this situation one might try to "prune" some of the elements of $|(\Pi\pi ST)_O(-)|$ and ensure that terms are to be defined in "nice enough" contexts.
Formalizing on these considerations, we introduce a new reduced function type, a derivative of the standard $\Pi\pi ST$ types with the main difference being that realisability is now internalised in the object assembly† part:

$(\Pi^R\pi ST)_O(\gamma_O) := (X, \vDash_X)$, where

$$\begin{aligned} X := \quad & \{(f_O, f_R) \quad | \quad f_O : \forall s \in |S_O(\gamma)|.T_O(\gamma, s), \quad \ldots \\ & \quad \exists a_f \in \mathscr{A}.\forall s \in |S_O(\gamma)|, a_s \in \mathscr{A}.a_s \vDash_{S_O(\gamma)} s \implies a_f \cdot !_\pi a_s \vDash_{T_O(\gamma,s)} f_O(s), \\ & \ldots\} \end{aligned}$$

A new operation on resourced contexts, $\mathbf{R}$, takes care of removing any nonrealisable element. The functor is defined on objects as:

$\mathbf{R} : \mathcal{ORGph}(\mathcal{A}sm^{\dagger}(\mathscr{A})) \to \mathcal{ORGph}(\mathcal{A}sm^{\dagger}(\mathscr{A}))$
$\mathbf{R}(X_O, X_R, X_{refl}, X_{src}, X_{tgt}) = ((\lfloor X_O \rfloor, \vDash_X), X_R, X_{refl}, X_{src}, X_{tgt})$

[3] it actually does not suffice, we implicitly assume a lemma that gives an isomorphism between terms of type U and small, discrete, p.i. r.g of assemblies†

and extends naturally to $\mathcal{O}RGph(\mathcal{A}sm^{\dagger}(\mathscr{A}))$ morphisms. It can be also endowed with comonadic structure by observing we have a canonical choice for the natural transformations $\mathbf{R}(X) \to X$ and $\mathbf{R}(X) \to \mathbf{R}(\mathbf{R}(X))$, namely the canonical injection and identity. Also observe that the functors $!_\pi$ commute with $\mathbf{R}$, particularly $!_0(\mathbf{R}(X)) = \mathbf{R}(!_0(X))$.

On the surface level, we have the usual type formation rule in the syntax:

$$\frac{0\Gamma \vdash S \quad 0\Gamma, x \overset{0}{:} S \vdash T}{0\Gamma \vdash \Pi^R \pi S T} \text{ reduced } \Pi$$

However, the abstraction rule differs from the standard Π-type. In general, given an unresourced term M, $\Gamma, x \overset{0}{:} S \vdash M \overset{0}{:} T$, one cannot find a corresponding λ term, $\Gamma \vdash (\lambda x \overset{\pi}{:} S.M) \overset{0}{:} \Pi^R \pi S T$. That is to say, no bijection on the object level exists between $f_M : \forall(\gamma, s).T(\gamma, s)$ and $f_\lambda : \forall\gamma.(\Pi^R \pi S T)_O(\gamma)$ as the latter function carries some notion of realisability while the former does not.
Such bijection exists in the 1 fragment, though. Thus, there is the reduced abstraction rule:

$$\frac{\Gamma, x \overset{\pi}{:} S \vdash M \overset{1}{:} T}{\mathbf{R}(\Gamma) \vdash (\lambda x \overset{\pi}{:} S.T) \overset{1}{:} \Pi^R \pi S T} \text{ reduced Lam}$$

To verify it is sound, suppose we are given a function $t_M : \forall(\gamma, s) \in |\Gamma.\pi S|_O.T(\gamma, s)$ and its tracker $a_M \in \mathscr{A}$. Unwinding the latter, we get :

$$\forall\gamma \in |\Gamma_O|, s \in |S_O(\gamma)|, a_\gamma, a_s \in \mathscr{A}, a_\gamma \vDash_{\Gamma_O} \gamma \wedge a_s \vDash_{S_O(\gamma)} s \implies a_M \cdot [a_\gamma, !_\pi a_s] \vDash_{T_O(\gamma,s)} t_m(\gamma, s)$$

Focus only on the object part and fix some $\gamma \in |\Gamma_O|$. Now we have to construct an inhabitant (f_O^γ, f_R^γ) of $|\Pi^R \pi S T_O(\gamma)|$. Again forgetting the relational part, let $f_O^\gamma := \lambda s \overset{\pi}{:} S.t_m(\gamma, s)$. Its realiser is a_f must satisfy

$$\forall s \in |S_O(\gamma)|, a_s \in \mathscr{A}.a_s \vDash_{S_O(\gamma)} s \implies a_f \cdot !_\pi a_s \vDash_{T(\gamma,s)} t_m(\gamma, s)$$

A plausible solution is $a_f := \lambda^* x.a_M \cdot [a_\gamma, x]$. Observe that in order a_f to be well-defined, a_γ must always exist. This is guaranteed by the application of $\mathbf{R}$ to the context Γ in the conclusion.
Finally, we can construct a term of Π^R type - namely let $t_\lambda := \lambda\gamma : |\mathbf{R}(\Gamma)|_O.f_O^\gamma$ with tracker $\lambda^* y\, x.a_m[y, x]$.
As a last step, we also have a non-altered application rule as well:

$$\frac{\Gamma_2 \vdash N \overset{\sigma'}{:} S \qquad \begin{matrix}\Gamma_1 \vdash M \overset{\sigma}{:} \Pi^R \pi S T \\ 0\Gamma_1 = 0\Gamma_2\end{matrix} \qquad \sigma' = 0 \iff (\pi = 0 \vee \sigma = 0)}{\Gamma_1 + \pi\Gamma_2 \vdash App_{(\Pi^R \pi S T)}(M, N) \overset{\sigma}{:} T[N/x]} \text{ App}$$

For brevity, let's verify only the case of $\sigma = \sigma' = 1$. Let $[\![\Gamma_1]\!] = (|\Gamma|, \vDash_1)$ and $[\![\Gamma_2]\!] = (|\Gamma|, \vDash_2)$, $(t_m, \_) : \forall \gamma \in |\Gamma_O|.(\Pi^R \pi ST)_O(\gamma)$, $t_n : \forall \gamma \in |\Gamma_O|.S_O(\gamma)$, a_m is a tracker for t_m, a_n - for t_n. Then we construct a function $t_{app} : \forall \gamma \in |\Gamma_O|.T[N/x]$, $t_{app} := \lambda \gamma.t_m(\gamma)(t_n(\gamma))$ with a realiser $a_{app} := \lambda^* x.\text{let } [a_1, a_2] = x \text{ in } a_m \cdot a_1 \cdot a_2$.

A subuniverse U^R Although introducing Π^R types was a step into right direction, we have not quite yet guaranteed bijection between elements of some assembly† and their realisers. To that end, we will semantically identify the types which enjoy that property (w.r.t their object part).

Definition 6 (Comodest set/Distinct realisers ?)**.** Let A be an assembly†. A has distinct realisers if for every x, y in $|A|$, $a \in \mathscr{A}$, the following holds

$$a \vDash_A x \wedge a \vDash_A y \implies x = y$$

Let S be a family of reflexive graphs with realisable objects over some reflexive graph Γ. S has distinct realisers if for every $\gamma \in \Gamma_O$, $S_O(\gamma)$ has distinct realisers.
A much weaker property is:

Definition 7 (No universal realiser)**.** Let A be an assembly†. A has no universar realiser if for every $a \in \mathscr{A}$, there is some $x \in \lfloor A \rfloor$, s.t. $a \nvDash_A x$.

Accordingly, a family of reflexive graphs with realisable objects S over reflixe graph Γ has no universal realiser if for every $\gamma \in \Gamma_O$, $S_O(\gamma)$ has no universal realiser.

Claim. *U^R is closed under Π^R types.*

4 Some (free) theorems

Definition 8 (No universal realizer)**.** Given a context Γ and a type T, s.t $\Gamma \vdash T$, the model $\mathcal{C}$ constructed so far has no universal realizer iff $\bigcap_{\gamma \in [\![\Gamma]\!]_O} \{a \in \mathscr{A} : \text{for every x} \in |[\![T]\!]_O(\gamma)|, a \vDash_{[\![T]\!]_O(\gamma_O)} x\}$ is empty.

Theorem 9. *Let Γ be a context and $T := \Pi a \overset{0}{:} \mathbf{U}.\Pi\_ \overset{0}{:} \mathbf{T}a.\mathbf{T}a$ - a type. Assume the model has no universal realizers. There is no resourced term M of that type - i.e. $\Gamma \vdash M \overset{1}{:} T$ does not hold in $\mathbb{M}$.*

Assume such term M, $\Gamma \vdash M \overset{1}{:} T$ exists. Fix some $\gamma \in \Gamma_O$ and consider the uncurried term M', s.t $\Gamma, a \overset{O}{:} \mathbf{U}, \_ \overset{0}{:} \mathbf{T}a \vdash M' \overset{1}{:} \mathbf{T}a$ and

$$M'_O(\gamma_O, a_O, \_) := \text{ } let \ ((f''_O, f''_R), f'_R) = M_O(\gamma_O) \ in \ f''_O(\gamma_O, a_O, \_)$$

$$M'_R(\gamma_R, a_R, \_) := \text{ } let \ (f'^{src}, f'^{tgt}, (f''^{src}, f''^{tgt}, r)) = M_R(\gamma_R) \ in \ r(\gamma_R, a_R, \_)$$

Fix some $\gamma_O \in \Gamma_O$. Spelling out explicitly the type of M'_R, "instantiated" at $\Gamma_{refl}(\gamma_O)$ (or equivalently, of $r(\Gamma_{refl}(\gamma_O), -, -)$ and suppressing the realizability information, we get that:

$$M'_R(\Gamma_{refl}(\gamma_O), -, -) : \forall a_R \in \mathbf{U}_R(\Gamma_{refl}(\gamma_O)).\mathbf{T}_R(\Gamma_{refl}(\gamma_O), a_R) \to \mathbf{T}_R(\Gamma_{refl}(\gamma_O), a_R)$$

Unpacking the definition of $\mathbf{U}_R(\Gamma_{refl}(\gamma_O))$, we get (by conditions in Definition 4):

$$\forall S, T \in \mathbf{U}_O(\gamma_O), \forall R \in \mathcal{U}, R_{src} : R \to S_O, R_{tgt} : R \to T_O :$$

$$R_{src}(M'_R(\Gamma_{refl}(\gamma_O), (S, T, R, R_{src}, R_{tgt}), (s, t))) = M'_O(\gamma_O, S, s)$$
$$R_{tgt}(M'_R(\Gamma_{refl}(\gamma_O), (S, T, R, R_{src}, R_{tgt}), (s, t))) = M'_O(\gamma_O, T, t)$$

Thus, we conclude that

$$\forall S, T \in \mathbf{U}_O(\gamma_O), \forall R \subseteq S \times T, \forall (s, t) \in R.\ (M'_O(\gamma_O, S, s), M'_O(\gamma_O, T, t)) \in R \tag{1}$$

Let X be some type, s.t $\Gamma \vdash X : \mathbf{U}$. Consider the term M instantiated at γ_O and X and $R^X := \{(x, x) | x : X\}$. Substituting X for S and T, R^X for R in (1) and applying currying, we get that for each $x : X$, $(M(\gamma_O, X(\gamma_O), x), M(\gamma_O, X(\gamma_O), x)) \in R$ holds. Hence $M(\gamma_O, X(\gamma_O), x) = x$.

Now since $M \in RTm(\Gamma, T)$, M is realizable - in particular, there exists an $a \in \mathscr{A}$ that tracks $M(\gamma_O X(\gamma_O), -)$. By def. we get that $\forall x \in X_O, \forall b \in \mathscr{A}.b \vDash_{\Gamma.0X} (\gamma_O, x) \implies a.!_0 b \vDash_{X(\gamma_O)} M(\gamma_O, X(\gamma_O), x)$ - a is a realizer for every element x in $X_O(\gamma_O)$. But that is a contradiction, as no universal realizer exists for X by assumption. Therefore no resourced term $M \overset{1}{:} T$ exists.

Definition 10 (Separability of realizers)**.** Let Γ be a context and $S \in Ty(\Gamma)$. S has separable realizers if for every $\gamma \in \Gamma_O$ and $x, y \in |S_O(\gamma)|$ and $a \in \mathscr{A}$, if $a \vDash_{S_O(\gamma)} x$ and $a \vDash_{S_O(\gamma)} y$, then $x = y$.

Theorem 11. *Let $\Gamma \vdash A : \mathbf{U}$ and $\Gamma, a \overset{\sigma}{:} A \vdash B : \mathbf{U}$. If A has a realisable inhabitant and B has separable realisers, then $[\![B]\!] \cong [\![\Pi x \overset{0}{:} A.\ B]\!]$ holds in $\mathbb{M}$.*

Let a^* be the realisable inhabitant of A, $a^* \in RTm(\Gamma, A)$, $a^*_o : \forall \gamma_o \in \Gamma_O.\lfloor A_O(\gamma_o) \rfloor$,
$a^*_r : \forall \gamma_r \in \Gamma_R.A_R(\gamma_r)$.
Define the morphisms $g : \Pi x \overset{0}{:} A.\ B \to B$ and $h : B \to \Pi x \overset{0}{:} A.\ B$ as [4]:

$$g_O(\gamma_o) := \lambda(f_O, f_R).f_O(a^*_o(\gamma)) \qquad h_O(\gamma_o) := \lambda b_O.(\lambda a_o.b_O, \lambda a_r.B_{refl}(b_O))$$

Let $V := \Pi x \overset{0}{:} A.\ B$, observe that if $(f_O, f'_R), (f_O, f''_R) \in |V_O|$, then $f'_R = f''_R$ using the proof-irrelevance of B. Hence, to show that two inhabitants of $|V_O|$ are equal, it suffices to prove it

[4] for the time being, focus only on the object part

for the first components only (1).

Fix some $(f_O, f_R) \in \lfloor V_O \rfloor$. To find a realiser of $g_O(\gamma_o)$, consider the realiser $a_f \vDash_{V_O(\gamma_o)} (f_O, f_R)$. By def. we have that $\forall x \in |A_O(\gamma_O)|, a \in \mathscr{A}.a \vDash_{A_O(\gamma_o)} x \implies a_f.!_0 a \vDash_{B_O(\gamma_o)} f_O(x)$. As B has separable realisers, it must be the case that f_O is a constant function (2). Define the realiser a_g of $g_O(\gamma_o)$ as $a_g := \lambda^* x.(x.I)$.
As for $h_O(\gamma_o)$, we must first ensure that h_O outputs well-defined inhabitants of $V_O(\gamma_o)$. Notice that both $f' := \lambda a_o.b_O$ and $f'' := \lambda a_r.B_{refl}(b_O)$ are well-typed and we can easily verify they satisfy the conditions in Definition 5 by direct substitution (where the extension of f' is the natural extension over $|A_O|$). To construct the realiser of f', let $a_b \vDash_{B_O(\gamma_o)} b_O$ (such a_b exists due to $b_O \in \lfloor B_O(\gamma_o) \rfloor$), then $K.a_b$ realises f'.

Let $b^* \in \lfloor B_O(\gamma_o) \rfloor$, by def. we have that $(g_O \circ h_O)(\gamma_o, b^*) = b^*$. As for the converse direction, let $(h_O \circ g_O)(\gamma_o)(f_O, f_R) = (f'_O, f'_R)$ for some $(f_O, f_R) \in V_O(\gamma_o)$. By (1), we show only that $f_O = f'_O$. As f_O is a constant function by (2), expand the definition of h_o to obtain immediately $f\varnothing = f'_O$.

References

[1] R. Atkey. "Syntax and Semantics of Quantitative Type Theory". In: *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science - LICS '18*. New York, New York, USA: ACM Press, 2018, pp. 56–65. URL: `http://dl.acm.org/citation.cfm?doid=3209108.3209189`.

[2] R. Atkey, N. Ghani, and P. Johann. "A relationally parametric model of dependent type theory". In: *Proceedings of the 41st ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages - POPL '14*. New York, New York, USA: ACM Press, 2014, pp. 503–515. URL: `http://dl.acm.org/citation.cfm?doid=2535838.2535852`.

[3] E. Engeler. "Algebras and combinators". In: *Algebra Universalis* 13.1 (Dec. 1981), pp. 389–392. URL: `https://link.springer.com/article/10.1007/BF02483849`.

[4] Q. Ma and J. C. Reynolds. "Types, abstraction, and parametric polymorphism, part 2". In: *Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics)*. Vol. 598 LNCS. 1992, pp. 1–40. URL: `http://link.springer.com/10.1007/3-540-55511-0%7B%5C_%7D1`.