

Relational realisability model for QTT

Our aim is to build a concrete realisability model for QTT which allows for meta-reasoning with results derived from parametricity. To that end, a construction of a quantitative category with families (QCwF, see [1]) is presented. We follow the relational approach to types introduced by Reynolds for typed lambda calculus [4] and later refined for dependent types theories [2].

Once and for all fix a usage semiring R and an R -linear combinatory algebra \mathcal{A} ¹.

0 Preliminaries

We recall some basic structures used in constructing the model.

Definition 1 (Assemblies[†]). An assembly[†] Γ is a pair $(|\Gamma|, e)$ where $|\Gamma|$ is a carrier set and e is a realisability function $|\Gamma| \rightarrow \mathcal{P}(\mathcal{A})$.

Given some $\gamma \in |\Gamma|$, $e(\gamma)$ is interpreted as the set of witnesses for the existence of γ . To emphasize on that aspect, we write $a \Vdash_{\Gamma} \gamma$ to denote $a \in e(\gamma)$. Moreover, let $[\Gamma]$ stand for the set of realisable elements - $\{\gamma \in |\Gamma| : e_{\Gamma}(\gamma) \neq \emptyset\} \subseteq |\Gamma|$.

A morphism between two assemblies[†] Γ and Δ is a function $f : |\Gamma| \rightarrow |\Delta|$ that is realisable - there exists a realiser $a_f \in \mathcal{A}$ that tracks the function f in the following sense:

$$\text{for every } \gamma \text{ in } |\Gamma| \text{ and } a_{\gamma} \text{ in } \mathcal{A}, a_{\gamma} \Vdash_{\Gamma} \gamma \implies a_f.a_{\gamma} \Vdash_{\Delta} f(\gamma) \text{ holds.}$$

Note that multiple realisers for the same function f do not induce multiple morphisms.

Using these notions we can construct a category $\mathcal{A}sm^{\dagger}(\mathcal{A})$.

Definition 2 (Reflexive graph). A reflexive graph (r.g.) G is a tuple $(G_O, G_R, G_{refl}, G_{src}, G_{tgt})$, where G_O and G_R are sets, $G_{src} : G_R \rightarrow G_O$, $G_{tgt} : G_R \rightarrow G_O$ and $G_{refl} : G_O \rightarrow G_R$ are functions, s.t. the identities hold:

$$G_{src} \circ G_{refl} = id_{G_O} = G_{tgt} \circ G_{refl}$$

G_O and G_R stand for a set of objects and a set of relations, respectively. In general, reflexive graphs are less structured than categories as relations need not compose.

A morphism between reflexive graphs G and H is a pair of functions $(f_o : G_O \rightarrow H_O, f_r : G_R \rightarrow H_R)$, s.t. all of the depicted squares commute:

¹In case some non-trivial properties of \mathcal{A} are required, we will assume that \mathcal{A} is a graph model (see [3])

$$\begin{array}{ccc}
G_O & \xrightarrow{f_o} & H_O \\
\begin{array}{c} \uparrow \\ G_{src} \end{array} \left(\begin{array}{c} \downarrow \\ G_{refl} \end{array} \right) \begin{array}{c} \uparrow \\ G_{tgt} \end{array} & & \begin{array}{c} \uparrow \\ H_{src} \end{array} \left(\begin{array}{c} \downarrow \\ H_{refl} \end{array} \right) \begin{array}{c} \uparrow \\ H_{tgt} \end{array} \\
G_R & \xrightarrow{f_r} & H_R
\end{array}$$

Reflexive graphs equipped with r.g. morphisms form a **category** \mathcal{RGph} . The terminal object $\mathbf{1}_{\mathcal{RGph}}$ is $(\{\star\}, \{\star\}, id, id, id)$.

Reflexive graphs provide enough structure for a dyadic interpretation of types in the spirit of [4].

1 Reflexive graphs with realisability information

One could easily generalize reflexive graphs by considering object and relation components from arbitrary categories instead of \mathcal{Set} . As our purpose is to build a relational model incorporating realisability information, we replace the set of objects with an assembly[†] and retain the \mathcal{Set} -based representation of relations.

Definition 3 (Reflexive graph with realisable objects). A reflexive graph with realisable objects G is a tuple $(G_O, G_R, G_{refl}, G_{src}, G_{tgt})$, where $G_O \in Ob(\mathcal{Asm}^\dagger(\mathcal{A}))$, G_R is a set and the functions $G_{refl} : |G_O| \rightarrow G_R$, $G_{src} : G_R \rightarrow |G_O|$, $G_{tgt} : G_R \rightarrow |G_O|$ are such that the identities in Definition 2 are satisfied.

Accordingly, a **morphism between reflexive graphs with realisable objects**, G and H , is a pair $(f_O : G_O \rightarrow H_O, f_R : G_R \rightarrow H_R)$, s.t. (f_O, f_R) is an \mathcal{RGph} morphism between the reflexive graphs $(|G_O|, G_R, G_{refl}, G_{src}, G_{tgt})$ and $(|H_O|, H_R, H_{refl}, H_{src}, H_{tgt})$.

With these components, we obtain a category $\mathcal{ORGph}(\mathcal{Asm}^\dagger(\mathcal{A}))$. By considering reflexive graphs with realisable objects of shape $(X, |X|, id_X, id_X, id_X)$, we identify an isomorphic copy of $\mathcal{Asm}^\dagger(\mathcal{A})$ inside $\mathcal{ORGph}(\mathcal{Asm}^\dagger(\mathcal{A}))$.

A terminal object $\mathbf{1}_{\mathcal{ORGph}(\mathcal{Asm}^\dagger(\mathcal{A}))}$ in $\mathcal{ORGph}(\mathcal{Asm}^\dagger(\mathcal{A}))$ is a tuple $(\mathbf{1}_{\mathcal{Asm}^\dagger(\mathcal{A})}, \{\star\}, id, id, id)$, where $\mathbf{1}_{\mathcal{Asm}^\dagger(\mathcal{A})}$ is the terminal assembly[†] $(\{\star\}, f)$, with f defined as $\star \mapsto \{I\}$.

Definition 4 (Family of reflexive graphs with realisable objects). Let \mathcal{C} be a category with a terminal object. Given a reflexive graph $\Gamma \in Ob(\mathcal{C})$, a family of reflexive graphs with realisable objects over Γ is a tuple $S := (S_O, S_R, S_{refl}, S_{src}, S_{tgt})$, where:

- $S_O : \Gamma_O \rightarrow \mathcal{Asm}^\dagger(\mathcal{A})$
- $S_R : \Gamma_R \rightarrow \mathcal{Set}$

- a Γ -indexed collection of functions $S_{refl} := \{f_\gamma : |S_O(\gamma)| \rightarrow S_R(\Gamma_{refl}(\gamma))\}_{\gamma \in \Gamma_O}$
- $S_{src} := \{f_\gamma : S_R(\gamma) \rightarrow |S_O(\Gamma_{src}(\gamma))|\}_{\gamma \in \Gamma_R}$
- $S_{tgt} := \{f_\gamma : S_R(\gamma) \rightarrow |S_O(\Gamma_{tgt}(\gamma))|\}_{\gamma \in \Gamma_R}$

such that

- each identity in the following collection is satisfied:

$$S_\sigma(\Gamma_{refl}(\gamma)) \circ S_{refl}(\gamma) = id \text{ for every } \gamma \in \Gamma_O, \sigma \in \{src, tgt\}$$

A morphism M between two families S and T of reflexive graphs with realisable objects over Γ is a pair (M_O, M_R) of Γ -indexed collections of morphisms:

- $M_O := \{f_\gamma : S_O(\gamma) \rightarrow T_O(\gamma)\}_{\gamma \in \Gamma_O}$
- $M_R := \{f_\gamma : S_R(\gamma) \rightarrow T_R(\gamma)\}_{\gamma \in \Gamma_R}$

s.t. the following identities are satisfied:

- $T_{refl}(M_O(\gamma)(s_o)) = M_R(\Gamma_{refl}(\gamma))(S_{refl}(\gamma)(s_o))$ for every $\gamma \in \Gamma_O, s_o \in |S_O(\gamma)|$
- $T_{src}(M_R(\gamma)(s_r)) = M_O(\Gamma_{src}(\gamma))(S_{src}(\gamma)(s_r))$ for every $\gamma \in \Gamma_R, s_r \in S_R(\gamma)$
- $T_{tgt}(M_R(\gamma)(s_r)) = M_O(\Gamma_{tgt}(\gamma))(S_{tgt}(\gamma)(s_r))$ for every $\gamma \in \Gamma_R, s_r \in S_R(\gamma)$

Families of reflexive graphs with realisable objects over Γ and their morphisms forms a category $\mathcal{Fam-ORG}(\mathcal{A}sm^\dagger(\mathcal{A}), \Gamma)$.

The terminal object $\mathbf{1}_{\mathcal{Fam-ORG}(\mathcal{A}sm^\dagger(\mathcal{A}), \Gamma)}$ is given by $(\lambda \gamma_r. \mathbf{1}_{\mathcal{A}sm^\dagger(\mathcal{A})}, \lambda \gamma_r. \{*\}, \mathbf{1}_{refl}, \mathbf{1}_{tgt}, \mathbf{1}_{src})$, where $\mathbf{1}_\sigma$ are the appropriate functions.

2 A CwF from families of reflexive graphs with realisable objects

As a first step toward obtaining a relationally parametric realisability model of QTT, we construct a concrete CwF using families of reflexive graphs with realisable objects.

Consider the category \mathcal{RGph} with terminal object $\mathbf{1}_{\mathcal{RGph}}$.

Let $\Gamma, \Delta \in Ob(\mathcal{RGph})$, define:

- the collection of semantic types $Ty(\Gamma)$ as the collection of families of reflexive graph with realisable objects over Γ .
- given a type $S \in Ty(\Gamma)$, an element $M \in Tm(\Gamma, S)$ is a pair of functions $(M_O : \forall \gamma \in \Gamma_O. |S_O(\gamma)|, M_R : \forall \gamma \in \Gamma_R. S_R(\gamma))$, s.t.

$$\begin{aligned} \forall \gamma \in \Gamma_O. S_{refl}(M_O(\gamma)) &= M_R(\Gamma_{refl}(\gamma)) \\ \forall \gamma \in \Gamma_R. S_{src}(M_R(\gamma)) &= M_O(\Gamma_{src}(\gamma)) \\ \forall \gamma \in \Gamma_R. S_{tgt}(M_R(\gamma)) &= M_O(\Gamma_{tgt}(\gamma)) \end{aligned}$$

- given $f : \Gamma \rightarrow \Delta$, substitutions in types and terms is a precomposition with f on the object and relation components of types and terms respectively. Clearly, these operations are compatible with identity and composition in \mathcal{RGph} .
- context extension: Suppose $S \in Ty(\Gamma)$, construct a r.g. $\Gamma.S$ as :

$$\begin{aligned} (\Gamma.S)_O &= \{(\gamma, x) : \gamma \in \Gamma_O, x \in |S_O(\gamma)|\} \\ (\Gamma.S)_R &= \{(\gamma, r) : \gamma \in \Gamma_R, r \in S_R(\gamma)\} \\ (\Gamma.S)_{refl}(\gamma, x) &= (\Gamma_{refl}(\gamma), S_{refl}(\gamma)(x)) \\ (\Gamma.S)_\sigma(\gamma, r) &= (\Gamma_\sigma(\gamma), S_\sigma(\gamma)(r)), \quad \sigma \in \{src, tgt\} \end{aligned}$$

Claim. $Hom_{\mathcal{RGph}}(\Delta, \Gamma.S) \cong \{(f, M) : f : \Delta \rightarrow \Gamma, M \in Tm(\Delta, S\{f\})\}$, natural in Δ .

Upgrading to a QCwF

Recall the definition of a QCwF from [1]. Given a usage semiring R , a R -QCwF consists of:

1. A CwF $(\mathcal{C}, 1, Ty, Tm, -. , \langle -. \rangle)$
2. A category \mathcal{L} with a faithful functor $U : \mathcal{L} \rightarrow \mathcal{C}$
3. A functor $(+) : \mathcal{L} \times_{\mathcal{C}} \mathcal{L} \rightarrow \mathcal{L}$, s.t $U(\Gamma_1 + \Gamma_2) = U(\Gamma_1) = U(\Gamma_2)^2$. $\mathcal{L} \times_{\mathcal{C}} \mathcal{L}$ denotes the pullback $\mathcal{L} \xrightarrow{U} \mathcal{C} \xleftarrow{U} \mathcal{L}$.

Additionally, there exists an object $\diamond \in \mathcal{L}$, s.t. $U\diamond = 1$.

4. A functor $\rho(-) : \mathcal{L} \rightarrow \mathcal{L}$ for each $\rho \in R$, s.t $U(\rho(-)) = U(-)$.
5. A collection $RTm(\Gamma, S)$ for each $\Gamma \in \mathcal{L}$ and $S \in Ty(U\Gamma)$, equipped with an injective function $U_{\Gamma, S} : RTm(\Gamma, S) \rightarrow Tm(U\Gamma, S)$.

For an \mathcal{L} morphisms $f : \Gamma \rightarrow \Delta$ and types $S \in Ty(U\Gamma)$, a function $-\{f\} : RTm(\Delta, S) \rightarrow RTm(\Gamma, S\{f\})$, s. t. $U(-\{f\}) = (U(-))\{Uf\}$.

6. Given $\Gamma \in \mathcal{L}$, $\rho \in R$ and $S \in Ty(U\Gamma)$, an object $\Gamma.\rho S$, s.t $U(\Gamma.\rho S) = U\Gamma.S$.

Additionally, the following natural transformations exist:

$$\begin{aligned} emp_\pi &: \diamond \rightarrow \pi\diamond \\ emp_+ &: \diamond \rightarrow \diamond + \diamond \\ ext_\pi &: \pi\Gamma.(\pi\rho S) \rightarrow \pi(\Gamma.\rho S), \text{ s.t. } U(ext_\pi) = id \\ ext_+ &: (\Gamma_1 + \Gamma_2).(\rho_1 + \rho_2)S \rightarrow \Gamma_1.\rho_1 S + \Gamma_2.\rho_2 S, \text{ s.t. } U(ext_+) = id \end{aligned}$$

7. Given $\Gamma \in \mathcal{L}$, $S \in Ty(U\Gamma)$, there exists :

$$\begin{aligned} &\text{a morphism } p_{\Gamma, S} : \Gamma.0S \rightarrow \Gamma, \text{ s.t. } U(p_{\Gamma, S}) = p_{U\Gamma, S} \\ &\text{an element } v_{\Gamma, S} \in RTm(0\Gamma.1S, S\{p_{U\Gamma, S}\}), \text{ s.t. } U(v_{\Gamma, S}) = v_{U\Gamma, S} \\ &\text{a morphism } wk(f, \rho S') : \Gamma.\rho S'\{Uf\} \rightarrow \Delta.\rho S' \text{ for each } f : \Gamma \rightarrow \Delta, S' \in Ty(U\Gamma, \Delta) \\ &\text{s.t. } U(wk(f, \rho S')) = wk(Uf, S') \end{aligned}$$

let $\Gamma_1, \Gamma_2 \in \mathcal{L}$, s.t $U\Gamma_1 = U\Gamma_2$ and $M \in RTm(\Gamma_2, S)$. There is a morphism $\overline{\rho M} : \Gamma_1 + \rho\Gamma_2 \rightarrow \Gamma_1.\rho S$, s.t $U(\overline{\rho M}) = \overline{UM}$

a morphism $\overline{M} : \Gamma \rightarrow \Gamma.0S$ for $M \in Tm(U\Gamma, S)$, s.t. $U(\overline{M}) = \overline{M}$.

²the second equality being trivially satisfied

Now to construct the concrete QCwF, consider the CwF from previous section as the underlying category.

Let $\mathcal{L} := \mathcal{ORGph}(\mathcal{Asm}^\dagger(\mathcal{A}))$, the category of reflexive graphs with realisable objects

For the addition structure, let Γ', Γ'' be reflexive graphs with realisable objects, s.t $|\Gamma'_O| = |\Gamma''_O|$ and $\Gamma'_R = \Gamma''_R$

- $\Gamma_O := (|\Gamma'_O|, \models_\Gamma)$ with $a \models_\Gamma \gamma$ iff there exist $x, y \in \mathcal{A}$, s.t. $a = [x, y]$ and $x \models_{\Gamma'} \gamma$ and $y \models_{\Gamma''} \gamma$.
- $\Gamma_R := \Gamma'_R (= \Gamma''_R)$.
- $\Gamma_\sigma := \Gamma'_\sigma$, where $\sigma \in \{src, tgt, refl\}$.

Define \diamond as the terminal object $\mathbf{1}_{\mathcal{ORGph}(\mathcal{Asm}^\dagger(\mathcal{A}))}$.

Consider the scaling structure and let $\Gamma := \rho(\Gamma')$:

- $\Gamma_\sigma = (|\Gamma'_\sigma|, \models_{\Gamma_\sigma})$ with $a \models_{\Gamma_\sigma} \gamma$ iff there is $x \in \mathcal{A}$, s.t $a = !_\rho x$ and $x \models_{\Gamma'_\sigma} \gamma$ for $\sigma \in \{O, R\}$
- again, scaling leaves unmodified Γ_σ for $\sigma \in \{src, tgt, refl\}$.

Let $RTm(\Gamma, S)$ be the collection of morphisms from the terminal family $\mathbf{1}_{\mathcal{Fam} - \mathcal{ORG}(\mathcal{Asm}^\dagger(\mathcal{A}), \Gamma)}$ to S . Spelling this out and simplifying it, an element of $RTm(\Gamma, S)$ is a tuple of functions $(M_O : \forall \gamma_o \in \Gamma_O. |S_O(\gamma_o)|, M_R : \forall \gamma_r \in \Gamma_R. S_R(\gamma_r))$, s.t. the conditions from Definition 4 are satisfied, namely

$$\begin{aligned} \forall \gamma_o \in \Gamma_O. S_{refl}(M_O(\gamma_o)) &= M_R(\Gamma_{refl}(\gamma_o)) \\ \forall \gamma_r \in \Gamma_R. S_{src}(M_R(\gamma_r)) &= M_O(\Gamma_{src}(\gamma_r)) \\ \forall \gamma_r \in \Gamma_R. S_{tgt}(M_R(\gamma_r)) &= M_O(\Gamma_{tgt}(\gamma_r)) \end{aligned}$$

such that M_O is tracked - $\exists a_M \in \mathcal{A}. \forall a_\gamma \in \mathcal{A}, \gamma \in \Gamma_O. a_\gamma \models_{\Gamma_O} \gamma \implies a_M \cdot a_\gamma \models_{S(\gamma)} M_O(\gamma)$.

$U_{\Gamma, S}$ is the just identity function.

Substitution in terms is given by precomposition with $f : \Gamma \rightarrow \Delta$, let $-\{f_O\} := \lambda M_O. \forall \gamma \in \Gamma. M_O(f(\gamma))$ and similarly, $-\{f_R\} := \lambda M_R. \forall \gamma \in \Gamma. M_R(f(\gamma))$. The functor U interacts nicely with the so-defined $-\{f\}$ as essentially the substitution in terms in the underlying CwF is defined in the same way.

Resourced context extension is given by $\Gamma. \rho S := (\Gamma'_O, \Gamma'_R, \Gamma'_\sigma)$, where

- $|\Gamma'_O| := \{(\gamma, x) : \gamma \in |\Gamma_O|, x \in S_O(\gamma)\}$
 $a \models_{\Gamma. \rho S} (\gamma, x)$ iff there exists $b, c \in \mathcal{A}$, s.t $a = [b, !_\rho c]$, $b \models_\Gamma \gamma$ and $c \models_{S(\gamma)} \pi_1((\check{\gamma}, x))$, where $(\check{-}) : \Gamma. S \rightarrow U(\Gamma. S)$, $(\check{-}) := id$ as the set-theoretic part of the extensions in the CwF and \mathcal{L} is the same by definition.

- $\Gamma'_R := \{(\gamma, r) : \gamma \in \Gamma_R, r \in S_R(\gamma)\}$
- Each Γ'_σ is defined pointwise.

The natural transformation $emp_\pi : \Diamond \rightarrow \pi\Diamond$ is given by the identity functions on both the object and relational part. It is realised by $K.!_\rho I$.

We list the realisers for the remaining transformations:

- $emp_+ - K.[I, I]$,
- $ext_\pi - \lambda^*q.let [x, y] = q \text{ in } F_\pi.(F_\pi.(!_\pi \lambda^*stu.ust).x).\delta_{\pi\rho}y$
- $ext_+ - \lambda^*q.let [[x, y], z] = q \text{ in } W_{\pi\rho}(\lambda^*ab.[[x, a], [y, b]]).z$

The underlying function part of the above-defined natural transformations is given by identity, hence naturality follows trivially.

Finally, we construct the morphisms, listed in Item 7:

- $p_{\Gamma.S} : \Gamma.0S \rightarrow \Gamma$ is the first projection of $(\Gamma.0S)_\sigma = \{(\gamma, s) : \gamma \in \Gamma_\sigma, s \in S(\gamma)\}$, ($\sigma \in \{O, R\}$) and is realized by $\lambda^*t.(t.K)$.
The equality $U(p_{\Gamma.S}) = p_{U\Gamma.S}$ holds trivially due to the identical structure of context extension in the underlying CwF and \mathcal{L} .
- define $v_{\Gamma.S} \in RTm(0\Gamma.1S, S\{p_{U\Gamma.S}\})$ as the second projection. $v_{\Gamma.S}$ is realized by $\lambda^*t.B.t.K.D$.
- Let a_f^σ realize f_σ , then $wk(f, \rho S') := \lambda(\gamma, s).(f(\gamma), s)$ is realized by $\lambda^*q.let [x, y] = q \text{ in } [a_f^\sigma.x, y]$
- given a $M_\sigma \in RTm(\Gamma, S)$, $M_\sigma : \forall \gamma \in U(\Gamma_\sigma).S_\sigma(\gamma)$ with realizers a_m^σ , let $\overline{\rho M_\sigma} := \lambda\gamma.(\gamma, M_\sigma(\gamma))$ realized by $\lambda^*q.let [x, y] = q \text{ in } [x, F_\rho.(!_\rho a_m^\sigma).y]$
- given a $M_\sigma \in Tm(U\Gamma, S) = M_\sigma : \forall \gamma \in U\Gamma_\sigma.S_\sigma(\gamma)$, let $\overline{M_\sigma} := \lambda\gamma.(\gamma, M_\sigma(\check{\gamma}))$ realized by the K combinator.

From now on, we refer to the constructed model as \mathbb{M} .

3 Type formers

Definition 5 (Dependent function types a QCwF). A QCwF supports dependent function types with usage information, if

- the underlying CwF \mathcal{C} supports them, namely, if for all $\Gamma \in Ob(\mathcal{C}), S \in Ty(\Gamma), T \in Ty(\Gamma.S), \pi \in R$, there exist type $\Pi\pi ST \in Ty(\Gamma)$ and a bijection

$$\Lambda : Tm(\Gamma.S, T) \cong Tm(\Gamma, \Pi\pi ST),$$

natural in Γ .

- for $\Gamma \in Ob(\mathcal{L})$, $S \in Ty(U\Gamma)$, $T \in Ty(U\Gamma.S)$, $\pi \in R$, there exists a bijection

$$\Lambda_{\mathcal{L}} : RTm(\Gamma.\pi S, T) \cong RTm(\Gamma, \Pi\pi ST),$$

natural in Γ such that $U \circ \Lambda_{\mathcal{L}} = \Lambda \circ U$ and $U \circ \Lambda_{\mathcal{L}}^{-1} = \Lambda^{-1} \circ U$.

To show that our model supports Π types, fix some $\pi \in R$, suppose Γ is a r.g in $Ob(\mathcal{C})$, $S \in Ty(\Gamma)$, $T \in Ty(\Gamma.S)$. Define the semantic type $\Pi\pi ST$ as the family of assemblies over Γ , consisting of:

- $(\Pi\pi ST)_O(\gamma) := (X, \models_X)$ for $\gamma \in \Gamma_O$, where

$$\begin{aligned} X := \{ & (f_O, f_R) \mid \\ & f_O : \forall s \in |S_O(\gamma)|. T_O(\gamma, s), \\ & f_R : \forall r \in S_R(\Gamma_{refl}(\gamma)). T_R(\Gamma_{refl}(\gamma), r), \\ & \forall s \in S_O(\gamma). T_{refl}(\gamma, s)(f_O(s)) = f_R(S_{refl}(\gamma)(s)), \\ & \forall r \in S_R(\Gamma_{refl}(\gamma)). T_{src}(\Gamma_{refl}(\gamma), r)(f_R(r)) = f_O(S_{src}(\Gamma_{refl}(\gamma))(r)), \\ & \forall r \in S_R(\Gamma_{refl}(\gamma)). T_{tgt}(\Gamma_{refl}(\gamma), r)(f_R(r)) = f_O(S_{tgt}(\Gamma_{refl}(\gamma))(r)) \} \end{aligned}$$

$$a \models_X (f_O, f_R) \text{ iff } \forall s \in |S_O(\gamma)|, a_s \in \mathcal{A}. a_s \models_{S_O(\gamma)} s \implies a \cdot !_\pi a_s \models_{T_O(\gamma, s)} f_O(s).$$

Note that f_R does not contribute any realisability information to \models_X .

- $(\Pi\pi ST)_R(\gamma) :=$

$$\begin{aligned} \{ & ((f_O^{src}, f_R^{src}), (f_O^{tgt}, f_R^{tgt}), r) \mid \\ & (f_O^{src}, f_R^{src}) \in (\Pi\pi ST)_O(\Gamma_{src}(\gamma)), \\ & (f_O^{tgt}, f_R^{tgt}) \in (\Pi\pi ST)_O(\Gamma_{tgt}(\gamma)), \\ & r : \forall s \in S_R(\gamma). T_R(\gamma, s), \\ & \forall s \in S_R(\gamma). T_{src}(\gamma, s)(r(s)) = f_O^{src}(S_{src}(\gamma)(s)), \\ & \forall s \in S_R(\gamma). T_{tgt}(\gamma, s)(r(s)) = f_O^{tgt}(S_{tgt}(\gamma)(s)) \} \end{aligned}$$

- $(\Pi\pi ST)_{refl}(\gamma) := \lambda(f_O, f_R). ((f_O, f_R), (f_O, f_R), f_R)$ for $\gamma \in \Gamma_O$.
- $(\Pi\pi ST)_{src}(\gamma) := \lambda(f_O^{src}, f_O^{tgt}, r). f_O^{src}$ for $\gamma \in \Gamma_R$.
- $(\Pi\pi ST)_{tgt}(\gamma) := \lambda(f_O^{src}, f_O^{tgt}, r). f_O^{tgt}$ for $\gamma \in \Gamma_R$.

Unwinding Definition 5, we get

$$\{(M_O, M_R) : \forall(\gamma, s) \in (\Gamma.\pi S). T(\gamma, s)\} \cong \{(N_O, N_R) : \forall\gamma \in \Gamma. (\Pi\pi ST)(\gamma)\}$$

where the terms have the following elaborated types:

$$\begin{aligned}
M_O &: \forall(\gamma, s) \in (\Gamma.\pi S)_O. T_O(\gamma, s) \\
M_R &: \forall(\gamma, r) \in (\Gamma.\pi S)_R. T_R(\gamma, s) \\
N_O &: \forall\gamma \in \Gamma_O. \\
&\quad \{(f_O, f_R) \mid f_O : \Pi S(\gamma)_O. T(\gamma)_O \\
&\quad \quad f_R : \Pi S_R(\Gamma_{refl}(\gamma)). T(\Gamma_{refl}(\gamma)))\} \\
N_R &: \forall\gamma \in \Gamma_R. \\
&\quad \{(f^{src}, f^{tgt}, r) \mid f^{src} \in (\Pi \pi ST)_O(\Gamma_{src}(\gamma)) \\
&\quad \quad f^{tgt} \in (\Pi \pi ST)_O(\Gamma_{tgt}(\gamma)) \\
&\quad \quad r : \Pi S_R(\gamma). T_R(\gamma)\}
\end{aligned}$$

Thus, we can define Λ as $\Lambda(M_O, M_R) = (N_O, N_R)$, where

$$\begin{aligned}
N_O &:= \lambda\gamma_o. (\lambda s. M_O(\gamma_o, s), \lambda s_r. M_R(\Gamma_{refl}(\gamma_o), s_r)) \\
N_R &:= \lambda\gamma_r. (N_O(\Gamma_{src}(\gamma)), N_O(\Gamma_{tgt}(\gamma)), \lambda s_r. M_R(\gamma, s_r))
\end{aligned}$$

For $\Lambda_{\mathcal{L}}$, a realizer a_m of M

(that is $\forall(\gamma, s), \forall(a_\gamma, a_s), [a_\gamma, a_s] \models_{\Gamma.\pi S} (\gamma, s) \implies a_m.[a_\gamma, a_s] \models_{T(\gamma, s)} M(\gamma, s)$)
can be transformed to a realizer a_n of N by:

$$a_n := \lambda^* y. (\lambda^* s. (a_m.[y, s]))$$

The conditions $U \circ \Lambda_{\mathcal{L}} = \Lambda \circ U$ and $U \circ \Lambda_{\mathcal{L}}^{-1} = \Lambda^{-1} \circ U$ follow trivially.

Universe of small types A plausible candidate for the universe U is given by the general construction:

$U_O :=$ the set of small reflexive graphs

$U_R := \{(A, B, R, R_{src}, R_{tgt}) : R \in \mathcal{U}, R_{src} : R \rightarrow A_O, R_{tgt} : R \rightarrow A_O, A, B \text{ are small r.g.}\}$

However, this universe turns out to be “too big” - types do not carry enough structure to model parametricity accurately. To remedy the definition, we impose restrictions laid out by the following characterization:

A family of reflexive graphs of assemblies S over Γ is

- small - if for all $\gamma_\sigma \in \Gamma_\sigma$, $S_R(\gamma_R) \in \mathcal{U}$ and $|S_O(\gamma_O)| \in \mathcal{U}$.
- discrete - if for every $\gamma \in \Gamma_O$, there exists $X \in \mathcal{A}sm^\dagger(\mathcal{A})$, s.t.

$$\begin{array}{ccc}
& S_O(\gamma) & \\
S_{src}(\Gamma_{refl}(\gamma)) \swarrow & \downarrow S_{refl}(\gamma) & \searrow S_{src}(\Gamma_{refl}(\gamma)) \\
& S_R(\Gamma_{refl}(\gamma)) & \\
\cong & & \\
& X & \\
id \swarrow & \downarrow id & \searrow id \\
& |X| &
\end{array}$$

- proof-irrelevant - if for all $\gamma \in \Gamma_R$, the function $\langle S_{src}(\gamma), S_{tgt}(\gamma) \rangle : S_R(\gamma) \rightarrow |S_O(\Gamma_{src}(\gamma))| \times |S_O(\Gamma_{tgt}(\gamma))|$ is injective.

For any reflexive graph Γ , define the small, discrete, proof-irrelevant universe $U \in Ty(\Gamma)$ and the type decoder $T \in Ty(\Gamma.U)$ as:

- $|U_O(\gamma)| :=$ the set of small, discrete r.g. of assemblies
 $a \models_{U_O(\gamma)} S$ for $a \in \mathcal{A}$, $a = I$ and $S \in |U_O(\gamma)|$.
- $U_R(\gamma_R) :=$
 $\{(S, T, R, R_{src}, R_{tgt}) \mid R \in \mathcal{U}$
 S, T are small discrete r.g. of assemblies
 $\langle R_{src}, R_{tgt} \rangle : R \rightarrow |S_O| \times |T_O|$ is injective $\}$
- $U_{refl}(\gamma_R)(S) := (S, |S|, S_{refl}, S_{src}, S_{tgt})$
- $U_{src}(\gamma_R)(S, T, R, R_{src}, R_{tgt}) = S$
- $U_{tgt}(\gamma_R)(S, T, R, R_{src}, R_{tgt}) = T$

and $T \in Ty(\Gamma.U)$ as:

- $T_O(\gamma_O, S) := S_O$
- $T_R(\gamma_R, (S, T, R, R_{src}, R_{tgt})) := R$
- $T_{refl}(\gamma_O, S) := S_{refl}$
- $T_{src}(\gamma_R, (S, T, R, R_{src}, R_{tgt})) := R_{src}$
- $T_{tgt}(\gamma_R, (S, T, R, R_{src}, R_{tgt})) := R_{tgt}$.

Claim. U is closed under Π types.

Given some r.g Γ and $S \in Ty(\Gamma)$, $T \in Ty(\Gamma.S)$, it suffices to show that $\Pi\pi TS \in Ty(\Gamma)$ is a small, discrete and proof-irrelevant family of r.g. of assemblies[†] ³ For brevity, let $V := \Pi\pi ST$.

Smallness follows by the closure under Π -types in the ambient set-theoretical universe \mathcal{U} .

For proof-irrelevance, take some $\gamma_R \in \Gamma$ and $f, g \in V_R(\gamma_R)$, s.t. $\langle V_{src}(\gamma_R), V_{tgt}(\gamma_R) \rangle(f) = \langle V_{src}(\gamma_R), V_{tgt}(\gamma_R) \rangle(g)$. WTP $f = g$, by def. we get immediately that $f^{src} = g^{src}$ and $f^{tgt} = g^{tgt}$. Given that $(f^{src}, f^{tgt}, r_f), (f^{src}, f^{tgt}, r_g) \in V_R(\gamma_R)$, note that ,

$$\begin{aligned} \forall s \in S_R(\gamma_R). \langle T_{src}(\gamma_R, s), T_{src}(\gamma_R, s) \rangle(r_f(s)) &= (f_O^{src}(S_{src}(\gamma_R)(s)), f_O^{tgt}(S_{tgt}(\gamma_R)(s))) \\ \forall s \in S_R(\gamma_R). \langle T_{src}(\gamma_R, s), T_{src}(\gamma_R, s) \rangle(r_g(s)) &= (f_O^{src}(S_{src}(\gamma_R)(s)), f_O^{tgt}(S_{tgt}(\gamma_R)(s))) \end{aligned}$$

Since T is proof-irrelevant, it follows directly that $r_f = r_g$ and thus $f = g$.

Reduced Π types - $\Pi^R\pi ST$ Our proposed definition of Π types give rise to few subtle issues with realisability. Ideally, the elements of an assembly[†] and their realisers are in 1:1 correspondence. But clearly this is not the case with Π types:

- $|(\Pi\pi ST)_O(-)|$ contains all set-theoretic functions $f : (s : S) \rightarrow T$, so some elements do not have realisers
- two distinct functions $f, f' : (s : S) \rightarrow T$ might have identical behaviour on $\lfloor S \rfloor$ but differ on some element of $|S| \setminus \lfloor S \rfloor$. Thus f and f' will share the same realiser a .

To remedy this situation one might try to “prune” some of the elements of $|(\Pi\pi ST)_O(-)|$ and ensure that terms are to be defined in “nice enough” contexts.

Formalizing on these considerations, we introduce a new reduced function type, a derivative of the standard $\Pi\pi ST$ types with the main difference being that realisability is now internalised in the object assembly[†] part:

$(\Pi^R\pi ST)_O(\gamma_O) := (X, \models_X)$, where

$$\begin{aligned} X := \{ & (f_O, f_R) \mid f_O : \forall s \in |S_O(\gamma)|. T_O(\gamma, s), \quad \dots \\ & \exists a_f \in \mathcal{A}. \forall s \in |S_O(\gamma)|, a_s \in \mathcal{A}. a_s \models_{S_O(\gamma)} s \implies a_f \cdot !_\pi a_s \models_{T_O(\gamma, s)} f_O(s), \\ & \dots \} \end{aligned}$$

A new operation on resourced contexts, \mathbf{R} , takes care of removing any nonrealisable element. The functor is defined on objects as:

$$\begin{aligned} \mathbf{R} : \mathcal{ORGph}(\mathcal{A}sm^\dagger(\mathcal{A})) &\rightarrow \mathcal{ORGph}(\mathcal{A}sm^\dagger(\mathcal{A})) \\ \mathbf{R}(X_O, X_R, X_{refl}, X_{src}, X_{tgt}) &= ((\lfloor X_O \rfloor, \models_X), X_R, X_{refl}, X_{src}, X_{tgt}) \end{aligned}$$

³it actually does not suffice, we implicitly assume a lemma that gives an isomorphism between terms of type U and small, discrete, p.i. r.g of assemblies[†]

and extends naturally to $\mathcal{ORGph}(\mathcal{Asm}^\dagger(\mathcal{A}))$ morphisms. It can be also endowed with comonadic structure by observing we have a canonical choice for the natural transformations $\mathbf{R}(X) \rightarrow X$ and $\mathbf{R}(X) \rightarrow \mathbf{R}(\mathbf{R}(X))$, namely the canonical injection and identity. Also observe that the functors $!_\pi$ commute with \mathbf{R} , particularly $!_0(\mathbf{R}(X)) = \mathbf{R}(!_0(X))$.

On the surface level, we have the usual type formation rule in the syntax:

$$\frac{0\Gamma \vdash S \quad 0\Gamma, x \overset{0}{:} S \vdash T}{0\Gamma \vdash \Pi^R \pi ST} \text{ reduced } \Pi$$

However, the abstraction rule differs from the standard Π -type. In general, given an unre-sourced term M , $\Gamma, x \overset{0}{:} S \vdash M \overset{0}{:} T$, one cannot find a corresponding λ term, $\Gamma \vdash (\lambda x \overset{\pi}{:} S.M) \overset{0}{:} \Pi^R \pi ST$. That is to say, no bijection on the object level exists between $f_M : \forall(\gamma, s). T(\gamma, s)$ and $f_\lambda : \forall \gamma. (\Pi^R \pi ST)_O(\gamma)$ as the latter function carries some notion of realisability while the former does not.

Such bijection exists in the 1 fragment, though. Thus, there is the reduced abstraction rule:

$$\frac{\Gamma, x \overset{\pi}{:} S \vdash M \overset{1}{:} T}{\mathbf{R}(\Gamma) \vdash (\lambda x \overset{\pi}{:} S.T) \overset{1}{:} \Pi^R \pi ST} \text{ reduced Lam}$$

To verify it is sound, suppose we are given a function $f_m : \forall(\gamma, s) \in |\Gamma.\pi S|_O. T(\gamma, s)$ and its tracker $a_m \in \mathcal{A}$. Unwinding the latter, we get :

$$\forall \gamma \in |\Gamma_O|, s \in |S_O(\gamma)|, a_\gamma, a_s \in \mathcal{A}, a_\gamma \models_{\Gamma_O} \gamma \wedge a_s \models_{S_O(\gamma)} s \implies a_m \cdot [a_\gamma, !_\pi a_s] \models_{T_O(\gamma, s)} f_m(\gamma, s)$$

Now we have to construct an inhabitant (f_O, f_R) , of $\Pi^R \pi ST$. Focus only on the object part and let $f_O := \lambda s \overset{\pi}{:} S. f_m(\gamma, s)$. Its realiser is a_f must satisfy

$$\forall s \in |S_O(\gamma)|, a_s \in \mathcal{A}. a_s \models_{S_O(\gamma)} s \implies a_f \cdot !_\pi a_s \models_{T(\gamma, s)} f_m(\gamma, s)$$

A plausible solution is $a_f := \lambda^* x. a_m \cdot [a_\gamma, a_s]$. Observe that in order a_f to be well-defined, a_γ must always exist. This is guaranteed by the application of \mathbf{R} to the context Γ in the conclusion.

Finally, we also have an altered application rule as well:

$$\frac{\Gamma_2 \vdash N \overset{\sigma'}{:} S \quad \Gamma_1 \vdash M \overset{\sigma}{:} \Pi^R \pi ST \quad 0\Gamma_1 = 0\Gamma_2 \quad \sigma' = 0 \iff (\pi = 0 \vee \sigma = 0)}{\Gamma_1 + \pi\Gamma_2 \vdash App_{(\Pi^R \pi ST)}(M, N) \overset{\sigma}{:} T[N/x]} \text{ App}$$

4 Some (free) theorems

Definition 6 (No universal realizer). Given a context Γ and a type T , s.t $\Gamma \vdash T$, the model \mathcal{C} constructed so far has no universal realizer iff $\bigcap_{\gamma \in \llbracket \Gamma \rrbracket_O} \{a \in \mathcal{A} : \text{for every } x \in \llbracket T \rrbracket_O(\gamma), a \models_{\llbracket T \rrbracket_O(\gamma_O)} x\}$ is empty.

Theorem 7. Let Γ be a context and $T := \Pi a : \mathbf{U}. \Pi_- : \mathbf{T}a. \mathbf{T}a$ - a type. Assume the model has no universal realizers. There is no resourced term M of that type - i.e. $\Gamma \vdash M : T$ does not hold in \mathbb{M} .

Assume such term M , $\Gamma \vdash M : T$ exists. Fix some $\gamma \in \Gamma_O$ and consider the uncurried term M' , s.t $\Gamma, a : \mathbf{U}, - : \mathbf{T}a \vdash M' : \mathbf{T}a$ and

$$M'_O(\gamma_O, a_O, -) := \text{let } ((f''_O, f''_R), f'_R) = M_O(\gamma_O) \text{ in } f''_O(\gamma_O, a_O, -)$$

$$M'_R(\gamma_R, a_R, -) := \text{let } (f'^{src}, f'^{tgt}, (f''^{src}, f''^{tgt}, r)) = M_R(\gamma_R) \text{ in } r(\gamma_R, a_R, -)$$

Fix some $\gamma_O \in \Gamma_O$. Spelling out explicitly the type of M'_R , “instantiated” at $\Gamma_{refl}(\gamma_O)$ (or equivalently, of $r(\Gamma_{refl}(\gamma_O), -, -)$ and suppressing the realizability information, we get that:

$$M'_R(\Gamma_{refl}(\gamma_O), -, -) : \forall a_R \in \mathbf{U}_R(\Gamma_{refl}(\gamma_O)). \mathbf{T}_R(\Gamma_{refl}(\gamma_O), a_R) \rightarrow \mathbf{T}_R(\Gamma_{refl}(\gamma_O), a_R)$$

Unpacking the definition of $\mathbf{U}_R(\Gamma_{refl}(\gamma_O))$, we get (by conditions in Definition 4):

$$\forall S, T \in \mathbf{U}_O(\gamma_O), \forall R \in \mathcal{U}, R_{src} : R \rightarrow S_O, R_{tgt} : R \rightarrow T_O :$$

$$R_{src}(M'_R(\Gamma_{refl}(\gamma_O), (S, T, R, R_{src}, R_{tgt}), (s, t))) = M'_O(\gamma_O, S, s)$$

$$R_{tgt}(M'_R(\Gamma_{refl}(\gamma_O), (S, T, R, R_{src}, R_{tgt}), (s, t))) = M'_O(\gamma_O, T, t)$$

Thus, we conclude that

$$\forall S, T \in \mathbf{U}_O(\gamma_O), \forall R \subseteq S \times T, \forall (s, t) \in R. (M'_O(\gamma_O, S, s), M'_O(\gamma_O, T, t)) \in R \quad (1)$$

Let X be some type, s.t $\Gamma \vdash X : \mathbf{U}$. Consider the term M instantiated at γ_O and X and $R^X := \{(x, x) | x : X\}$. Substituting X for S and T , R^X for R in (1) and applying currying, we get that for each $x : X$, $(M(\gamma_O, X(\gamma_O), x), M(\gamma_O, X(\gamma_O), x)) \in R$ holds. Hence $M(\gamma_O, X(\gamma_O), x) = x$.

Now since $M \in RTm(\Gamma, T)$, M is realizable - in particular, there exists an $a \in \mathcal{A}$ that tracks $M(\gamma_O X(\gamma_O), -)$. By def. we get that $\forall x \in X_O, \forall b \in \mathcal{A}. b \models_{\Gamma.OX} (\gamma_O, x) \implies a.!_0 b \models_{X(\gamma_O)} M(\gamma_O, X(\gamma_O), x)$ - a is a realizer for every element x in $X_O(\gamma_O)$. But that is a contradiction, as no universal realizer exists for X by assumption. Therefore no resourced term $M : T$ exists.

Definition 8 (Separability of realizers). Let Γ be a context and $S \in Ty(\Gamma)$. S has separable realizers if for every $\gamma \in \Gamma_O$ and $x, y \in |S_O(\gamma)|$ and $a \in \mathcal{A}$, if $a \models_{S_O(\gamma)} x$ and $a \models_{S_O(\gamma)} y$, then $x = y$.

Theorem 9. Let $\Gamma \vdash A : \mathbf{U}$ and $\Gamma, a \overset{\sigma}{:} A \vdash B : \mathbf{U}$. If A has a realisable inhabitant and B has separable realisers, then $\llbracket B \rrbracket \cong \llbracket \Pi x \overset{0}{:} A. B \rrbracket$ holds in \mathbb{M} .

Let a^* be the realisable inhabitant of A , $a^* \in RTm(\Gamma, A)$, $a_o^* : \forall \gamma_o \in \Gamma_O. \llbracket A_O(\gamma_o) \rrbracket$, $a_r^* : \forall \gamma_r \in \Gamma_R. A_R(\gamma_r)$.

Define the morphisms $g : \Pi x \overset{0}{:} A. B \rightarrow B$ and $h : B \rightarrow \Pi x \overset{0}{:} A. B$ as ⁴:

$$g_O(\gamma_o) := \lambda(f_O, f_R). f_O(a_o^*(\gamma)) \quad h_O(\gamma_o) := \lambda b_O. (\lambda a_o. b_O, \lambda a_r. B_{refl}(b_O))$$

Let $V := \Pi x \overset{0}{:} A. B$, observe that if $(f_O, f'_R), (f_O, f''_R) \in |V_O|$, then $f'_R = f''_R$ using the proof-irrelevance of B . Hence, to show that two inhabitants of $|V_O|$ are equal, it suffices to prove it for the first components only (1).

Fix some $(f_O, f_R) \in \llbracket V_O \rrbracket$. To find a realiser of $g_O(\gamma_o)$, consider the realiser $a_f \models_{V_O(\gamma_o)} (f_O, f_R)$. By def. we have that $\forall x \in \llbracket A_O(\gamma_o) \rrbracket, a \in \mathcal{A}. a \models_{A_O(\gamma_o)} x \implies a_f. !_0 a \models_{B_O(\gamma_o)} f_O(x)$. As B has separable realisers, it must be the case that f_O is a constant function (2). Define the realiser a_g of $g_O(\gamma_o)$ as $a_g := \lambda^* x. (x. I)$.

As for $h_O(\gamma_o)$, we must first ensure that h_O outputs well-defined inhabitants of $V_O(\gamma_o)$. Notice that both $f' := \lambda a_o. b_O$ and $f'' := \lambda a_r. B_{refl}(b_O)$ are well-typed and we can easily verify they satisfy the conditions in Definition 5 by direct substitution (where the extension of f' is the natural extension over $\llbracket A_O \rrbracket$). To construct the realiser of f' , let $a_b \models_{B_O(\gamma_o)} b_O$ (such a_b exists due to $b_O \in \llbracket B_O(\gamma_o) \rrbracket$), then $K.a_b$ realises f' .

Let $b^* \in \llbracket B_O(\gamma_o) \rrbracket$, by def. we have that $(g_O \circ h_O)(\gamma_o, b^*) = b^*$. As for the converse direction, let $(h_O \circ g_O)(\gamma_o)(f_O, f_R) = (f'_O, f'_R)$ for some $(f_O, f_R) \in V_O(\gamma_o)$. By (1), we show only that $f_O = f'_O$. As f_O is a constant function by (2), expand the definition of h_o to obtain immediately $f_O = f'_O$.

⁴for the time being, focus only on the object part

References

- [1] R. Atkey. “Syntax and Semantics of Quantitative Type Theory”. In: *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science - LICS '18*. New York, New York, USA: ACM Press, 2018, pp. 56–65. URL: <http://dl.acm.org/citation.cfm?doid=3209108.3209189>.
- [2] R. Atkey, N. Ghani, and P. Johann. “A relationally parametric model of dependent type theory”. In: *Proceedings of the 41st ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages - POPL '14*. New York, New York, USA: ACM Press, 2014, pp. 503–515. URL: <http://dl.acm.org/citation.cfm?doid=2535838.2535852>.
- [3] E. Engeler. “Algebras and combinators”. In: *Algebra Universalis* 13.1 (Dec. 1981), pp. 389–392. URL: <https://link.springer.com/article/10.1007/BF02483849>.
- [4] Q. Ma and J. C. Reynolds. “Types, abstraction, and parametric polymorphism, part 2”. In: *Lecture Notes in Computer Science (including subseries Lecture Notes in Artificial Intelligence and Lecture Notes in Bioinformatics)*. Vol. 598 LNCS. 1992, pp. 1–40. URL: http://link.springer.com/10.1007/3-540-55511-0%7B%5C_%7D1.