Relational realizability model for QTT

Our aim is to build a concrete realizability model for QTT which allows for metareasoning with results derived from parametricity. To that end, a construction of a quantified category with families (QCwF[**fill**]) is presented. We follow the relational approach to types introduced by Reynolds for typed lambda calculus[**fill**] and later refined for dependent types theories.[**fill**] Once and for all fix a usage semiring R and an R-linear combinatory algebra $\mathscr{A}$[1].

Taking stocks

Definition 1 (Assembly)**.** An assembly Γ is a pair $(|\Gamma|, e)$ where $|\Gamma|$ is a carrier set and e is a realizability function $|\Gamma| \to \mathcal{P}(\mathscr{A})$, s.t $e(\gamma)$ is nonempty for every $\gamma \in |\Gamma|$.

We interpret $E(\gamma)$ as a set of witnesses for the existence of γ and write $a \vDash_\Gamma \gamma$ to denote $a \in e(\gamma)$.
A morphism between two assemblies $(|\Gamma|, E_\Gamma)$ and $(|\Delta|, E_\Delta)$ is a function $f : |\Gamma| \to |\Delta|$ that is realizable - there exists $a_f \in \mathscr{A}$, s.t the following holds for every $\gamma \in |\Gamma|$ and $a_\gamma \in \mathscr{A}$:

$$a_\gamma \vDash_\Gamma \gamma \implies a_f.a_\gamma \vDash_\Delta f(\gamma)$$

We say that a_f tracks f. Note that only an existence of a realizer for f is stipulated in the definition - multiple realizers do not induce multiple morphisms.
Using these notions we can construct a category $\mathcal{A}sm(\mathscr{A})$.

Definition 2 (Reflexive graph)**.** A reflexive graph (r.g.) G is a tuple $(G_O, G_R, G_{refl}, G_{src}, G_{tgt})$, where G_O and G_R are sets, G_{src} and G_{tgt} are functions $G_R \to G_O$ and G_{refl} is a function $G_O \to G_R$, s.t $G_{src} \circ G_{refl} = id_{G_O} = G_{tgt} \circ G_{refl}$.

G_O and G_R stand for a set of objects and a set of relations, respectively. In general, reflexive graphs are less structured than categories as relations need not compose.

A morphism between reflexive graphs G and H is a pair of functions $(f_o : G_O \to H_O, f_r : G_R \to H_R)$, s.t all of the depicted squares commute:

$$\begin{tikzcd}
G_O \arrow[rr, "f_o"] \arrow[d, "G_{refl}"] & & H_O \arrow[d, "H_{refl}"] \\
G_R \arrow[rr, "f_r"] \arrow[u, bend left, "G_{src}"] \arrow[u, bend right, "G_{tgt}"'] & & H_R \arrow[u, bend left, "H_{src}"] \arrow[u, bend right, "H_{tgt}"']
\end{tikzcd}$$

Reflexive graphs equipped with r.g. morphisms form a category $\mathcal{R}Gph(\mathcal{S}et)$.
We use reflexive graphs to give a dyadic interpretation of types in the spirit of [**fill**].

[1] In case some non-trivial properties of $\mathscr{A}$ are required, we will tacitly assume that $\mathscr{A}$ is a graph model(see [**fill**]) - **also to fix**

Assembling a reflexive graph

One could easily generalize reflexive graphs by considering object and relation components from an arbitrary category $\mathcal{C}$ instead of $\mathcal{S}et$. As our purpose is to build a realizability model, we pick $\mathcal{A}sm(\mathscr{A})$ as the category of interest and identify two appropriate notions of reflexive graph of assemblies and a family of reflexive graphs of assemblies.

Definition 3 (Reflixive graph of assemblies)**.** A reflexive graph of assemblies G is a pair of assemblies (G_O, G_R) and a triple of set-theoretic $(G_{refl} : |G_O| \to |G_R|,\ G_{src} : |G_R| \to |G_O|,\ G_{tgt} : |G_R| \to |G_O|)$, such that the identities in Definition 2 are satisfied.

In essence, sets are directly replaced by the objects from $\mathcal{A}sm(\mathscr{A})$, while the morphism are kept intact - plain set-theoretic functions.
However, we require realizable functions only in the definition morphism between reflexive graph of assemblies. With these components, we obtain a category $\mathcal{R}Gph(\mathscr{A})$. By considering r.g. of assemblies of shape (X, X, id_X, id_X, id_X), we identify isomorphic copy of $\mathcal{A}sm(\mathscr{A})$ inside $\mathcal{R}Gph(\mathscr{A})$.
A terminal object in $\mathcal{R}Gph(\mathscr{A})$ is a tuple $(1, 1, id, id, id)$, where 1 is the terminal assembly $(\{\star\}, f)$, with f defined as $\star \mapsto \{I\}$.

Definition 4 (Family of reflexive graphs of assemblies)**.** Let $\mathcal{C}$ be a category with a terminal object. Given a reflexive graph $\Gamma \in Ob(\mathcal{C})$, a family of internal r.g. over Γ is a tuple $S := (S_O, S_R, S_{refl}, S_{src}, S_{tgt})$, where:

- $S_O : \Gamma_O \to \mathcal{A}sm(\mathscr{A})$
- $S_R : \Gamma_R \to \mathcal{A}sm(\mathscr{A})$
- a Γ-indexed collection of functions $S_{refl} := \{f : |S_O(\gamma)| \to |S_R(\Gamma_{refl}(\gamma))|\}_{\gamma\in\Gamma_O}$
- $S_{src} := \{f : |S_R(\gamma)| \to |S_O(\Gamma_{src}(\gamma))|\}_{\gamma\in\Gamma_R}$
- $S_{tgt} := \{f : |S_R(\gamma)| \to |S_O(\Gamma_{tgt}(\gamma))|\}_{\gamma\in\Gamma_R}$

such that

- each identity in the following collection is satisfied:

$$S_\sigma(\Gamma_{refl}(\gamma)) \circ S_{refl}(\gamma) = id \text{ for every } \gamma \in \Gamma_O, \sigma \in \{src, tgt\}$$

We are only interested in cases when $\mathcal{C} = \mathcal{S}et$ or $\mathcal{C} = \mathcal{A}sm(\mathscr{A})$.
A morphism M between two families S and T of internal r.g. over Γ is a pair of Γ-indexed collection of functions:

- $M_O := \{f : |S_O(\gamma)| \to |T_O(\gamma)|\}_{\gamma\in\Gamma_O}$
- $M_R := \{f : |S_R(\gamma)| \to |T_R(\gamma)|\}_{\gamma\in\Gamma_O}$

such that:

- $T_{refl}(M_O(\gamma)(x)) = M_R(\Gamma_{refl}(\gamma))(S_{refl}(\gamma)(x))$ for every $\gamma \in \Gamma_O$, $x \in S_O(\gamma)$
- $T_{src}(M_R(\gamma)(r)) = M_O(\Gamma_{src}(\gamma))(S_{src}(\gamma)(r))$ for every $\gamma \in \Gamma_R$, $r \in S_R(\gamma)$
- $T_{tgt}(M_R(\gamma)(r)) = M_O(\Gamma_{tgt}(\gamma))(S_{tgt}(\gamma)(r))$ for every $\gamma \in \Gamma_R$, $r \in S_R(\gamma)$

A terminal family of r.g. over Γ, 1_Γ, consists of two constant functions, mapping $\gamma \in \Gamma$ to a terminal assembly 1, and three Γ-indexed collections with a sole element id_1.

A CwF from families of reflexive graphs of assemblies

As a first step toward obtaining a relationally parametric realizability model of QTT, we construct a concrete CwF using families of internal graphs of assemblies.
Consider the category $\mathcal{R}Gph$ with terminal object $1 := (\{\star\}, \{\star\}, id, id, id)$.
Let $\Gamma, \Delta \in Ob(\mathcal{R}Gph)$, define:

- the collection of semantic types $Ty(\Gamma)$ as the collection of families of reflexive graphs of assemblies Γ.
- given a type $S \in Ty(\Gamma)$, let $Tm(\Gamma, S) := Hom(1_\Gamma, S)$.
  Spelling this out and ignoring the contribution of the terminal family, we get:
  An element $M \in Tm(\Gamma, S)$ is a pair of functions $(M_O : \forall\gamma \in \Gamma_O.S_O(\gamma), M_R : \forall\gamma \in \Gamma_R.S_R(\gamma))$, such that

$$\forall\gamma \in \Gamma_O.S_{refl}(M_O(\gamma)) = M_R(\Gamma_{refl}(\gamma))$$
$$\forall\gamma \in \Gamma_R.S_{src}(M_R(\gamma)) = M_O(\Gamma_{src}(\gamma))$$
$$\forall\gamma \in \Gamma_R.S_{tgt}(M_R(\gamma)) = M_O(\Gamma_{tgt}(\gamma))$$

- given $f : \Gamma \to \Delta$, substitutions in types and terms is a precomposition with f on the object and relation components of types and terms respectively. Clearly, these operations are compatible with identity and composition in $\mathcal{R}Gph$
- context extension : Suppose $S \in Ty(\Gamma)$, construct a r.g. $\Gamma.S$ as :

$$\begin{aligned}(\Gamma.S)_O &= \{(\gamma, x) : \gamma \in \Gamma_O, x \in S_O(\gamma)\}\\ (\Gamma.S)_R &= \{(\gamma, r) : \gamma \in \Gamma_R, r \in S_R(\gamma)\}\\ (\Gamma.S)_{refl}(\gamma, x) &= (\Gamma_{refl}(\gamma), S_{refl}(\gamma)(x))\\ (\Gamma.S)\sigma(\gamma, r) &= (\Gamma\sigma(\gamma), S\sigma(\gamma)(r)), \quad \sigma \in \{src, tgt\}\end{aligned}$$

Claim. *$Hom_{\mathcal{R}Gph}(\Delta, \Gamma.S) \cong \{(f, M) : f : \Delta \to \Gamma, M \in Tm(\Delta, S\{f\}\}$, natural in Δ.*

Upgrading to a QCwF

Recall the definition of a QCwF from [**fill**]. Given a usage semiring R, a R-QCwF consists of:

1. A CwF $(\mathcal{C}, 1, Ty, Tm, -.-, \langle -.- \rangle)$
2. A category $\mathcal{L}$ with a faithful functor $U : \mathcal{L} \to \mathcal{C}$
3. A functor $(+) : \mathcal{L} \times_{\mathcal{C}} \mathcal{L} \to \mathcal{L}$, s.t $U(\Gamma_1 + \Gamma_2) = U(\Gamma_1) = U(\Gamma_2)$[2]. $\mathcal{L} \times_{\mathcal{C}} \mathcal{L}$ denotes the pullback $\mathcal{L} \xrightarrow{U} \mathcal{C} \xleftarrow{U} \mathcal{L}$.
   Additionally, there exists an object $\Diamond \in \mathcal{L}$, s.t. $U\Diamond = 1$.
4. A functor $\rho(-) : \mathcal{L} \to \mathcal{L}$ for each $\rho \in R$, s.t $U(\rho(-)) = U(-)$.
5. A collection $RTm(\Gamma, S)$ for each $\Gamma \in \mathcal{L}$ and $S \in Ty(U\Gamma)$, equipped with an injective function $U_{\Gamma.S} : RTm(\Gamma, S) \to Tm(U\Gamma, S)$.
   For an $\mathcal{L}$ morphisms $f : \Gamma \to \Delta$ and types $S \in Ty(U\Gamma)$, a function $-\{f\} : RTm(\Delta, S) \to RTm(\Gamma, S\{f\})$, s. t. $U(-\{f\}) = (U(-))\{Uf\}$.
6. Given $\Gamma \in \mathcal{L}$, $\rho \in R$ and $S \in Ty(U\Gamma)$, an object $\Gamma.\rho S$, s.t $U(\Gamma.\rho S) = U\Gamma.S$.
   Additionally, there exist the following natural transformations:
   $emp_\pi : \Diamond \to \pi\Diamond$, s.t. $U(emp_\pi) = id_1$
   $emp_+ : \Diamond \to \Diamond + \Diamond$, s.t. $U(emp_+) = id_1$
   $ext_\pi : \pi\Gamma.(\pi\rho S) \to \pi(\Gamma.\rho S)$, s.t. $U(ext_\pi) = id$
   $ext_+ : (\Gamma_1 + \Gamma_2).(\rho_1 + \rho_2)S \to \Gamma_1.\rho_1 S + \Gamma_2.\rho_2 S$, s.t. $U(ext_+) = id$
7. Given $\Gamma \in \mathcal{L}$, $S \in Ty(U\Gamma)$, there exists:
   a morphism $p_{\Gamma.S} : \Gamma.0S \to \Gamma$, s.t. $U(p_{\Gamma.S}) = p_{U\Gamma.S}$
   an element $v_{\Gamma.S} \in RTm(0\Gamma.1S, S\{p_{U\Gamma.S}\})$, s.t. $U(v_{\Gamma.S}) = v_{U\Gamma.S}$
   a morphism $wk(f, \rho S') : \Gamma.\rho S'\{Uf\} \to \Delta.\rho S'$ for each $f : \Gamma \to \Delta$, $S' \in Ty(U\Gamma, \Delta)$ s.t. $U(wk(f, \rho S')) = wk(Uf, S')$
   let $\Gamma_1, \Gamma_2 \in \mathcal{L}$, s.t $U\Gamma_1 = U\Gamma_2$ and $M \in RTm(\Gamma_2, S)$. There is a morphism $\overline{\rho M} : \Gamma_1 + \rho\Gamma_2 \to \Gamma_1.\rho S$, s.t $U(\overline{\rho M}) = \overline{UM}$
   a morphism $\overline{M} : \Gamma \to \Gamma.0S$ for $M \in Tm(U\Gamma, S)$, s.t. $U(\overline{M}) = \overline{M}$.

Now to construct the concrete QCwF, consider the CwF from previous section as the underlying category.
Take $\mathcal{L} := \mathcal{R}Gph(\mathscr{A})$ and let U be the functor $\mathcal{R}Gph(\mathscr{A}) \to \mathcal{R}Gph$, sending an assembly to its underlying set, forgetting the realizability function.
For the addition structure, let Γ', Γ'' be r.g. of assemblies, s.t $|\Gamma'_O| = |\Gamma''_O|$ and $|\Gamma'_R| = |\Gamma''_R|$. Construct the r.g. of assemblies $\Gamma := \Gamma' + \Gamma''$, where:

- $\Gamma_O := (|\Gamma'_O|, \vDash_\Gamma)$ with $a \vDash_\Gamma \gamma$ iff there exist $x, y \in \mathscr{A}$, s.t. $a = [x, y]$ and $x \vDash_{\Gamma'} \gamma$ and $y \vDash_{\Gamma''} \gamma$.
- define Γ_R similarly as Γ_O.
- Γ_{refl}, Γ_{src}, Γ_{tgt} are the same as their Γ' counterparts (or Γ'').

Let $\Diamond$ be the terminal object in $\mathcal{R}Gph(\mathscr{A})$
Consider the scaling structure and let $\Gamma := \rho(\Gamma')$:

[2] the second equality being trivially satisfied

- $\Gamma_\sigma = (|\Gamma'_\sigma|, \vDash_{\Gamma_\sigma})$ with $a \vDash_{\Gamma_\sigma} \gamma$ iff there is $x \in \mathscr{A}$, s.t $a =!_\rho x$ and $x \vDash_{\Gamma'_\sigma} \gamma$ for $\sigma \in \{O, R\}$
- again, scaling leaves unmodified Γ_σ for $\sigma \in \{src, tgt, rfl\}$.

Let $RTm(\Gamma, S)$ be the collection of assembly morphisms from the terminal object to S (note any set-theoretic function from the terminal object is realizable). Spelling this out, $RTm(\Gamma, S)$ consists of tuples (M_O, M_R), s.t. the conditions from $Definition$ 4 are satisfied. $U_{\Gamma.S}$ just forgets the realizability information and is trivially injective. Substitution in terms is given by precomposition with $f : \Gamma \to \Delta$ - $-\{f_O\} := \lambda M_O.\forall\gamma \in \Gamma.M_O(f(\gamma))$ and similarly, $-\{f_R\} := \lambda M_R.\forall\gamma \in \Gamma.M_R(f(\gamma))$. The functor U interacts nicely with the so-defined $-\{f\}$ as essentially the substitution in terms in the underlying CwF is defined in the same way.

Let $\Gamma.\rho S := (\Gamma'_O, \Gamma'_R, \Gamma'_\sigma)$, where

- $|\Gamma'_O| := \{(\gamma, x) : \gamma \in \Gamma_O, x \in S_O(\gamma)\}$ and $a \vDash_{\Gamma.\rho S} (\gamma, x)$ iff there exists $b, c \in \mathscr{A}$, s.t $a = [b, !_\rho c]$, $b \vDash_\Gamma \gamma$ and $c \vDash_{S(\gamma)} \pi_1(\check{(\gamma, x)})$, where $\check{(-)} : \Gamma.S \to U(\Gamma.S)$, $\check{(-)} := id$ as the set-theoretic part of the extensions in the CwF and $\mathcal{L}$ is the same by definition.
- $|\Gamma'_R|$ is defined analogously.
- Γ'_σ is defined pointwise.

$emp_\pi : \Diamond \to \pi\Diamond$ is given by identity function on both the object and relational part. It is realized by $K.!_\rho I$. Similarly, emp_+ is realized by $K.[I, I]$,
ext_π - by $\lambda^* q.let\ [x, y] = q\ in\ F_\pi.(F_\pi(!_\pi\lambda^* stu.ust).x).\delta_{\pi\rho} y$ and
ext_+ - by $\lambda^* q.let\ [[x, y], z] = q\ in\ W_{\pi\rho}.(\lambda^* ab.[[x, a], [y, b]]).z$

The underlying function part of the above-defined natural transformations is given by identity, hence naturality follows trivially.
Finally, we construct the morphisms, listed in 7:

- $p_{\Gamma.S} : \Gamma.0S \to \Gamma$ is the first projection of $(\Gamma.0S)_\sigma = \{(\gamma, s) : \gamma \in \Gamma_\sigma.s \in S(\gamma)\}$, $(\sigma \in \{_{O},_{R}\})$ and is realized by $\lambda^* t.(t.K)$.
  The equality $U(p_{\Gamma.S}) = p_{U\Gamma.S}$ holds trivially due to the identical structure of context extension in the underlying CwF and $\mathcal{L}$.
- define $v_{\Gamma.S} \in RTm(0\Gamma.1S, S\{p_{U\Gamma.S}\})$ as the second projection. $v_{\Gamma.S}$ is realized by $\lambda^* t.B.t.K.D$.
- Let a^σ_f realizes f_σ, then $wk(f, \rho S') := \lambda(\gamma, s).(f(\gamma), s)$ is realized by $\lambda^* q.let\ [x, y] = q\ in\ [a^\sigma_f.x, y]$
- given a $M_\sigma \in RTm(\Gamma, S) = M_\sigma : \forall\gamma \in U\Gamma_\sigma.S_\sigma(\gamma)$ with realizers a^σ_m, let $\overline{\rho M}_\sigma := \lambda\gamma.(\gamma, M_\sigma(\gamma))$ realized by $\lambda^* q.let\ [x, y] = q\ in\ [x, F_\rho.(!_\rho a^\sigma_m).y]$
- given a $M_\sigma \in Tm(U\Gamma, S) = M_\sigma : \forall\gamma \in U\Gamma_\sigma.S_\sigma(\gamma)$, let $\overline{M}_\sigma := \lambda\gamma.(\gamma, M_\sigma(\check{\gamma}))$ realized by the K combinator.

Type formers

Dependent function types in 1-fragment Given an internal reflexive graph Γ, $S \in Ty(\Gamma)$, $T \in Ty(\Gamma.S)$, $\rho \in R$, define $\Pi\rho ST \in Ty(\Gamma)$ as the tuple of:

- Object assembly. Let $\gamma \in |\Gamma_O|$

$$
\begin{aligned}
|(\Pi\rho ST)_O|(\gamma) := \{(f_O,f_R)|\ & \\
& f_O : \forall s \in S_O(\gamma).T_O(\gamma, s), \\
& \exists a \in \mathscr{A}\, \forall s \in |S_O(\gamma)|, b \in \mathscr{A}. \\
& \qquad b \vDash_{S_O(\gamma)} s \implies a.!_\rho b \vDash_{T_O(\gamma,s)} f_O(s) \\
& f_R : \forall r \in S_R(\Gamma_{refl}(\gamma)).T_R(\Gamma_{refl}(\gamma), r), \\
& \exists a \in \mathscr{A}\, \forall r \in |S_R(\Gamma_{refl}(\gamma))|, b \in \mathscr{A}. \\
& \qquad b \vDash_{S_R(\Gamma_{refl}(\gamma))} r \implies a.!_\rho b \vDash_{T_R(\Gamma_{refl}(\gamma),r)} f_R(r), \\
& \forall s \in S_O(\gamma).T_{refl}(\gamma, s)(f_O(s)) = f_R(S_{refl}(\gamma)(s)), \\
& \forall r \in S_R(\Gamma_{refl}(\gamma)).T_{src}(\Gamma_{refl}(\gamma), r)(f_R(r)) = f_O(S_{src}(\Gamma_{refl}(\gamma))(r)), \\
& \forall r \in S_R(\Gamma_{refl}(\gamma)).T_{tgt}(\Gamma_{refl}(\gamma), r)(f_R(r)) = f_O(S_{tgt}(\Gamma_{refl}(\gamma))(r))\}
\end{aligned}
$$

  $a \vDash_{(\Pi\rho ST)_O(\gamma)} (f_O, f_R)$ iff $a = [a_o, a_r]$ where the realizers a_o, a_r are given by the existential statements above.

- Relation assembly. Let $\gamma \in |\Gamma_R|$

$$
\begin{aligned}
|(\Pi\rho ST)_R|(\gamma) := \{((f_O^{src}, f_R^{src}), (f_O^{tgt}, f_O^{tgt}), r)|\ & \\
& (f_O^{src}, f_R^{src}) \in (\Pi\rho ST)_O(\Gamma_{src}(\gamma)), \\
& (f_O^{tgt}, f_R^{tgt}) \in (\Pi\rho ST)_O(\Gamma_{tgt}(\gamma)), \\
& r : \forall s \in S_R(\gamma).T_R(\gamma, s), \\
& \exists a \in \mathscr{A}\, \forall s \in |S_R(\gamma)|, b \in \mathscr{A}. b \vDash_{S_R(\gamma)} s \implies a.!_\rho b \vDash_{T_R(\gamma,s)} r(s), \\
& \forall s \in S_R(\gamma).T_{src}(\gamma, s)(r(s)) = f_O^{src}(S_{src}(\gamma)(s)), \\
& \forall s \in S_R(\gamma).T_{tgt}(\gamma, s)(r(s)) = f_O^{tgt}(S_{tgt}(\gamma)(s))\}
\end{aligned}
$$

- collections of morphisms:
  $(\Pi\rho ST)_{refl}(\gamma)(f_O, f_R) := ((f_O, f_R), (f_O, f_R), f_R)$ for $\gamma \in \Gamma_O$
  $(\Pi\rho ST)_{src}(\gamma)(f^{src}, f^{tgt}, r) := f^{src}$ for $\gamma \in \Gamma_R$
  $(\Pi\rho ST)_{src}(\gamma)(f^{src}, f^{tgt}, r) := f^{tgt}$ for $\gamma \in \Gamma_R$
  These collections satisfy the identity conditions in Definition 4.