Outline

1 Quantitative polynomial functors

1.1 Category of closed types and linear functions

Let $\mathcal{C}$ be the category of closed types and linear functions $f : (x \overset{1}{:} X) \to Y$ for which derivations in QTT exist. Composition of morphisms $\Gamma \vdash f \overset{\sigma}{:} (x \overset{1}{:} X) \to Y$ and $\Gamma \vdash g \overset{\sigma}{:} (y \overset{1}{:} Y) \to Z$ is given by ordinary function composition $\Gamma \vdash \lambda x \overset{1}{:} X.g(f(x)) \overset{\sigma}{:} (x \overset{1}{:} X) \to Z$.

The linearity restriction on the morphisms does not lead to loss of expressiveness - a function with arbitrary resource annotations can be represented as linear one via the exponential type $!_\rho A := (a \overset{\rho}{:} A) \otimes I$.
For an arbitrary $\Gamma \vdash f \overset{1}{:} X \xrightarrow{\rho} Y$, the embedding is given by:

$$
\begin{aligned}
f \mapsto \lambda z \overset{1}{:} (a \overset{\rho}{:} X) \otimes I.\ &\text{let } (x, i) = z \text{ in} \\
&\qquad \text{let } * = i \text{ in } f(x) \overset{1}{:} ((a \overset{\rho}{:} X) \otimes I) \xrightarrow{1} Y
\end{aligned}
$$

Suppose $\mathcal{D}$ stands for the derivation of $\Gamma \vdash f \overset{1}{:} X \xrightarrow{\rho} Y$ and $\mathcal{D}_x$, $\mathcal{D}_z$ and $\mathcal{D}_i$ for the derivations of $0\Gamma, x \overset{1}{:} X \vdash x \overset{1}{:} X$, $0\Gamma, z \overset{1}{:} (a \overset{\rho}{:} X) \otimes I \vdash z \overset{1}{:} (a \overset{\rho}{:} X) \otimes I$ and $0\Gamma, i \overset{1}{:} I \vdash i \overset{1}{:} I$ obtained by the applications of the `Var` rule.

$$
\dfrac{\mathcal{D}_z \quad \dfrac{\mathcal{D}_i \quad \dfrac{\mathcal{D}_x \quad \mathcal{D}}{\Gamma, x \overset{\rho}{:} X \vdash f(x) \overset{1}{:} Y}}{\Gamma, x \overset{\rho}{:} X, i \overset{1}{:} I \vdash \text{let } * = i \text{ in } f(x) \overset{1}{:} Y}}{\dfrac{\Gamma, z \overset{1}{:} (a \overset{\rho}{:} X) \otimes I) \vdash \text{let } (x, z) = z \text{ in let } * = i \text{ in } f(x) \overset{1}{:} Y}{\Gamma \vdash \lambda z \overset{1}{:} (a \overset{\rho}{:} X) \otimes I).\text{let } (x, z) = z \text{ in let } * = i \text{ in } f(x) \overset{1}{:} (a \overset{\rho}{:} X) \otimes I) \xrightarrow{1} Y}}
$$

1.2 Category of closed types and linear functions in a nonempty context

Let Δ be an "underlying context", i.e. a context of the form $\Delta = 0\Gamma_0$ for some Γ_0. There is a category $\mathcal{C}_\Delta$ where the objects are types X such that $\Delta \vdash X$ type, and a morphism X to Y consists of pair (Γ, f), where Γ is a context such that $0\Gamma = \Delta$, and $\Gamma \vdash f : X \xrightarrow{1} Y$.

- The identity morphism is given by $(\Delta, \lambda x.x)$;
- Composition of (Γ_2, g) and (Γ_1, f) is given by $(\Gamma_1 + \Gamma_2, \lambda x.f(g(x)))$.

This is a category since $0\Delta = \Delta$, and context addition is associative, and with $\Gamma + \Delta = \Delta + \Gamma = \Gamma$. Note that the category of $\mathcal{C}$ closed types from Section 1.1 is a special case $\mathcal{C} = \mathcal{C}_\diamond$ where $\Delta = \diamond$, because the only Γ with $0\Gamma = \diamond$ is $\Gamma = \diamond$.

Lemma 1. *Fix Δ as above, and let $\Delta \vdash A$ type and $\Delta, x \overset{0}{:} A \vdash B[x]$ type. The operation $F(X) = (a \overset{1}{:} A) \otimes (Ba \xrightarrow{1} X)$ is a functor $\mathcal{C}_\Delta \to \mathcal{C}_\Delta$.*

Proof. If $\Delta \vdash X$ type then $\Delta \vdash F(X)$ type. On morphisms, we can define

$$F(\Gamma, f) = (\Gamma, \lambda z.\ \text{let } (a, h) = z \text{ in } (a, \lambda b. f(h(b))))$$

as the following derivation shows:

$$\dfrac{\dfrac{\dfrac{}{0\Gamma, z \overset{0}{:} F(X), a \overset{1}{:} A, h \overset{0}{:} B[a] \xrightarrow{1} X \vdash a : A} \qquad \begin{matrix}\vdots \\ \mathcal{D}\end{matrix}}{\Gamma, z \overset{0}{:} F(X), a \overset{1}{:} A, h \overset{1}{:} B[a] \xrightarrow{1} X \vdash (a, \lambda b. f(h(b))) : F(Y)} \qquad \dfrac{}{0\Gamma, z \overset{1}{:} F(X) \vdash z : F(X)}}{\dfrac{\Gamma, z \overset{1}{:} F(X) \vdash \text{ let } (a, h) = z \text{ in } (a, \lambda b. f(h(b))) : F(Y)}{\Gamma \vdash \lambda z.\ \text{let } (a, h) = z \text{ in } (a, \lambda b. f(h(b))) : F(X) \xrightarrow{1} F(Y)}}$$

where $\mathcal{D}$ is the derivation

$$\dfrac{\dfrac{\Gamma, a \overset{0}{:} A, h \overset{0}{:} B[a] \xrightarrow{1} X, b \overset{0}{:} B[a] \vdash f : X \xrightarrow{1} Y \qquad \begin{matrix}\vdots \\ \mathcal{D}'\end{matrix}}{\Gamma, a \overset{0}{:} A, h \overset{1}{:} B[a] \xrightarrow{1} X, b \overset{1}{:} B[a] \vdash f(h(b)) : Y}}{\Gamma, a \overset{0}{:} A, h \overset{1}{:} B[a] \xrightarrow{1} X \vdash \lambda b. f(h(b)) : B[a] \xrightarrow{1} Y}$$

weakened by $z \overset{0}{:} F(X)$, where again $\mathcal{D}'$ is the derivation

$$\dfrac{\dfrac{}{0\Gamma, a \overset{0}{:} A, h \overset{1}{:} B[a] \xrightarrow{1} X, b \overset{0}{:} B[a] \vdash h : B[a] \xrightarrow{1} X} \qquad \dfrac{}{0\Gamma, a \overset{0}{:} A, h \overset{1}{:} B[a] \xrightarrow{0} X, b \overset{1}{:} B[a] \vdash b : B[a]}}{0\Gamma, a \overset{0}{:} A, h \overset{1}{:} B[a] \xrightarrow{1} X, b \overset{1}{:} B[a] \vdash h(b) : X}$$

similarly weakened. This is functorial by the η-rules for functions and pairs. $\square$

1.3 Internal representation

Let $F_0 : Obj(\mathcal{C}) \to Obj(\mathcal{C})$ be the function mapping a type X to the type $(a \overset{1}{:} A) \otimes (Ba \xrightarrow{1} X)$.

Claim. *F_0 can be extended to an endofunctor $F : \mathcal{C} \to \mathcal{C}$.*

Suppose $\Gamma \vdash f \overset{1}{:} X \xrightarrow{1} Y$ is some morphism with a derivation $\mathcal{D}_f$, then $Ff : (a \overset{1}{:} A) \otimes (Ba \xrightarrow{1} X) \xrightarrow{1} (a \overset{1}{:} A) \otimes (Ba \xrightarrow{1} Y)$ is defined by precomposition with f on the second component.

By applying `Var` rule, we get the following:

- a derivation $\mathcal{D}_a$ of $0\Gamma, a \overset{1}{:} A, g \overset{0}{:} Ba \xrightarrow{1} X \vdash a \overset{1}{:} A$
- a derivation $\mathcal{D}_g$ of $0\Gamma, a \overset{0}{:} A, g \overset{1}{:} Ba \to X \vdash g \overset{1}{:} Ba \xrightarrow{1} X$
- a derivation $\mathcal{D}_b$ of $0\Gamma, a \overset{0}{:} A, g \overset{0}{:} Ba \to X, b \overset{1}{:} Ba \vdash b \overset{1}{:} Ba$
- a derivation $\mathcal{D}_z$ of $0\Gamma, z \overset{1}{:} (a \overset{1}{:} A) \otimes (Ba \xrightarrow{1} X) \vdash z \overset{1}{:} (a \overset{1}{:} A) \otimes (Ba \xrightarrow{1} X)$

Form an intermediate derivation $\mathcal{D}$ by :

$$\dfrac{\mathcal{D}_a \quad \dfrac{\dfrac{D_f \quad \dfrac{D_g \quad D_b}{0\Gamma, a \overset{0}{:} A, g \overset{1}{:} Ba \to X, b \overset{1}{:} Ba \vdash g(b) \overset{1}{:} X}\,App}{0\Gamma, a \overset{0}{:} A, g \overset{1}{:} Ba \to X, b \overset{1}{:} Ba, f \overset{1}{:} X \xrightarrow{1} Y \vdash f(g(b)) \overset{1}{:} Y}\,App}{0\Gamma, a \overset{0}{:} A, g \overset{1}{:} Ba \to X, f \overset{1}{:} X \xrightarrow{1} Y \vdash \lambda b.f(g(b)) \overset{1}{:} Ba \xrightarrow{1} Y}\,Abs}{\Gamma, a \overset{1}{:} A, g \overset{1}{:} Ba \xrightarrow{1} X \vdash (a, \lambda b \overset{1}{:} Ba.f(g(b))) \overset{1}{:} F(Y)}\,\otimes - intro$$

and use it to get a final one for Ff:

$$\dfrac{\dfrac{\mathcal{D}_z \quad \mathcal{D}}{\Gamma, z \overset{1}{:} F(X)) \vdash \text{let } (x, u) = a \text{ in } (x, \lambda b \overset{1}{:} Ba.f(u(b))) \overset{1}{:} F(Y)}\,\oplus - E}{\Gamma \vdash \lambda z.\text{let } (x, u) = a \text{ in } (x, \lambda b \overset{1}{:} Ba.f(u(b))) \overset{1}{:} F(X) \xrightarrow{1} F(Y)}\,Lam$$

Definition 2. We will call any functor isomorphic to F a quantitative polynomial functor.

1.4 External representation (using adjoints)

1.5 Generalising to non-empty contexts

1.6 Properties of quantitative polynomial functors

2 Algebras for QPFs

Recall that an algebra for an endofunctor $F : \mathcal{C} \to \mathcal{C}$ is a pair (X, a), where X is an object of $\mathcal{C}$ and a - a morphism $F(X) \to X$. A morphism between F-algebras is a map $f : X \to Y$, making the square commute:

$$\begin{tikzcd}
F(X) \arrow[r, "F(f)"] \arrow[d, "a"'] & F(Y) \arrow[d, "b"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

2.1 $\mathbb{N}$

Fix $A := \mathbf{Bool}$ and B, such that $B(false) := \emptyset$ and $B(true) := I$ where $x \overset{1}{:} A \vdash B$ type.
Sketch:
Observe that a function $\Gamma \vdash f \overset{1}{:} F_{A,B}(X) \xrightarrow{1} X$ is equivalent to functions $\Gamma \vdash f_l \overset{1}{:} (\emptyset \xrightarrow{1} X) \to X$ and $\Gamma \vdash f_r : (I \xrightarrow{1} X) \to X$.
Assuming that the type $\emptyset \xrightarrow{1} X$ is a (sub?)singleton (or $\emptyset \xrightarrow{1} X \cong I$, but maybe too strong), f_l just encodes a choice of an element of X. Similarly, $I \xrightarrow{1} X$ also encodes the same data, that is $I \xrightarrow{1} X \cong X$ (provable in QTT?).
Essentially, an algebra for $F_{A,B}$ is a diagram $I \to X \to X$.
Now, we prove the following propositions:

(i) Suppose $\Gamma \vdash \mathbf{Nat}$ type, then there exists an initial algebra for $F_{A,B}(\mathbf{Nat})$

(ii) Suppose $\Gamma \vdash X$ type, such that there exists an initial algebra for $F_{A,B}(X)$, then $F_{A,B}(X) \cong \mathbf{Nat}$.

Before embarking on proving that, let's examine a simpler construction:

Interlude : Pseudobooleans as a constant QPF Let $A := \mathbf{Bool}$, $\Gamma, a \overset{1}{:} A \vdash B$ type, $B(a) := \emptyset$. An algebra for this $F_{A,B}(X)$ is a diagram $X \leftarrow \{*\} \to X$.

Assume $\Gamma \vdash \mathbf{Bool}$ type. Then define $t(i) := true$ and $f(i) := false$ using the unit elimination. Let $(F_{A,B}(Y), m, n)$ be another algebra. Define $\varphi(x) := Elim(m(*), n(*), k)$, where $k : F_{A,B}(\mathbf{Bool}) \xrightarrow{1} \mathbf{Bool}$ is the structure map. The squares commute by the computation rules for $\mathbf{Bool}$.

$$\begin{tikzcd}
\mathbf{Bool} \arrow[d, "\varphi"'] & \{*\} \arrow[l, "t"'] \arrow[r, "f"] \arrow[d, equal] & \mathbf{Bool} \arrow[d, "\varphi"] \\
Y & \{*\} \arrow[l, "m"] \arrow[r, "n"'] & Y
\end{tikzcd}$$

Conversely, assume that there is an initial algebra $(X, k : F_{A,B}X \to X)$. Designate some elements $t, f \in X$, such that $t \neq f$. For any other algebra represented diagrammatically $(Y, m : I \xrightarrow{1} Y, n : I \xrightarrow{1} Y)$, we have that $\varphi(t) = m$ and $\varphi(f) = n$ by initiality. Define $Elim(m, n, x) := \varphi(x)$.

Back to $\mathbb{N}$, let **Nat** be defined as:

$$\frac{0\Gamma \vdash A}{0\Gamma \vdash \mathbf{Nat}}\ \text{Nat} \qquad \frac{0\Gamma \vdash}{0\Gamma \vdash 0 \overset{\sigma}{:} \mathbf{Nat}} \qquad \frac{\Gamma \vdash N \overset{\sigma}{:} \mathbf{Nat}}{\Gamma \vdash suc(N) \overset{\sigma}{:} \mathbf{Nat}}\ \text{suc}$$

$$\dots$$

$$\begin{tikzcd} & X \arrow[r, "\sigma"] \arrow[dd, "\varphi"] & X \arrow[dd, "\varphi"] \\ I \arrow[ur, "\mu"] \arrow[dr, "z"'] & & \\ & Y \arrow[r, "s"] & Y \end{tikzcd}$$

2.2 Lists

2.3 Trees

2.4 Induction principle

3 Rules for W-types in QTT

4 Parametricity and W-types

5 Appendix

5.1 (Stand-alone) Sum types

$$\frac{0\Gamma \vdash A \quad 0\Gamma \vdash B}{0\Gamma \vdash \rho A \oplus \pi B}\ \oplus\text{-type} \qquad \frac{\Gamma \vdash S_1 \overset{\sigma}{:} A}{\rho\Gamma \ \vdash \mathbf{inl}\, S_1 \overset{\sigma}{:} \rho A \oplus \pi B}\ \text{inl} \qquad \frac{\Gamma \vdash S_2 \overset{\sigma}{:} B}{\pi\Gamma \vdash \mathbf{inr}\, S_2 \overset{\sigma}{:} \rho A \oplus \pi B}\ \text{inr}$$

$$\frac{\begin{array}{c} 0\Gamma, x \overset{0}{:} \rho A \oplus \pi B \vdash C \\ \Gamma \vdash M \overset{\sigma}{:} \rho A \oplus \pi B \qquad \Gamma', a \overset{\rho}{:} A \vdash T_1 \overset{\sigma}{:} C[\mathbf{inl}\, a/x] \qquad \Gamma', b \overset{\pi}{:} B \vdash T_2 \overset{\sigma}{:} C[\mathbf{inr}\, b/x] \qquad 0\Gamma = 0\Gamma' \end{array}}{\Gamma' + \Gamma \vdash \mathbf{case}(M, T_1, T_2) \overset{\sigma}{:} C[M/x]}\ \oplus\text{-elim}$$

$$\frac{\Gamma \vdash S_1 \overset{\sigma}{:} A \qquad \Gamma \vdash M \overset{\sigma}{:} \rho A \oplus \pi B \qquad \Gamma', a \overset{\rho}{:} A \vdash T_1 \overset{\sigma}{:} C[\mathbf{inl}\, a/x] \qquad 0\Gamma = 0\Gamma'}{\Gamma' + \rho\Gamma \vdash \mathbf{case}(\mathbf{inl}(S_1), T_1, T_2) \equiv T_1[S_1/a]}\ \oplus\text{-comp}$$

Figure 1: Rules for $\oplus$-type

We give the following semantics for the $\oplus$-type:
$|\rho A \oplus \pi B\ (\gamma)| := |A(\gamma)| \sqcup |B(\gamma)|$
$a \vDash_{\rho A \oplus \pi B\ (\gamma)} (i, x)$ iff $(\exists b. a = [!_\rho b, \ulcorner true \urcorner] \wedge b \vDash_{A(\gamma)} x \wedge i = 0) \vee$
$(\exists c. a = [!_\pi c, \ulcorner false \urcorner] \wedge c \vDash_{B(\gamma)} x \wedge i = 1)$

Claim. *The rules are sound when interpreted wrt to the given semantics for $\oplus$-types and realisability model.*

Proof. The underlying set-theoretic functions are immediate. For the realisers, let

- $a_{\mathbf{inl}} := \lambda^* x. [F_\rho \cdot\ !_\rho a_{s_1} \cdot x, \ulcorner true \urcorner]$

  if $a_{s_1} \cdot a_\gamma \vDash s_1$*, then* $a_{\mathbf{inl}} \cdot !_\rho a_\gamma \vDash \mathbf{inl} s_1$*;* $a_{\mathbf{inl}} \cdot !_\rho a_\gamma \rightsquigarrow [!_\rho (a_{s_1} \cdot a_\gamma), \ulcorner true \urcorner]$

- $a_{\mathbf{case}} := \begin{array}{l} \lambda^* x.\ \text{let } [a'_\gamma, a_\gamma] = x, \\ \qquad [a, b] = a_m \cdot a_\gamma \text{ in} \\ \qquad E(b, a_{T_1}, a_{T_2}) \cdot [a'_\gamma, a] \end{array}$

  assuming $a_m \cdot a_\gamma \vDash M$*,* $a_{T_1} \cdot [a'_\gamma, !_\rho a_a] \vDash T_1$*,* $a_{T_2} \cdot [a'_\gamma, !_\pi a_b] \vDash T_2$*, then we want to find* $a_{\mathbf{case}}$*, s.t.* $a_{\mathbf{case}} \cdot [a'_\gamma, a_\gamma] \vDash \mathbf{case}(M, T_1, T_2)$*.*

if $a_m \cdot a_\gamma = [!_\rho a_a, \ulcorner true \urcorner]$, *then* $a_{\mathbf{case}} \cdot [a_\gamma, a'_\gamma] \rightsquigarrow E(\ulcorner true \urcorner, a_{T_1}, a_{T_2}) \cdot [a'_\gamma, !_\rho a_a] \rightsquigarrow a_{T_1} \cdot [a'_\gamma, !_\rho a_a]$

if $a_m \cdot a_\gamma = [!_\pi a_b, \ulcorner false \urcorner]$, *then* $\ldots$

$\square$

Claim. *There is a bijection:*

$$RTm(\Gamma, \Pi(x \overset{\tau}{:} \rho A \oplus \pi B)\, C) \cong RTm(\Gamma, \Pi(y \overset{\tau\rho}{:} A)\, C[\mathbf{inl}y/x]) \times RTm(\Gamma, \Pi(z \overset{\tau\pi}{:} B)\, C[\mathbf{inl}z/x]))$$

(natural in Γ*).*

Proof. Given a term $\Gamma \vdash f \overset{1}{:} (x \overset{\tau}{:} \rho A \oplus \pi B) \to C$, we can derive another term $\Gamma \vdash f^l \overset{1}{:} (y \overset{\tau\rho}{:} A) \to C[\mathbf{inl}y/x]$:

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash f : (x \overset{\tau}{:} \rho A \oplus \pi B) \to C}{\Gamma, y \overset{0}{:} A \vdash f \overset{1}{:} (x \overset{\tau}{:} \rho A \oplus \pi B) \to C}\;Weak \qquad \dfrac{\dfrac{\vdash 0\Gamma, y \overset{1}{:} A}{0\Gamma, y \overset{1}{:} \vdash y \overset{1}{:} A}\;var}{0\Gamma, y \overset{\rho}{:} A \vdash \mathbf{inl}y \overset{1}{:} \rho A \oplus \pi B}\;inl}{\Gamma, y \overset{\tau\rho}{:} A \vdash f(\mathbf{inl}y) \overset{1}{:} C[\mathbf{inl}y/x]}\;App}{\Gamma \vdash \lambda y \overset{\tau\rho}{:} A.\, f(\mathbf{inl}y) : (y \overset{\tau\rho}{:} A) \to C[\mathbf{inl}y/x]}\;Lam$$

Analogously, we can obtain $\Gamma \vdash f^r \overset{1}{:} (z \overset{\tau\pi}{:} B) \to C[\mathbf{inr}y/x]$.
Now suppose we have terms $\Gamma \vdash l \overset{1}{:} (y \overset{\tau\rho}{:} A) \to C[\mathbf{inl}y/x]$ and $\Gamma \vdash r \overset{1}{:} (z \overset{\tau\pi}{:} B) \to C[\mathbf{inr}z/x]$.
Using the isomorphism $\Lambda^{\mathcal{L}}$, we get judgements $\Gamma, y \overset{\tau\rho}{:} A \vdash l^* : C[\mathbf{inl}y/x]$ and $\Gamma, z \overset{\tau\pi}{:} B \vdash r^* : C[\mathbf{inr}z/x]$. $\square$