

# Relational realizability model for QTT

Our aim is to build a concrete realizability model for QTT which allows for metareasoning with results derived from parametricity. To that end, a construction of a quantified category with families (QCwF[**fill**]) is presented. We follow the relational approach to types introduced by Reynolds for typed lambda calculus[**fill**] and later refined for dependent types theories.[**fill**] Once and for all fix a usage semiring  $R$  and an  $R$ -linear combinatory algebra  $\mathcal{A}$ <sup>1</sup>.

## Taking stocks

**Definition 1** (Assembly). An assembly  $\Gamma$  is a pair  $(|\Gamma|, e)$  where  $|\Gamma|$  is a carrier set and  $e$  is a realizability function  $|\Gamma| \rightarrow \mathcal{P}(\mathcal{A})$ , s.t  $e(\gamma)$  is nonempty for every  $\gamma \in |\Gamma|$ .

We interpret  $E(\gamma)$  as a set of witnesses for the existence of  $\gamma$  and write  $a \Vdash_{\Gamma} \gamma$  to denote  $a \in e(\gamma)$ .

A morphism between two assemblies  $(|\Gamma|, E_{\Gamma})$  and  $(|\Delta|, E_{\Delta})$  is a function  $f : |\Gamma| \rightarrow |\Delta|$  that is realizable - there exists  $a_f \in \mathcal{A}$ , s.t the following holds for every  $\gamma \in |\Gamma|$  and  $a_{\gamma} \in \mathcal{A}$ :

$$a_{\gamma} \Vdash_{\Gamma} \gamma \implies a_f.a_{\gamma} \Vdash_{\Delta} f(\gamma)$$

We say that  $a_f$  tracks  $f$ . Note that only an existence of a realizer for  $f$  is stipulated in the definition - multiple realizers do not induce multiple morphisms.

Using these notions we can construct a category  $\mathcal{A}sm(\mathcal{A})$ .

**Definition 2** (Reflexive graph). A reflexive graph (r.g.)  $G$  is a tuple  $(G_O, G_R, G_{refl}, G_{src}, G_{tgt})$ , where  $G_O$  and  $G_R$  are sets,  $G_{src}$  and  $G_{tgt}$  are functions  $G_R \rightarrow G_O$  and  $G_{refl}$  is a function  $G_O \rightarrow G_R$ , s.t  $G_{src} \circ G_{refl} = id_{G_O} = G_{tgt} \circ G_{refl}$ .

$G_O$  and  $G_R$  stand for a set of objects and a set of relations, respectively. In general, reflexive graphs are less structured than categories as relations need not compose.

A morphism between reflexive graphs  $G$  and  $H$  is a pair of functions  $(f_o : G_O \rightarrow H_O, f_r : G_R \rightarrow H_R)$ , s.t all of the depicted squares commute:

$$\begin{array}{ccc} G_O & \xrightarrow{f_o} & H_O \\ \begin{array}{c} \uparrow | \downarrow \\ G_{src} \left( G_{refl} \right) G_{tgt} \end{array} & & \begin{array}{c} \uparrow | \downarrow \\ H_{src} \left( H_{refl} \right) H_{tgt} \end{array} \\ G_R & \xrightarrow{f_r} & H_R \end{array}$$

Reflexive graphs equipped with r.g. morphisms form a category  $\mathcal{RGph}(\mathcal{Set})$ .

We use reflexive graphs to give a dyadic interpretation of types in the spirit of [**fill**].

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<sup>1</sup>In case some non-trivial properties of  $\mathcal{A}$  are required, we will tacitly assume that  $\mathcal{A}$  is a graph model(see [**fill**]) - also to fix

## Assembling a reflexive graph

One could easily generalize reflexive graphs by considering object and relation components from an arbitrary category  $\mathcal{C}$  instead of  $\mathcal{Set}$ . As our purpose is to build a realizability model, we pick  $\mathcal{Asm}(\mathcal{A})$  as the category of interest and identify two appropriate notions of reflexive graph of assemblies and a family of reflexive graphs of assemblies.

**Definition 3** (Reflexive graph of assemblies). A reflexive graph of assemblies  $G$  is a pair of assemblies  $(G_O, G_R)$  and a triple of set-theoretic  $(G_{refl} : |G_O| \rightarrow |G_R|, G_{src} : |G_R| \rightarrow |G_O|, G_{tgt} : |G_R| \rightarrow |G_O|)$ , such that the identities in Definition 2 are satisfied.

In essence, sets are directly replaced by the objects from  $\mathcal{Asm}(\mathcal{A})$ , while the morphism are kept intact - plain set-theoretic functions.

However, we require realizable functions only in the definition morphism between reflexive graph of assemblies. With these components, we obtain a category  $\mathcal{RGph}(\mathcal{A})$ . By considering r.g. of assemblies of shape  $(X, X, id_X, id_X, id_X)$ , we identify isomorphic copy of  $\mathcal{Asm}(\mathcal{A})$  inside  $\mathcal{RGph}(\mathcal{A})$ .

A terminal object in  $\mathcal{RGph}(\mathcal{A})$  is a tuple  $(1, 1, id, id, id)$ , where 1 is the terminal assembly  $(\{\star\}, f)$ , with  $f$  defined as  $\star \mapsto \{I\}$ .

**Definition 4** (Family of reflexive graphs of assemblies). Let  $\mathcal{C}$  be a category with a terminal object. Given a reflexive graph  $\Gamma \in Ob(\mathcal{C})$ , a family of internal r.g. over  $\Gamma$  is a tuple  $S := (S_O, S_R, S_{refl}, S_{src}, S_{tgt})$ , where:

- $S_O : \Gamma_O \rightarrow \mathcal{Asm}(\mathcal{A})$
- $S_R : \Gamma_R \rightarrow \mathcal{Asm}(\mathcal{A})$
- a  $\Gamma$ -indexed collection of functions  $S_{refl} := \{f : |S_O(\gamma)| \rightarrow |S_R(\Gamma_{refl}(\gamma))|\}_{\gamma \in \Gamma_O}$
- $S_{src} := \{f : |S_R(\gamma)| \rightarrow |S_O(\Gamma_{src}(\gamma))|\}_{\gamma \in \Gamma_R}$
- $S_{tgt} := \{f : |S_R(\gamma)| \rightarrow |S_O(\Gamma_{tgt}(\gamma))|\}_{\gamma \in \Gamma_R}$

such that

- each identity in the following collection is satisfied:

$$S_\sigma(\Gamma_{refl}(\gamma)) \circ S_{refl}(\gamma) = id \text{ for every } \gamma \in \Gamma_O, \sigma \in \{src, tgt\}$$

We are only interested in cases when  $\mathcal{C} = \mathcal{Set}$  or  $\mathcal{C} = \mathcal{Asm}(\mathcal{A})$ .

A morphism  $M$  between two families  $S$  and  $T$  of internal r.g. over  $\Gamma$  is a pair of  $\Gamma$ -indexed collection of functions:

- $M_O := \{f : |S_O(\gamma)| \rightarrow |T_O(\gamma)|\}_{\gamma \in \Gamma_O}$
- $M_R := \{f : |S_R(\gamma)| \rightarrow |T_R(\gamma)|\}_{\gamma \in \Gamma_O}$

such that:

- $T_{refl}(M_O(\gamma)(x)) = M_R(\Gamma_{refl}(\gamma))(S_{refl}(\gamma)(x))$  for every  $\gamma \in \Gamma_O$ ,  $x \in S_O(\gamma)$
- $T_{src}(M_R(\gamma)(r)) = M_O(\Gamma_{src}(\gamma))(S_{src}(\gamma)(r))$  for every  $\gamma \in \Gamma_R$ ,  $r \in S_R(\gamma)$
- $T_{tgt}(M_R(\gamma)(r)) = M_O(\Gamma_{tgt}(\gamma))(S_{tgt}(\gamma)(r))$  for every  $\gamma \in \Gamma_R$ ,  $r \in S_R(\gamma)$

A terminal family of r.g. over  $\Gamma$ ,  $1_\Gamma$ , consists of two constant functions, mapping  $\gamma \in \Gamma$  to a terminal assembly 1, and three  $\Gamma$ -indexed collections with a sole element  $id_1$ .

## A CwF from families of reflexive graphs of assemblies

As a first step toward obtaining a relationally parametric realizability model of QTT, we construct a concrete CwF using families of internal graphs of assemblies.

Consider the category  $\mathcal{RGph}$  with terminal object  $1 := (\{\star\}, \{\star\}, id, id, id)$ .

Let  $\Gamma, \Delta \in Ob(\mathcal{RGph})$ , define:

- the collection of semantic types  $Ty(\Gamma)$  as the collection of families of reflexive graphs of assemblies  $\Gamma$ .
- given a type  $S \in Ty(\Gamma)$ , let  $Tm(\Gamma, S) := Hom(1_\Gamma, S)$ .  
Spelling this out and ignoring the contribution of the terminal family, we get:  
An element  $M \in Tm(\Gamma, S)$  is a pair of functions  $(M_O : \forall \gamma \in \Gamma_O. S_O(\gamma), M_R : \forall \gamma \in \Gamma_R. S_R(\gamma))$ , such that

$$\begin{aligned} \forall \gamma \in \Gamma_O. S_{refl}(M_O(\gamma)) &= M_R(\Gamma_{refl}(\gamma)) \\ \forall \gamma \in \Gamma_R. S_{src}(M_R(\gamma)) &= M_O(\Gamma_{src}(\gamma)) \\ \forall \gamma \in \Gamma_R. S_{tgt}(M_R(\gamma)) &= M_O(\Gamma_{tgt}(\gamma)) \end{aligned}$$

- given  $f : \Gamma \rightarrow \Delta$ , substitutions in types and terms is a precomposition with  $f$  on the object and relation components of types and terms respectively. Clearly, these operations are compatible with identity and composition in  $\mathcal{RGph}$
- context extension : Suppose  $S \in Ty(\Gamma)$ , construct a r.g.  $\Gamma.S$  as :

$$\begin{aligned} (\Gamma.S)_O &= \{(\gamma, x) : \gamma \in \Gamma_O, x \in S_O(\gamma)\} \\ (\Gamma.S)_R &= \{(\gamma, r) : \gamma \in \Gamma_R, r \in S_R(\gamma)\} \\ (\Gamma.S)_{refl}(\gamma, x) &= (\Gamma_{refl}(\gamma), S_{refl}(\gamma)(x)) \\ (\Gamma.S)\sigma(\gamma, r) &= (\Gamma\sigma(\gamma), S\sigma(\gamma)(r)), \quad \sigma \in \{src, tgt\} \end{aligned}$$

**Claim.**  $Hom_{\mathcal{RGph}}(\Delta, \Gamma.S) \cong \{(f, M) : f : \Delta \rightarrow \Gamma, M \in Tm(\Delta, S\{f\})\}$ , natural in  $\Delta$ .

## Upgrading to a QCwF

We simultaneously recall the definition of a QCwF and add structure to our CwF.

1. A CwF  $(\mathcal{C}, 1, Ty, Tm, \dashv, \langle -, - \rangle)$ .

Consider the CwF built in the previous section

2. Category  $\mathcal{L}$  for context with resource annotation, equipped with a faithful functor  $U : \mathcal{L} \rightarrow \mathcal{C}$ .

Take  $cL := \mathcal{R}Gph(\mathcal{A})$  and let  $U$  be the functor  $\mathcal{R}Gph(\mathcal{A}) \rightarrow \mathcal{R}Gph$ , sending an assembly to its underlying set, forgetting the realizability function.

3. Addition structure - let  $\mathcal{L} \times_{\mathcal{C}} \mathcal{L}$  denote the pullback  $\mathcal{L} \xrightarrow{U} \mathcal{C} \xleftarrow{U} \mathcal{L}$ , we stipulate the existence of a functor  $(+) : \mathcal{L} \times_{\mathcal{C}} \mathcal{L} \rightarrow \mathcal{L}$ , s.t.  $U(\Gamma' + \Gamma'') = U(\Gamma') = U(\Gamma'')$ . Moreover, there is a distinguished object  $\Diamond \in \mathcal{L}$ , s.t.  $U(\Diamond) = 1$ .

Suppose  $\Gamma', \Gamma''$  are r.g. of assemblies, such that  $|\Gamma'_O| = |\Gamma''_O|$  and  $|\Gamma'_R| = |\Gamma''_R|$ . Construct the internal r.g.  $\Gamma := \Gamma' + \Gamma''$ , where:

- $\Gamma_O := (|\Gamma'_O|, \models_{\Gamma})$  with  $a \models_{\Gamma} \gamma$  iff there exist  $x, y \in \mathcal{A}$ , s.t.  $a = [x, y]$  and  $x \models_{\Gamma'} \gamma$  and  $y \models_{\Gamma''} \gamma$ .
- define  $\Gamma_R$  similarly as  $\Gamma_O$ .
- $\Gamma_{refl}, \Gamma_{src}, \Gamma_{tgt}$  are all inherited from  $\Gamma'$  (or  $\Gamma''$ ). *(should check this is indeed well-defined)*

Pick a terminal internal r.g. as an interpretation for  $\Diamond$ .

4. Scaling with  $\rho \in R$  - there is a functor  $\rho(-) : \mathcal{L} \rightarrow \mathcal{L}$ , s.t.  $U(\rho(-)) = U(-)$ .

Again we define an internal r.g.  $\Gamma := \rho(\Gamma')$  as :

- $\Gamma_{\sigma} = (|\Gamma'_{\sigma}|, \models_{\Gamma_{\sigma}})$  with  $a \models_{\Gamma_{\sigma}} \gamma$  iff there is  $x \in \mathcal{A}$ , s.t.  $a = !_\rho x$  and  $x \models_{\Gamma'} \gamma$  for  $\sigma \in \{O, R\}$
- $\Gamma_{\sigma}$  is just  $\Gamma'_{\sigma}$  for  $\sigma \in \{src, tgt, rfl\}$

As the functor  $\rho$  modifies only realizability information, it is clear that  $U(\rho(-)) = U(-)$  holds.

## Type formers

### Universe of small types

**Dependent function types in 1-fragment** Given an internal reflexive graph  $\Gamma$ ,  $S \in Ty(\Gamma)$ ,  $T \in Ty(\Gamma.S)$ ,  $\rho \in R$ , define  $\Pi_{\rho}ST \in Ty(\Gamma)$  as the tuple of:

- Object assembly. Let  $\gamma \in |\Gamma_O|$

$$|(\Pi\rho ST)_O|(\gamma) := \{(f_O, f_R)|$$

$$f_O : \forall s \in S_O(\gamma). T_O(\gamma, s),$$

$$\exists a \in \mathcal{A} \forall s \in |S_O(\gamma)|, b \in \mathcal{A}.$$

$$b \models_{S_O(\gamma)} s \implies a.!\rho b \models_{T_O(\gamma, s)} f_O(s)$$

$$f_R : \forall r \in S_R(\Gamma_{refl}(\gamma)). T_R(\Gamma_{refl}(\gamma), r),$$

$$\exists a \in \mathcal{A} \forall r \in |S_R(\Gamma_{refl}(\gamma))|, b \in \mathcal{A}.$$

$$b \models_{S_R(\Gamma_{refl}(\gamma))} r \implies a.!\rho b \models_{T_R(\Gamma_{refl}(\gamma), r)} f_R(r),$$

$$\forall s \in S_O(\gamma). T_{refl}(\gamma, s)(f_O(s)) = f_R(S_{refl}(\gamma)(s)),$$

$$\forall r \in S_R(\Gamma_{refl}(\gamma)). T_{src}(\Gamma_{refl}(\gamma), r)(f_R(r)) = f_O(S_{src}(\Gamma_{refl}(\gamma))(r)),$$

$$\forall r \in S_R(\Gamma_{refl}(\gamma)). T_{tgt}(\Gamma_{refl}(\gamma), r)(f_R(r)) = f_O(S_{tgt}(\Gamma_{refl}(\gamma))(r))\}$$

$a \models_{(\Pi\rho ST)_O(\gamma)} (f_O, f_R)$  iff  $a = [a_o, a_r]$  where the realizers  $a_o, a_r$  are given by the existential statements above.

- Relation assembly. Let  $\gamma \in |\Gamma_R|$

$$|(\Pi\rho ST)_R|(\gamma) := \{((f_O^{src}, f_R^{src}), (f_O^{tgt}, f_R^{tgt}), r)|$$

$$(f_O^{src}, f_R^{src}) \in (\Pi\rho ST)_O(\Gamma_{src}(\gamma)),$$

$$(f_O^{tgt}, f_R^{tgt}) \in (\Pi\rho ST)_O(\Gamma_{tgt}(\gamma)),$$

$$r : \forall s \in S_R(\gamma). T_R(\gamma, s),$$

$$\exists a \in \mathcal{A} \forall s \in |S_R(\gamma)|, b \in \mathcal{A}. b \models_{S_R(\gamma)} s \implies a.!\rho b \models_{T_R(\gamma, s)} r(s),$$

$$\forall s \in S_R(\gamma). T_{src}(\gamma, s)(r(s)) = f_O^{src}(S_{src}(\gamma)(s)),$$

$$\forall s \in S_R(\gamma). T_{tgt}(\gamma, s)(r(s)) = f_O^{tgt}(S_{tgt}(\gamma)(s))\}$$

- collections of morphisms:

$$(\Pi\rho ST)_{refl}(\gamma)(f_O, f_R) := ((f_O, f_R), (f_O, f_R), f_R) \text{ for } \gamma \in \Gamma_O$$

$$(\Pi\rho ST)_{src}(\gamma)(f^{src}, f^{tgt}, r) := f^{src} \text{ for } \gamma \in \Gamma_R$$

$$(\Pi\rho ST)_{tgt}(\gamma)(f^{src}, f^{tgt}, r) := f^{tgt} \text{ for } \gamma \in \Gamma_R$$

These collections satisfy the identity conditions in Definition 4.

## Dependent pairs