Outline

1 Quantitative polynomial functors

1.1 Category of closed types and linear functions

Let $\mathcal{C}$ be the category of closed types and linear functions $f : (x \overset{1}{:} X) \to Y$ for which derivations in QTT exist. Composition of morphisms $\Gamma \vdash f \overset{\sigma}{:} (x \overset{1}{:} X) \to Y$ and $\Gamma \vdash g \overset{\sigma}{:} (y \overset{1}{:} Y) \to Z$ is given by ordinary function composition $\Gamma \vdash \lambda x \overset{1}{:} X.g(f(x)) \overset{\sigma}{:} (x \overset{1}{:} X) \to Z$.
The linearity restriction on the morphisms does not lead to loss of expressiveness - functions with arbitrary resource annotations can be represented as linear ones via:

1.2 Category of closed types and linear functions in a nonempty context

Let Δ be an "underlying context", i.e. a context of the form $\Delta = 0\Gamma_0$ for some Γ_0. There is a category $\mathcal{C}_\Delta$ where the objects are types X such that $\Delta \vdash X$ type, and a morphism X to Y consists of pair (Γ, f), where Γ is a context such that $0\Gamma = \Delta$, and $\Gamma \vdash f : X \xrightarrow{1} Y$.

- The identity morphism is given by $(\Delta, \lambda x.x)$;
- Composition of (Γ_2, g) and (Γ_1, f) is given by $(\Gamma_1 + \Gamma_2, \lambda x.f(g(x)))$.

This is a category since $0\Delta = \Delta$, and context addition is associative, and with $\Gamma + \Delta = \Delta + \Gamma = \Gamma$. Note that the category of $\mathcal{C}$ closed types from Section 1.1 is a special case $\mathcal{C} = \mathcal{C}_\diamond$ where $\Delta = \diamond$, because the only Γ with $0\Gamma = \diamond$ is $\Gamma = \diamond$.

Lemma 1. *Fix Δ as above, and let $\Delta \vdash A$ type and $\Delta, x \overset{0}{:} A \vdash B[x]$ type. The operation $F(X) = (a \overset{1}{:} A) \otimes (Ba \xrightarrow{1} X)$ is a functor $\mathcal{C}_\Delta \to \mathcal{C}_\Delta$.*

Proof. If $\Delta \vdash X$ type then $\Delta \vdash F(X)$ type. On morphisms, we can define

$$F(\Gamma, f) = (\Gamma, \lambda z. \text{ let } (a, h) = z \text{ in } (a, \lambda b.f(h(b))))$$

as the following derivation shows:

$$\dfrac{\dfrac{\dfrac{}{0\Gamma, z \overset{0}{:} F(X), a \overset{1}{:} A, h \overset{0}{:} B[a] \xrightarrow{1} X \vdash a : A} \qquad \begin{array}{c}\vdots \\ \mathcal{D}\end{array}}{\Gamma, z \overset{0}{:} F(X), a \overset{1}{:} A, h \overset{1}{:} B[a] \xrightarrow{1} X \vdash (a, \lambda b.f(h(b))) : F(Y)} \qquad \dfrac{}{0\Gamma, z \overset{1}{:} F(X) \vdash z : F(X)}}{\dfrac{\Gamma, z \overset{1}{:} F(X) \vdash \text{ let } (a, h) = z \text{ in } (a, \lambda b.f(h(b))) : F(Y)}{\Gamma \vdash \lambda z. \text{ let } (a, h) = z \text{ in } (a, \lambda b.f(h(b))) : F(X) \xrightarrow{1} F(Y)}}$$

where $\mathcal{D}$ is the derivation

$$\dfrac{\dfrac{\Gamma, a \overset{0}{:} A, h \overset{0}{:} B[a] \xrightarrow{1} X, b \overset{0}{:} B[a] \vdash f : X \xrightarrow{1} Y \qquad \overset{\vdots}{\mathcal{D}'}}{\Gamma, a \overset{0}{:} A, h \overset{1}{:} B[a] \xrightarrow{1} X, b \overset{1}{:} B[a] \vdash f(h(b)) : Y}}{\Gamma, a \overset{0}{:} A, h \overset{1}{:} B[a] \xrightarrow{1} X \vdash \lambda b.f(h(b)) : B[a] \xrightarrow{1} Y}$$

weakened by $z \overset{0}{:} F(X)$, where again $\mathcal{D}'$ is the derivation

$$\dfrac{\dfrac{}{0\Gamma, a \overset{0}{:} A, h \overset{1}{:} B[a] \xrightarrow{1} X, b \overset{0}{:} B[a] \vdash h : B[a] \xrightarrow{1} X} \qquad \dfrac{}{0\Gamma, a \overset{0}{:} A, h \overset{1}{:} B[a] \xrightarrow{0} X, b \overset{1}{:} B[a] \vdash b : B[a]}}{0\Gamma, a \overset{0}{:} A, h \overset{1}{:} B[a] \xrightarrow{1} X, b \overset{1}{:} B[a] \vdash h(b) : X}$$

similarly weakened. This is functorial by the η-rules for functions and pairs. $\square$

1.3 Internal representation

Let F be the polynomial functor mapping a type X to $(a \overset{1}{:} A) \otimes (Ba \xrightarrow{1} X)$ and a function $f : X \xrightarrow{1} Y$ to $Ff : (a \overset{1}{:} A) \otimes (Ba \xrightarrow{1} X) \to (a \overset{1}{:} A) \otimes (Ba \xrightarrow{1} Y)$

If D_f is the derivation of f, we show how to derive Ff. Given some $a : A$ and $h : Ba \xrightarrow{1} X$, start by composing with f and constructing a tensor product:

$$\dfrac{\overset{\vdots}{0\Gamma, a \overset{1}{:}, h \overset{0}{:} \cdots \vdash a \overset{1}{:} A} \qquad \overset{\dfrac{\dfrac{\vdash \Gamma, a \overset{1}{:} A, h \overset{1}{:} Ba \xrightarrow{1} X}{0\Gamma, a \overset{0}{:} A, h \overset{1}{:} Ba \xrightarrow{1} X \vdash h \overset{1}{:} Ba \xrightarrow{1} X}\,Var \qquad \dfrac{D_f}{\Gamma \vdash f \overset{1}{:} X \xrightarrow{1} Y}}{\vdots}}{\Gamma, a \overset{0}{:} A, h \overset{1}{:} Ba \xrightarrow{1} X \vdash \lambda b \overset{1}{:} Ba.f(g(b)) \overset{1}{:} (Ba \xrightarrow{1} Y)}}{\Gamma, a \overset{1}{:} A, h \overset{1}{:} Ba \xrightarrow{1} X \vdash (a, \lambda b \overset{1}{:} Ba.f(g(b))) \overset{1}{:} (a \overset{1}{:} A) \otimes (Ba \xrightarrow{1} Y)}$$

$$\dfrac{\dfrac{\dfrac{\vdash \Gamma, z \overset{1}{:} (a \overset{1}{:} A) \otimes (Ba \xrightarrow{1} X)}{0\Gamma, z \overset{1}{:} (a \overset{1}{:} A) \otimes (Ba \xrightarrow{1} X) \vdash z \overset{1}{:} \ldots} \qquad \Gamma, \cdots \vdash (a, \lambda b \overset{1}{:} Ba.f(g(b))) \overset{1}{:} (a \overset{1}{:} A) \otimes (Ba \xrightarrow{1} Y)}{\Gamma, z \overset{1}{:} (a \overset{1}{:} A) \otimes (Ba \xrightarrow{1} X) \vdash \text{let } (x, u) = a \text{ in } (x, \lambda b \overset{1}{:} Ba.f(u(b))) \overset{1}{:} (a \overset{1}{:} A) \otimes (Ba \xrightarrow{1} Y)}\,\oplus - E}{\Gamma \vdash \lambda z \overset{1}{:} (a \overset{1}{:} A) \otimes (Ba \to X).\ \text{let } \ldots \overset{1}{:} (a \overset{1}{:} A) \otimes (Ba \xrightarrow{1} X) \overset{1}{\longrightarrow} (a \overset{1}{:} A) \otimes (Ba \xrightarrow{1} Y)}\,Lam$$

1.4 External representation (using adjoints)

$(s \overset{0}{:} S) \otimes ((t \overset{0}{:} T) \otimes Id_{f(s),g(t)}))$ Use QCwF structure...

1.5 Generalising to non-empty contexts

1.6 Properties of quantitative polynomial functors

2 Algebras for QPFs

2.1 $\mathbb{N}$

$f : 1 \to Bool$, Let $A := Bool$ and $B = \mathbf{1}$, P_f.
$P_f(X) := (a \overset{1}{:} Bool) \otimes ((b \in f^{-1}(a)) \to X)$ Assume N, prove initiality.
If there is an initial algebra for P_f there is a type N that satisfies n.n. induction (requires dependent function).

2.2 Lists

2.3 Trees

2.4 Induction principle

3 Rules for W-types in QTT

4 Parametricity and W-types

5 Appendix

5.1 (Stand-alone) Sum types

$$\frac{0\Gamma \vdash A \quad 0\Gamma \vdash B}{0\Gamma \vdash \rho A \oplus \pi B}\ \oplus\text{-type} \qquad \frac{\Gamma \vdash S_1 \overset{\sigma}{:} A}{\rho\Gamma \ \vdash \mathbf{inl}\, S_1 \overset{\sigma}{:} \rho A \oplus \pi B}\ \text{inl} \qquad \frac{\Gamma \vdash S_2 \overset{\sigma}{:} B}{\pi\Gamma \vdash \mathbf{inr}\, S_2 \overset{\sigma}{:} \rho A \oplus \pi B}\ \text{inr}$$

$$\frac{\Gamma \vdash M \overset{\sigma}{:} \rho A \oplus \pi B \qquad \begin{array}{c} 0\Gamma, x \overset{0}{:} \rho A \oplus \pi B \vdash C \\ \Gamma', a \overset{\rho}{:} A \vdash T_1 \overset{\sigma}{:} C[\mathbf{inl}\, a/x] \end{array} \qquad \Gamma', b \overset{\pi}{:} B \vdash T_2 \overset{\sigma}{:} C[\mathbf{inr}\, b/x] \qquad 0\Gamma = 0\Gamma'}{\Gamma' + \Gamma \vdash \mathbf{case}(M, T_1, T_2) \overset{\sigma}{:} C[M/x]}\ \oplus\text{-elim}$$

$$\frac{\Gamma \vdash S_1 \overset{\sigma}{:} A \qquad \Gamma \vdash M \overset{\sigma}{:} \rho A \oplus \pi B \qquad \Gamma', a \overset{\rho}{:} A \vdash T_1 \overset{\sigma}{:} C[\mathbf{inl}\, a/x] \qquad 0\Gamma = 0\Gamma'}{\Gamma' + \rho\Gamma \vdash \mathbf{case}(\mathbf{inl}(S_1), T_1, T_2) \equiv T_1[S_1/a]}\ \oplus\text{-comp}$$

Figure 1: Rules for $\oplus$-type

We give the following semantics for the $\oplus$-type:
$|\rho A \oplus \pi B\ (\gamma)| := |A(\gamma)| \sqcup |B(\gamma)|$
$a \vDash_{\rho A \oplus \pi B\ (\gamma)} (i, x)$ iff $(\exists b. a = [!_\rho b, \ulcorner true \urcorner] \wedge b \vDash_{A(\gamma)} x \wedge i = 0) \vee$
$(\exists c. a = [!_\pi c, \ulcorner false \urcorner] \wedge c \vDash_{B(\gamma)} x \wedge i = 1)$

Claim. *The rules are sound when interpreted wrt to the given semantics for $\oplus$-types and realisability model.*

Proof. The underlying set-theoretic functions are immediate. For the realisers, let

- $a_{\mathbf{inl}} := \lambda^* x. [F_\rho \cdot\ !_\rho a_{s_1} \cdot x, \ulcorner true \urcorner]$

  if $a_{s_1} \cdot a_\gamma \vDash s_1$, *then* $a_{\mathbf{inl}} \cdot !_\rho a_\gamma \vDash \mathbf{inl} s_1$; $a_{\mathbf{inl}} \cdot !_\rho a_\gamma \rightsquigarrow [!_\rho (a_{s_1} \cdot a_\gamma), \ulcorner true \urcorner]$

- $a_{\mathbf{case}} := \begin{array}{l} \lambda^* x.\ \text{let } [a'_\gamma, a_\gamma] = x, \\ \qquad [a, b] = a_m \cdot a_\gamma \text{ in} \\ \qquad E(b, a_{T_1}, a_{T_2}) \cdot [a'_\gamma, a] \end{array}$

  assuming $a_m \cdot a_\gamma \vDash M$, $a_{T_1} \cdot [a'_\gamma, !_\rho a_a] \vDash T_1$, $a_{T_2} \cdot [a'_\gamma, !_\pi a_b] \vDash T_2$, *then we want to find* $a_{\mathbf{case}}$, *s.t.* $a_{\mathbf{case}} \cdot [a'_\gamma, a_\gamma] \vDash \mathbf{case}(M, T_1, T_2)$.

if $a_m \cdot a_\gamma = [!_\rho a_a, \ulcorner true \urcorner]$, *then* $a_{\mathbf{case}} \cdot [a_\gamma, a'_\gamma] \rightsquigarrow E(\ulcorner true \urcorner, a_{T_1}, a_{T_2}) \cdot [a'_\gamma, !_\rho a_a] \rightsquigarrow a_{T_1} \cdot [a'_\gamma, !_\rho a_a]$

if $a_m \cdot a_\gamma = [!_\pi a_b, \ulcorner false \urcorner]$, *then* ...

$\square$

Claim. *There is a bijection:*

$$RTm(\Gamma, \Pi(x \overset{\tau}{:} \rho A \oplus \pi B)\, C) \cong RTm(\Gamma, \Pi(y \overset{\tau\rho}{:} A)\, C[\mathbf{inl} y/x]) \times RTm(\Gamma, \Pi(z \overset{\tau\pi}{:} B)\, C[\mathbf{inl} z/x]))$$

(natural in Γ).

Proof. Given a term $\Gamma \vdash f \overset{1}{:} (x \overset{\tau}{:} \rho A \oplus \pi B) \to C$, we can derive another term $\Gamma \vdash f^l \overset{1}{:} (y \overset{\tau\rho}{:} A) \to C[\mathbf{inl} y/x]$:

$$\dfrac{\dfrac{\dfrac{\Gamma \vdash f : (x \overset{\tau}{:} \rho A \oplus \pi B) \to C}{\Gamma, y \overset{0}{:} A \vdash f \overset{1}{:} (x \overset{\tau}{:} \rho A \oplus \pi B) \to C}\,Weak \quad \dfrac{\dfrac{\vdash 0\Gamma, y \overset{1}{:} A}{0\Gamma, y \overset{1}{:} \vdash y \overset{1}{:} A}\,var}{0\Gamma, y \overset{\rho}{:} A \vdash \mathbf{inl} y \overset{1}{:} \rho A \oplus \pi B}\,inl}{\Gamma, y \overset{\tau\rho}{:} A \vdash f(\mathbf{inl} y) \overset{1}{:} C[\mathbf{inl} y/x]}\,App}{\Gamma \vdash \lambda y \overset{\tau\rho}{:} A.\, f(\mathbf{inl} y) : (y \overset{\tau\rho}{:} A) \to C[\mathbf{inl} y/x]}\,Lam$$

Analogously, we can obtain $\Gamma \vdash f^r \overset{1}{:} (z \overset{\tau\pi}{:} B) \to C[\mathbf{inr} y/x]$.
Now suppose we have terms $\Gamma \vdash l \overset{1}{:} (y \overset{\tau\rho}{:} A) \to C[\mathbf{inl} y/x]$ and $\Gamma \vdash r \overset{1}{:} (z \overset{\tau\pi}{:} B) \to C[\mathbf{inr} z/x]$.
Using the isomorphism $\Lambda^{\mathcal{L}}$, we get judgements $\Gamma, y \overset{\tau\rho}{:} A \vdash l^* : C[\mathbf{inl} y/x]$ and $\Gamma, z \overset{\tau\pi}{:} B \vdash r^* : C[\mathbf{inr} z/x]$. $\square$

Now we can focus on the relational part

$$\begin{aligned}
(\rho A \oplus \pi B)(\gamma)_R &:= A(\gamma)_R \sqcup B(\gamma)_R \\
(\rho A \oplus \pi B)(\gamma)_{refl} &:= A(\gamma)_{refl} \sqcup B(\gamma)_{refl} \\
(\rho A \oplus \pi B)(\gamma)_\sigma &:= A(\gamma)_\sigma \sqcup B(\gamma)_\sigma, \quad \sigma \in \{\text{src}, \text{tgt}\}
\end{aligned}$$

Suppose A, B are discrete. Then $\rho A \oplus \pi B$ is also discrete. (Since coproducts preserve isos).