

# Outline

## 1 Quantitative polynomial functors

### 1.1 Category of closed types and linear functions

Let  $\mathcal{C}$  be the category of closed types and linear functions  $f : (x \vdash^1 X) \rightarrow Y$  for which derivations in QTT exist. Composition of morphisms  $\Gamma \vdash f \vdash^\sigma (x \vdash^1 X) \rightarrow Y$  and  $\Gamma \vdash g \vdash^\sigma (y \vdash^1 Y) \rightarrow Z$  is given by ordinary function composition  $\Gamma \vdash \lambda x \vdash^1 X. g(f(x)) \vdash^\sigma (x \vdash^1 X) \rightarrow Z$ .

The linearity restriction on the morphisms does not lead to loss of expressiveness - a function with arbitrary resource annotations can be represented as linear one via the exponential type  $!_\rho A := (a \vdash^\rho A) \otimes I$ .

For an arbitrary  $\Gamma \vdash f \vdash^1 X \xrightarrow{\rho} Y$ , the embedding is given by:

$$f \mapsto \lambda z \vdash^1 (a \vdash^\rho X) \otimes I. \text{ let } (x, i) = z \text{ in} \\ \text{ let } * = i \text{ in } f(x) \vdash^1 ((a \vdash^\rho X) \otimes I) \xrightarrow{1} Y$$

Suppose  $\mathcal{D}$  stands for the derivation of  $\Gamma \vdash f \vdash^1 X \xrightarrow{\rho} Y$  and  $\mathcal{D}_x$ ,  $\mathcal{D}_z$  and  $\mathcal{D}_i$  for the derivations of  $0\Gamma, x \vdash^1 X \vdash x \vdash^1 X$ ,  $0\Gamma, z \vdash^1 (a \vdash^\rho X) \otimes I \vdash z \vdash^1 (a \vdash^\rho X) \otimes I$  and  $0\Gamma, i \vdash^1 I \vdash i \vdash^1 I$  obtained by the applications of the **Var** rule.

$$\frac{\frac{\frac{\mathcal{D}_x \quad \mathcal{D}}{\mathcal{D}_i \quad \Gamma, x \vdash^\rho X \vdash f(x) \vdash^1 Y}}{\mathcal{D}_z \quad \Gamma, x \vdash^\rho X, i \vdash^1 I \vdash \text{let } * = i \text{ in } f(x) \vdash^1 Y}}{\Gamma, z \vdash^1 (a \vdash^\rho X) \otimes I \vdash \text{let } (x, z) = z \text{ in let } * = i \text{ in } f(x) \vdash^1 Y}}{\Gamma \vdash \lambda z \vdash^1 (a \vdash^\rho X) \otimes I. \text{let } (x, z) = z \text{ in let } * = i \text{ in } f(x) \vdash^1 (a \vdash^\rho X) \otimes I \xrightarrow{1} Y}$$

### 1.2 Category of closed types and linear functions in a nonempty context

Let  $\Delta$  be an “underlying context”, i.e. a context of the form  $\Delta = 0\Gamma_0$  for some  $\Gamma_0$ . There is a category  $\mathcal{C}_\Delta$  where the objects are types  $X$  such that  $\Delta \vdash X$  type, and a morphism  $X$  to  $Y$  consists of pair  $(\Gamma, f)$ , where  $\Gamma$  is a context such that  $0\Gamma = \Delta$ , and  $\Gamma \vdash f : X \xrightarrow{1} Y$ .

- The identity morphism is given by  $(\Delta, \lambda x. x)$ ;
- Composition of  $(\Gamma_2, g)$  and  $(\Gamma_1, f)$  is given by  $(\Gamma_1 + \Gamma_2, \lambda x. f(g(x)))$ .

This is a category since  $0\Delta = \Delta$ , and context addition is associative, and with  $\Gamma + \Delta = \Delta + \Gamma = \Gamma$ . Note that the category of  $\mathcal{C}$  closed types from Section 1.1 is a special case  $\mathcal{C} = \mathcal{C}_\diamond$  where  $\Delta = \diamond$ , because the only  $\Gamma$  with  $0\Gamma = \diamond$  is  $\Gamma = \diamond$ .

**Lemma 1.** *Fix  $\Delta$  as above, and let  $\Delta \vdash A$  type and  $\Delta, x : A \vdash B[x]$  type. The operation  $F(X) = (a : A) \otimes (Ba \xrightarrow{1} X)$  is a functor  $\mathcal{C}_\Delta \rightarrow \mathcal{C}_\Delta$ .*

*Proof.* If  $\Delta \vdash X$  type then  $\Delta \vdash F(X)$  type. On morphisms, we can define

$$F(\Gamma, f) = (\Gamma, \lambda z. \text{let } (a, h) = z \text{ in } (a, \lambda b. f(h(b))))$$

as the following derivation shows:

$$\frac{\frac{\frac{}{0\Gamma, z : F(X), a : A, h : B[a] \xrightarrow{1} X \vdash a : A} \quad \dot{\mathcal{D}}}{\Gamma, z : F(X), a : A, h : B[a] \xrightarrow{1} X \vdash (a, \lambda b. f(h(b))) : F(Y)} \quad \frac{}{0\Gamma, z : F(X) \vdash z : F(X)}}{\Gamma, z : F(X) \vdash \text{let } (a, h) = z \text{ in } (a, \lambda b. f(h(b))) : F(Y)} \quad \frac{}{\Gamma \vdash \lambda z. \text{let } (a, h) = z \text{ in } (a, \lambda b. f(h(b))) : F(X) \xrightarrow{1} F(Y)}$$

where  $\mathcal{D}$  is the derivation

$$\frac{\frac{\frac{}{\Gamma, a : A, h : B[a] \xrightarrow{1} X, b : B[a] \vdash f : X \xrightarrow{1} Y} \quad \dot{\mathcal{D}}'}{\Gamma, a : A, h : B[a] \xrightarrow{1} X, b : B[a] \vdash f(h(b)) : Y}}{\Gamma, a : A, h : B[a] \xrightarrow{1} X \vdash \lambda b. f(h(b)) : B[a] \xrightarrow{1} Y}$$

weakened by  $z : F(X)$ , where again  $\mathcal{D}'$  is the derivation

$$\frac{\frac{}{0\Gamma, a : A, h : B[a] \xrightarrow{1} X, b : B[a] \vdash h : B[a] \xrightarrow{1} X} \quad \frac{}{0\Gamma, a : A, h : B[a] \xrightarrow{0} X, b : B[a] \vdash b : B[a]}}{0\Gamma, a : A, h : B[a] \xrightarrow{1} X, b : B[a] \vdash h(b) : X}$$

similarly weakened. This is functorial by the  $\eta$ -rules for functions and pairs.  $\square$

### 1.3 Internal representation

Let  $F_0 : \text{Obj}(\mathcal{C}) \rightarrow \text{Obj}(\mathcal{C})$  be the function mapping a type  $X$  to the type  $(a : A) \otimes (Ba \xrightarrow{1} X)$ .

**Claim.**  $F_0$  can be extended to an endofunctor  $F : \mathcal{C} \rightarrow \mathcal{C}$ .

Suppose  $\Gamma \vdash f : X \xrightarrow{1} Y$  is some morphism with a derivation  $\mathcal{D}_f$ , then  $Ff : (a : A) \otimes (Ba \xrightarrow{1} X) \xrightarrow{1} (a : A) \otimes (Ba \xrightarrow{1} Y)$  is defined by precomposition with  $f$  on the second component.

By applying **Var** rule, we get the following:

- a derivation  $\mathcal{D}_a$  of  $0\Gamma, a : A, g : Ba \xrightarrow{1} X \vdash a : A$
- a derivation  $\mathcal{D}_g$  of  $0\Gamma, a : A, g : Ba \rightarrow X \vdash g : Ba \xrightarrow{1} X$
- a derivation  $\mathcal{D}_b$  of  $0\Gamma, a : A, g : Ba \rightarrow X, b : Ba \vdash b : Ba$
- a derivation  $\mathcal{D}_z$  of  $0\Gamma, z : (a : A) \otimes (Ba \xrightarrow{1} X) \vdash z : (a : A) \otimes (Ba \xrightarrow{1} X)$

Form an intermediate derivation  $\mathcal{D}$  by :

$$\frac{\frac{\frac{\frac{D_g \quad D_b}{0\Gamma, a : A, g : Ba \rightarrow X, b : Ba \vdash g(b) : X} \text{App}}{0\Gamma, a : A, g : Ba \rightarrow X, b : Ba, f : X \xrightarrow{1} Y \vdash f(g(b)) : Y} \text{App}}{0\Gamma, a : A, g : Ba \rightarrow X, f : X \xrightarrow{1} Y \vdash \lambda b.f(g(b)) : Ba \xrightarrow{1} Y} \text{Abs}}{\Gamma, a : A, g : Ba \xrightarrow{1} X \vdash (a, \lambda b : Ba.f(g(b))) : F(Y)} \otimes - \text{intro}$$

and use it to get a final one for  $Ff$ :

$$\frac{\frac{\frac{D_z \quad \mathcal{D}}{\Gamma, z : F(X)) \vdash \text{let } (x, u) = a \text{ in } (x, \lambda b : Ba.f(u(b))) : F(Y)} \oplus - E}{\Gamma \vdash \lambda z.\text{let } (x, u) = a \text{ in } (x, \lambda b : Ba.f(u(b))) : F(X) \xrightarrow{1} F(Y)} \text{Lam}$$

**Definition 2.** We will call any functor isomorphic to  $F$  a quantitative polynomial functor.

## 1.4 External representation (using adjoints)

## 1.5 Generalising to non-empty contexts

## 1.6 Properties of quantitative polynomial functors

# 2 Algebras for QPFs

Recall that an algebra for an endofunctor  $F : \mathcal{C} \rightarrow \mathcal{C}$  is a pair  $(X, a)$ , where  $X$  is an object of  $\mathcal{C}$  and  $a$  - a morphism  $F(X) \rightarrow X$ . A morphism between  $F$ -algebras is a map  $f : X \rightarrow Y$ ,

making the square commute:

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ \downarrow a & & \downarrow b \\ X & \xrightarrow{f} & Y \end{array}$$

## 2.1 $\mathbb{N}$

Fix  $A := \mathbf{Bool}$  and  $B$ , such that  $B(false) := \emptyset$  and  $B(true) := I$  where  $x \vdash^1 A \vdash B$  type.

**Sketch:**

Observe that a function  $\Gamma \vdash f \vdash^1 F_{A,B}(X) \xrightarrow{1} X$  is equivalent to functions  $\Gamma \vdash f_l \vdash^1 (\emptyset \xrightarrow{1} X) \rightarrow X$  and  $\Gamma \vdash f_r \vdash^1 (I \xrightarrow{1} X) \rightarrow X$ .

Assuming that the type  $\emptyset \xrightarrow{1} X$  is a (sub?)singleton (or  $\emptyset \xrightarrow{1} X \cong I$ , but maybe too strong),  $f_l$  just encodes a choice of an element of  $X$ . Similarly,  $I \xrightarrow{1} X$  also encodes the same data, that is  $I \xrightarrow{1} X \cong X$  (provable in QTT?).

Essentially, an algebra for  $F_{A,B}$  is a diagram  $I \rightarrow X \rightarrow X$ .

Now, we prove the following propositions:

- (i) Suppose  $\Gamma \vdash \mathbf{Nat}$  type, then there exists an initial algebra for  $F_{A,B}(\mathbf{Nat})$
- (ii) Suppose  $\Gamma \vdash X$  type, such that there exists an initial algebra for  $F_{A,B}(X)$ , then  $F_{A,B}(X) \cong \mathbf{Nat}$ .

Before embarking on proving that, let's examine a simpler construction:

**Interlude : Pseudobooleans as a constant QPF** Let  $A := \mathbf{Bool}$ ,  $\Gamma, a \vdash^1 A \vdash B$  type,  $B(a) := \emptyset$ . An algebra for this  $F_{A,B}(X)$  is a diagram  $X \leftarrow \{*\} \rightarrow X$ .

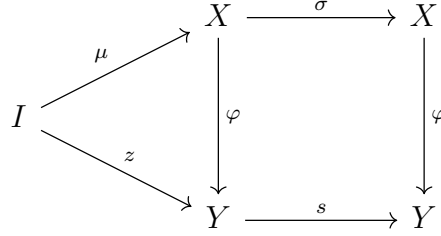
Assume  $\Gamma \vdash \mathbf{Bool}$  type. Then define  $t(i) := true$  and  $f(i) := false$  using the unit elimination. Let  $(F_{A,B}(Y), m, n)$  be another algebra. Define  $\varphi(x) := Elim(m(*), n(*), k)$ , where  $k : F_{A,B}(\mathbf{Bool}) \xrightarrow{1} \mathbf{Bool}$  is the structure map. The squares commute by the computation rules for  $\mathbf{Bool}$ .

$$\begin{array}{ccccc} \mathbf{Bool} & \xleftarrow{t} & \{*\} & \xrightarrow{f} & \mathbf{Bool} \\ \downarrow \varphi & & \parallel & & \downarrow \varphi \\ Y & \xleftarrow{m} & \{*\} & \xrightarrow{n} & Y \end{array}$$

Conversely, assume that there is an initial algebra  $(X, k : F_{A,B}X \rightarrow X)$ . Designate some elements  $t, f \in X$ , such that  $t \neq f$ . For any other algebra represented diagrammatically  $(Y, m : I \xrightarrow{1} Y, n : I \xrightarrow{1} Y)$ , we have that  $\varphi(t) = m$  and  $\varphi(f) = n$  by initiality. Define  $Elim(m, n, x) := \varphi(x)$ .

Back to  $\mathbb{N}$ , let **Nat** be defined as:

$$\begin{array}{ccc} \frac{0\Gamma \vdash A}{0\Gamma \vdash \mathbf{Nat}} \text{Nat} & \frac{0\Gamma \vdash}{0\Gamma \vdash 0 : \mathbf{Nat}} & \frac{\Gamma \vdash N : \mathbf{Nat}}{\Gamma \vdash suc(N) : \mathbf{Nat}} \text{suc} \\ \dots & & \end{array}$$



## 2.2 Lists

## 2.3 Trees

## 2.4 Induction principle

# 3 Rules for W-types in QTT

# 4 Parametricity and W-types

## 5 Appendix

### 5.1 (Stand-alone) Sum types

$$\begin{array}{c}
\frac{0\Gamma \vdash A \quad 0\Gamma \vdash B}{0\Gamma \vdash \rho A \oplus \pi B} \oplus\text{-type} \quad \frac{\Gamma \vdash S_1 \overset{\sigma}{:} A}{\rho\Gamma \vdash \mathbf{inl} S_1 \overset{\sigma}{:} \rho A \oplus \pi B} \text{inl} \quad \frac{\Gamma \vdash S_2 \overset{\sigma}{:} B}{\pi\Gamma \vdash \mathbf{inr} S_2 \overset{\sigma}{:} \rho A \oplus \pi B} \text{inr} \\
\\
\frac{\Gamma \vdash M \overset{\sigma}{:} \rho A \oplus \pi B \quad \Gamma', a \overset{\rho}{:} A \vdash T_1 \overset{\sigma}{:} C[\mathbf{inl} a/x] \quad \Gamma', b \overset{\pi}{:} B \vdash T_2 \overset{\sigma}{:} C[\mathbf{inr} b/x] \quad 0\Gamma = 0\Gamma'}{\Gamma' + \Gamma \vdash \mathbf{case}(M, T_1, T_2) \overset{\sigma}{:} C[M/x]} \oplus\text{-elim} \\
\\
\frac{\Gamma \vdash S_1 \overset{\sigma}{:} A \quad \Gamma \vdash M \overset{\sigma}{:} \rho A \oplus \pi B \quad \Gamma', a \overset{\rho}{:} A \vdash T_1 \overset{\sigma}{:} C[\mathbf{inl} a/x] \quad 0\Gamma = 0\Gamma'}{\Gamma' + \rho\Gamma \vdash \mathbf{case}(\mathbf{inl}(S_1), T_1, T_2) \equiv T_1[S_1/a]} \oplus\text{-comp}
\end{array}$$

Figure 1: Rules for  $\oplus$ -type

We give the following semantics for the  $\oplus$ -type:

$$\begin{aligned}
|\rho A \oplus \pi B(\gamma)| &:= |A(\gamma)| \sqcup |B(\gamma)| \\
a \models_{\rho A \oplus \pi B(\gamma)} (i, x) &\text{ iff } (\exists b. a = [!_{\rho} b, \ulcorner \text{true} \urcorner] \wedge b \models_{A(\gamma)} x \wedge i = 0) \vee \\
&\quad (\exists c. a = [!_{\pi} c, \ulcorner \text{false} \urcorner] \wedge c \models_{B(\gamma)} x \wedge i = 1)
\end{aligned}$$

**Claim.** *The rules are sound when interpreted wrt to the given semantics for  $\oplus$ -types and realisability model.*

*Proof.* The underlying set-theoretic functions are immediate. For the realisers, let

- $a_{\mathbf{inl}} := \lambda^* x. [F_{\rho} \cdot !_\rho a_{s_1} \cdot x, \ulcorner \text{true} \urcorner]$

$$\text{if } a_{s_1} \cdot a_{\gamma} \models s_1, \text{ then } a_{\mathbf{inl}} \cdot !_\rho a_{\gamma} \models \mathbf{inl} s_1; a_{\mathbf{inl}} \cdot !_\rho a_{\gamma} \rightsquigarrow [!_{\rho}(a_{s_1} \cdot a_{\gamma}), \ulcorner \text{true} \urcorner]$$

$$\lambda^* x. \text{ let } [a'_{\gamma}, a_{\gamma}] = x,$$

- $a_{\mathbf{case}} := \begin{array}{l} [a, b] = a_m \cdot a_{\gamma} \text{ in} \\ E(b, a_{T_1}, a_{T_2}) \cdot [a'_{\gamma}, a] \end{array}$

assuming  $a_m \cdot a_{\gamma} \models M$ ,  $a_{T_1} \cdot [a'_{\gamma}, !_\rho a_a] \models T_1$ ,  $a_{T_2} \cdot [a'_{\gamma}, !_\pi a_b] \models T_2$ , then we want to find  $a_{\mathbf{case}}$ , s.t.  $a_{\mathbf{case}} \cdot [a'_{\gamma}, a_{\gamma}] \models \mathbf{case}(M, T_1, T_2)$ .

if  $a_m \cdot a_\gamma = [!_\rho a_a, \ulcorner true \urcorner]$ , then  $a_{\text{case}} \cdot [a_\gamma, a'_\gamma] \rightsquigarrow E(\ulcorner true \urcorner, a_{T_1}, a_{T_2}) \cdot [a'_\gamma, !_\rho a_a] \rightsquigarrow$   
 $a_{T_1} \cdot [a'_\gamma, !_\rho a_a]$   
 if  $a_m \cdot a_\gamma = [!_\pi a_b, \ulcorner false \urcorner]$ , then ...

□

**Claim.** *There is a bijection:*

$$RTm(\Gamma, \Pi(x \overset{\tau}{:} \rho A \oplus \pi B) C) \cong RTm(\Gamma, \Pi(y \overset{\tau\rho}{:} A) C[\mathbf{inl}y/x]) \times RTm(\Gamma, \Pi(z \overset{\tau\pi}{:} B) C[\mathbf{inl}z/x])$$

(natural in  $\Gamma$ ).

*Proof.* Given a term  $\Gamma \vdash f \overset{1}{:} (x \overset{\tau}{:} \rho A \oplus \pi B) \rightarrow C$ , we can derive another term  $\Gamma \vdash f^l \overset{1}{:} (y \overset{\tau\rho}{:} A) \rightarrow C[\mathbf{inl}y/x]$ :

$$\frac{\frac{\Gamma \vdash f : (x \overset{\tau}{:} \rho A \oplus \pi B) \rightarrow C}{\Gamma, y \overset{0}{:} A \vdash f \overset{1}{:} (x \overset{\tau}{:} \rho A \oplus \pi B) \rightarrow C} \text{Weak} \quad \frac{\frac{\vdash 0\Gamma, y \overset{1}{:} A}{0\Gamma, y \overset{1}{:} \vdash y \overset{1}{:} A} \text{var}}{0\Gamma, y \overset{\rho}{:} A \vdash \mathbf{inl}y \overset{1}{:} \rho A \oplus \pi B} \text{inl}}{\Gamma, y \overset{\tau\rho}{:} A \vdash f(\mathbf{inl}y) \overset{1}{:} C[\mathbf{inl}y/x]} \text{App}$$

$$\frac{\Gamma, y \overset{\tau\rho}{:} A \vdash f(\mathbf{inl}y) \overset{1}{:} C[\mathbf{inl}y/x]}{\Gamma \vdash \lambda y \overset{\tau\rho}{:} A. f(\mathbf{inl}y) : (y \overset{\tau\rho}{:} A) \rightarrow C[\mathbf{inl}y/x]} \text{Lam}$$

Analogously, we can obtain  $\Gamma \vdash f^r \overset{1}{:} (z \overset{\tau\pi}{:} B) \rightarrow C[\mathbf{inr}y/x]$ .

Now suppose we have terms  $\Gamma \vdash l \overset{1}{:} (y \overset{\tau\rho}{:} A) \rightarrow C[\mathbf{inl}y/x]$  and  $\Gamma \vdash r \overset{1}{:} (z \overset{\tau\pi}{:} B) \rightarrow C[\mathbf{inr}z/x]$ .

Using the isomorphism  $\Lambda^{\mathcal{L}}$ , we get judgements  $\Gamma, y \overset{\tau\rho}{:} A \vdash l^* : C[\mathbf{inl}y/x]$  and

$\Gamma, z \overset{\tau\pi}{:} B \vdash r^* : C[\mathbf{inr}z/x]$ .

□