Relational realisability model for QTT

Our aim is to build a concrete realisability model for QTT which allows for meta-reasoning with results derived from parametricity. To that end, a construction of a quantitative category with families (QCwF, see [1]) is presented. We follow the relational approach to types introduced by Reynolds for typed lambda calculus [4] and later refined for dependent types theories [2].

Once and for all fix a usage semiring R and an R-linear combinatory algebra $\mathscr{A}$[1].

0 Preliminaries

We recall some basic structures used in constructing the model.

Definition 1 (Assemblies†)**.** An assembly† Γ is a pair $(|\Gamma|, e)$ where $|\Gamma|$ is a carrier set and e is a realisability function $|\Gamma| \to \mathcal{P}(\mathscr{A})$.

Given some $\gamma \in |\Gamma|$, $e(\gamma)$ is interpreted as the set of witnesses for the existence of γ. To emphasize on that aspect, we write $a \vDash_\Gamma \gamma$ to denote $a \in e(\gamma)$. Moreover, let $\lfloor \Gamma \rfloor$ stand for the set of realisable elements - $\{\gamma \in |\Gamma| : e_\Gamma(\gamma) \neq \emptyset\} \subseteq |\Gamma|$.

A morphism between two assemblies† Γ **and** Δ is a function $f : |\Gamma| \to |\Delta|$ that is realisable - there exists a realiser $a_f \in \mathscr{A}$ that tracks the function f in the following sense:

$$\text{for every } \gamma \text{ in } |\Gamma| \text{ and } a_\gamma \text{ in } \mathscr{A},\ a_\gamma \vDash_\Gamma \gamma \implies a_f.a_\gamma \vDash_\Delta f(\gamma) \text{ holds.}$$

Note that multiple realisers for the same function f do not induce multiple morphisms.
Using these notions we can construct a category $\mathcal{A}sm^\dagger(\mathscr{A})$.

Definition 2 (Reflexive graph)**.** A reflexive graph (r.g.) G is a tuple $(G_O, G_R, G_{refl}, G_{src}, G_{tgt})$, where G_O and G_R are sets, $G_{src} : G_R \to G_O$, $G_{tgt} : G_R \to G_O$ and $G_{refl} : G_O \to G_R$ are functions, s.t. the identities hold:

$$G_{src} \circ G_{refl} = id_{G_O} = G_{tgt} \circ G_{refl}$$

G_O and G_R stand for a set of objects and a set of relations, respectively. In general, reflexive graphs are less structured than categories as relations need not compose.

A morphism between reflexive graphs G **and** H is a pair of functions $(f_o : G_O \to H_O, f_r : G_R \to H_R)$, s.t. all of the depicted squares commute:

[1] In case some non-trivial properties of $\mathscr{A}$ are required, we will assume that $\mathscr{A}$ is a graph model (see [3])

$$
\begin{tikzcd}
G_O \arrow[r, "f_o"] \arrow[d, "G_{refl}"] & H_O \arrow[d, "H_{refl}"] \\
G_R \arrow[r, "f_r"] \arrow[u, bend left, "G_{src}"] \arrow[u, bend right, "G_{tgt}"'] & H_R \arrow[u, bend left, "H_{src}"] \arrow[u, bend right, "H_{tgt}"']
\end{tikzcd}
$$

Reflexive graphs equipped with r.g. morphisms form **a category** $\mathcal{R}Gph$. The terminal object $\mathbf{1}_{\mathcal{R}Gph}$ is $(\{\star\}, \{\star\}, id, id, id)$.

Reflexive graphs provide enough structure for a dyadic interpretation of types in the spirit of [4].

1 Reflexive graphs with realisability information

One could easily generalize reflexive graphs by considering object and relation components from arbitrary categories instead of $\mathcal{S}et$. As our purpose is to build a relational model incorporating realisability information, we replace the set of objects with an assembly† and retain the $\mathcal{S}et$-based representation of relations.

Definition 3 (Reflexive graph with realisable objects)**.** A reflexive graph with realisable objects G is a tuple $(G_O, G_R, G_{refl}, G_{src}, G_{tgt})$, where $G_O \in Ob(\mathcal{A}sm^\dagger(\mathscr{A}))$, G_R is a set and the functions $G_{refl} : |G_O| \to G_R$, $G_{src} : G_R \to |G_O|$, $G_{tgt} : G_R \to |G_O|$ are such that the identities in Definition 2 are satisfied.

Accordingly, **a morphism between reflexive graphs with realisable objects,** G **and** H, is a pair $(f_O : G_O \to H_O, f_R : G_R \to H_R)$, s.t. (f_O, f_R) is an $\mathcal{R}Gph$ morphism between the reflexive graphs $(|G_O|, G_R, G_{refl}, G_{src}, G_{tgt})$ and $(|H_O|, H_R, H_{refl}, H_{src}, H_{tgt})$.

With these components, we obtain a category $\mathcal{O}RGph(\mathcal{A}sm^\dagger(\mathscr{A}))$. By considering reflexive graphs with realisable objects of shape $(X, |X|, id_X, id_X, id_X)$, we identify an isomorphic copy of $\mathcal{A}sm^\dagger(\mathscr{A})$ inside $\mathcal{O}RGph(\mathcal{A}sm^\dagger(\mathscr{A}))$.

A terminal object $\mathbf{1}_{\mathcal{O}RGph(\mathcal{A}sm^\dagger(\mathscr{A}))}$ in $\mathcal{O}RGph(\mathcal{A}sm^\dagger(\mathscr{A}))$ is a tuple $(\mathbf{1}_{\mathcal{A}sm^\dagger(\mathscr{A})}, \{*\}, id, id, id)$, where $\mathbf{1}_{\mathcal{A}sm^\dagger(\mathscr{A})}$ is the terminal assembly† $(\{\star\}, f)$, with f defined as $\star \mapsto \{I\}$.

Definition 4 (Family of reflexive graphs with realisable objects)**.** Let $\mathcal{C}$ be a category with a terminal object. Given a reflexive graph $\Gamma \in Ob(\mathcal{C})$, a family of reflexive graphs with realisable objects over Γ is a tuple $S := (S_O, S_R, S_{refl}, S_{src}, S_{tgt})$, where:

- $S_O : \Gamma_O \to \mathcal{A}sm^\dagger(\mathscr{A})$
- $S_R : \Gamma_R \to \mathcal{S}et$

- a Γ-indexed collection of functions $S_{refl} := \{f_\gamma : |S_O(\gamma)| \to S_R(\Gamma_{refl}(\gamma))\}_{\gamma \in \Gamma_O}$
- $S_{src} := \{f_\gamma : S_R(\gamma) \to |S_O(\Gamma_{src}(\gamma))|\}_{\gamma \in \Gamma_R}$
- $S_{tgt} := \{f_\gamma : S_R(\gamma) \to |S_O(\Gamma_{tgt}(\gamma))|\}_{\gamma \in \Gamma_R}$

such that

- each identity in the following collection is satisfied:

$$S_\sigma(\Gamma_{refl}(\gamma)) \circ S_{refl}(\gamma) = id \text{ for every } \gamma \in \Gamma_O, \sigma \in \{src, tgt\}$$

A morphism M between two families S and T of reflexive graphs with realisable objects over Γ is a pair (M_O, M_R) of Γ-indexed collections of morphisms:

- $M_O := \{f_\gamma : S_O(\gamma) \to T_O(\gamma)\}_{\gamma \in \Gamma_O}$
- $M_R := \{f_\gamma : S_R(\gamma) \to T_R(\gamma)\}_{\gamma \in \Gamma_R}$

s.t. the following identities are satisfied:

- $T_{refl}(M_O(\gamma)(s_o)) = M_R(\Gamma_{refl}(\gamma))(S_{refl}(\gamma)(s_o))$ for every $\gamma \in \Gamma_O$, $s_o \in |S_O(\gamma)|$
- $T_{src}(M_R(\gamma)(s_r)) = M_O(\Gamma_{src}(\gamma))(S_{src}(\gamma)(s_r))$ for every $\gamma \in \Gamma_R$, $s_r \in S_R(\gamma)$
- $T_{tgt}(M_R(\gamma)(s_r)) = M_O(\Gamma_{tgt}(\gamma))(S_{tgt}(\gamma)(s_r))$ for every $\gamma \in \Gamma_R$, $s_r \in S_R(\gamma)$

Families of reflexive graphs with realisable objects over Γ and their morphisms forms a category $\mathcal{F}am{-}\mathcal{ORG}(\mathcal{A}sm^\dagger(\mathscr{A}), \Gamma)$.

The terminal object $\mathbf{1}_{\mathcal{F}am-\mathcal{ORG}(\mathcal{A}sm^\dagger(\mathscr{A}),\mathbf{\Gamma})}$ is given by $(\lambda\gamma_r.\mathbf{1}_{\mathcal{A}sm^\dagger(\mathscr{A})}, \lambda\gamma_r.\{*\}, \mathbf{1_{refl}}, \mathbf{1_{tgt}}, \mathbf{1_{src}})$, where $\mathbf{1}_\sigma$ are the appropriate functions.

2 A CwF from families of reflexive graphs with realisable objects

As a first step toward obtaining a relationally parametric realisability model of QTT, we construct a concrete CwF using families of reflexive graphs with realisable objects.

Consider the category $\mathcal{R}Gph$ with terminal object $\mathbf{1}_{\mathcal{R}Gph}$.
Let $\Gamma, \Delta \in Ob(\mathcal{R}Gph)$, define:

- the collection of semantic types $Ty(\Gamma)$ as the collection of families of reflexive graph with realisable objects over Γ.
- given a type $S \in Ty(\Gamma)$, an element $M \in Tm(\Gamma, S)$ is a pair of functions
$(M_O : \forall\gamma \in \Gamma_O.|S_O(\gamma)|,\ M_R : \forall\gamma \in \Gamma_R.S_R(\gamma))$, s.t.

$$\begin{aligned}
\forall\gamma \in \Gamma_O.S_{refl}(M_O(\gamma)) &= M_R(\Gamma_{refl}(\gamma)) \\
\forall\gamma \in \Gamma_R.S_{src}(M_R(\gamma)) &= M_O(\Gamma_{src}(\gamma)) \\
\forall\gamma \in \Gamma_R.S_{tgt}(M_R(\gamma)) &= M_O(\Gamma_{tgt}(\gamma))
\end{aligned}$$

- given $f : \Gamma \to \Delta$, substitutions in types and terms is a precomposition with f on the object and relation components of types and terms respectively. Clearly, these operations are compatible with identity and composition in $\mathcal{R}Gph$.
- context extension: Suppose $S \in Ty(\Gamma)$, construct a r.g. $\Gamma.S$ as :

$$\begin{aligned}
(\Gamma.S)_O &= \{(\gamma, x) : \gamma \in \Gamma_O, x \in |S_O(\gamma)|\} \\
(\Gamma.S)_R &= \{(\gamma, r) : \gamma \in \Gamma_R, r \in S_R(\gamma)\} \\
(\Gamma.S)_{refl}(\gamma, x) &= (\Gamma_{refl}(\gamma), S_{refl}(\gamma)(x)) \\
(\Gamma.S)_\sigma(\gamma, r) &= (\Gamma_\sigma(\gamma), S_\sigma(\gamma)(r)), \quad \sigma \in \{src, tgt\}
\end{aligned}$$

Claim. *$Hom_{\mathcal{R}Gph}(\Delta, \Gamma.S) \cong \{(f, M) : f : \Delta \to \Gamma, M \in Tm(\Delta, S\{f\}\}$, natural in Δ.*

Upgrading to a QCwF

Recall the definition of a QCwF from [1]. Given a usage semiring R, a R-QCwF consists of:

1. A CwF $(\mathcal{C}, 1, Ty, Tm, -.-, \langle -.- \rangle)$
2. A category $\mathcal{L}$ with a faithful functor $U : \mathcal{L} \to \mathcal{C}$
3. A functor $(+) : \mathcal{L} \times_\mathcal{C} \mathcal{L} \to \mathcal{L}$, s.t $U(\Gamma_1 + \Gamma_2) = U(\Gamma_1) = U(\Gamma_2)$[2]. $\mathcal{L} \times_\mathcal{C} \mathcal{L}$ denotes the pullback $\mathcal{L} \xrightarrow{U} \mathcal{C} \xleftarrow{U} \mathcal{L}$.
   Additionally, there exists an object $\Diamond \in \mathcal{L}$, s.t. $U\Diamond = 1$.
4. A functor $\rho(-) : \mathcal{L} \to \mathcal{L}$ for each $\rho \in R$, s.t $U(\rho(-)) = U(-)$.
5. A collection $RTm(\Gamma, S)$ for each $\Gamma \in \mathcal{L}$ and $S \in Ty(U\Gamma)$, equipped with an injective function $U_{\Gamma,S} : RTm(\Gamma, S) \to Tm(U\Gamma, S)$.
   For an $\mathcal{L}$ morphisms $f : \Gamma \to \Delta$ and types $S \in Ty(U\Gamma)$, a function $-\{f\} : RTm(\Delta, S) \to RTm(\Gamma, S\{f\})$, s. t. $U(-\{f\}) = (U(-))\{Uf\}$.
6. Given $\Gamma \in \mathcal{L}$, $\rho \in R$ and $S \in Ty(U\Gamma)$, an object $\Gamma.\rho S$, s.t $U(\Gamma.\rho S) = U\Gamma.S$.
   Additionally, the following natural transformations exist:
   $emp_\pi : \Diamond \to \pi\Diamond$
   $emp_+ : \Diamond \to \Diamond + \Diamond$
   $ext_\pi : \pi\Gamma.(\pi\rho S) \to \pi(\Gamma.\rho S)$, s.t. $U(ext_\pi) = id$
   $ext_+ : (\Gamma_1 + \Gamma_2).(\rho_1 + \rho_2)S \to \Gamma_1.\rho_1 S + \Gamma_2.\rho_2 S$, s.t. $U(ext_+) = id$
7. Given $\Gamma \in \mathcal{L}$, $S \in Ty(U\Gamma)$, there exists :
   a morphism $p_{\Gamma.S} : \Gamma.0S \to \Gamma$, s.t. $U(p_{\Gamma.S}) = p_{U\Gamma.S}$
   an element $v_{\Gamma.S} \in RTm(0\Gamma.1S, S\{p_{U\Gamma.S}\})$, s.t. $U(v_{\Gamma.S}) = v_{U\Gamma.S}$
   a morphism $wk(f, \rho S') : \Gamma.\rho S'\{Uf\} \to \Delta.\rho S'$ for each $f : \Gamma \to \Delta$, $S' \in Ty(U\Gamma, \Delta)$ s.t. $U(wk(f, \rho S')) = wk(Uf, S')$
   let $\Gamma_1, \Gamma_2 \in \mathcal{L}$, s.t $U\Gamma_1 = U\Gamma_2$ and $M \in RTm(\Gamma_2, S)$. There is a morphism $\overline{\rho M}$: $\Gamma_1 + \rho\Gamma_2 \to \Gamma_1.\rho S$, s.t $U(\overline{\rho M}) = \overline{UM}$
   a morphism $\overline{M} : \Gamma \to \Gamma.0S$ for $M \in Tm(U\Gamma, S)$, s.t. $U(\overline{M}) = \overline{M}$.

[2] the second equality being trivially satisfied

Now to construct the concrete QCwF, consider the CwF from previous section as the underlying category.

Let $\mathcal{L} := \mathcal{ORGph}(\mathcal{A}sm^{\dagger}(\mathscr{A}))$, the category of reflexive graphs with realisable objects
For the addition structure, let Γ', Γ'' be reflexive graphs with realisable objects, s.t $|\Gamma'_O| = |\Gamma''_O|$ and $\Gamma'_R = \Gamma''_R$

- $\Gamma_O := (|\Gamma'_O|, \vDash_\Gamma)$ with $a \vDash_\Gamma \gamma$ iff there exist $x, y \in \mathscr{A}$, s.t. $a = [x, y]$ and $x \vDash_{\Gamma'} \gamma$ and $y \vDash_{\Gamma''} \gamma$.
- $\Gamma_R := \Gamma'_R (= \Gamma''_R)$.
- $\Gamma_\sigma := \Gamma'_\sigma$, where $\sigma \in \{src, tgt, refl\}$.

Define $\Diamond$ as the terminal object $\mathbf{1}_{\mathcal{ORGph}(\mathcal{A}sm^{\dagger}(\mathscr{A}))}$.

Consider the scaling structure and let $\Gamma := \rho(\Gamma')$:

- $\Gamma_\sigma = (|\Gamma'_\sigma|, \vDash_{\Gamma_\sigma})$ with $a \vDash_{\Gamma_\sigma} \gamma$ iff there is $x \in \mathscr{A}$, s.t $a =!_\rho x$ and $x \vDash_{\Gamma'_\sigma} \gamma$ for $\sigma \in \{O, R\}$
- again, scaling leaves unmodified Γ_σ for $\sigma \in \{src, tgt, refl\}$.

Let $RTm(\Gamma, S)$ be the collection of morphisms from the terminal family $\mathbf{1}_{\mathcal{F}am-\mathcal{ORG}(\mathcal{A}sm^{\dagger}(\mathscr{A}),\mathbf{\Gamma})}$ to S. Spelling this out and simplifying it, an element of $RTm(\Gamma, S)$ is a tuple of functions $(M_O : \forall \gamma_o \in \Gamma_O.|S_O(\gamma_o)|, M_R : \forall \gamma_r \in \Gamma_R.S_R(\gamma_r))$, s.t. the conditions from Definition 4 are satisfied, namely

$$\forall \gamma_o \in \Gamma_O.S_{refl}(M_O(\gamma_o)) = M_R(\Gamma_{refl}(\gamma_o))$$
$$\forall \gamma_r \in \Gamma_R.S_{src}(M_R(\gamma_r)) = M_O(\Gamma_{src}(\gamma_r))$$
$$\forall \gamma_r \in \Gamma_R.S_{tgt}(M_R(\gamma_r)) = M_O(\Gamma_{tgt}(\gamma_r))$$

such that M_O is tracked - $\exists a_M \in \mathscr{A}.\forall a_\gamma \in \mathscr{A}, \gamma \in \Gamma_O.\, a_\gamma \vDash_{\Gamma_O} \gamma \implies a_m \cdot a_\gamma \vDash_{S(\gamma)} M_O(\gamma)$.
$U_{\Gamma,S}$ is the just identity function.
Substitution in terms is given by precomposition with $f : \Gamma \to \Delta$, let $-\{f_O\} := \lambda M_O.\forall \gamma \in \Gamma.M_O(f(\gamma))$ and similarly, $-\{f_R\} := \lambda M_R.\forall \gamma \in \Gamma.M_R(f(\gamma))$. The functor U interacts nicely with the so-defined $-\{f\}$ as essentially the substitution in terms in the underlying CwF is defined in the same way.

Resourced context extension is given by $\Gamma.\rho S := (\Gamma'_O, \Gamma'_R, \Gamma'_\sigma)$, where

- $|\Gamma'_O| := \{(\gamma, x) : \gamma \in |\Gamma_O|, x \in S_O(\gamma)\}$
  $a \vDash_{\Gamma.\rho S} (\gamma, x)$ iff there exists $b, c \in \mathscr{A}$, s.t $a = [b, !_\rho c]$, $b \vDash_\Gamma \gamma$ and $c \vDash_{S(\gamma)} \pi_1(\widecheck{(\gamma, x)})$, where $\widecheck{(-)} : \Gamma.S \to U(\Gamma.S)$, $\widecheck{(-)} := id$ as the set-theoretic part of the extensions in the CwF and $\mathcal{L}$ is the same by definition.

- $\Gamma'_R := \{(\gamma, r) : \gamma \in \Gamma_R, r \in S_R(\gamma)\}$
- Each Γ'_σ is defined pointwise.

The natural transformation $emp_\pi : \Diamond \to \pi\Diamond$ is given by the identity functions on both the object and relational part. It is realised by $K.!_\rho I$.
We list the realisers for the remaining transformations:

- emp_+ - $K.[I, I]$,
- ext_π - $\lambda^* q.let\ [x, y] = q\ in\ F_\pi.(F_\pi.(!_\pi \lambda^* stu.ust).x).\delta_{\pi\rho} y$
- ext_+ - $\lambda^* q.let\ [[x, y], z] = q\ in\ W_{\pi\rho}.(\lambda^* ab.[[x, a], [y, b]]).z$

The underlying function part of the above-defined natural transformations is given by identity, hence naturality follows trivially.

Finally, we construct the morphisms, listed in Item 7:

- $p_{\Gamma.S} : \Gamma.0S \to \Gamma$ is the first projection of $(\Gamma.0S)_\sigma = \{(\gamma, s) : \gamma \in \Gamma_\sigma.s \in S(\gamma)\}$, $(\sigma \in \{_{O}, _{R}\})$ and is realized by $\lambda^* t.(t.K)$.
  The equality $U(p_{\Gamma.S}) = p_{U\Gamma.S}$ holds trivially due to the identical structure of context extension in the underlying CwF and $\mathcal{L}$.

- define $v_{\Gamma.S} \in RTm(0\Gamma.1S, S\{p_{U\Gamma.S}\})$ as the second projection. $v_{\Gamma.S}$ is realized by $\lambda^* t.\ B.t.K.D$.

- Let a^σ_f realize f_σ, then $wk(f, \rho S') := \lambda(\gamma, s).(f(\gamma), s)$ is realized by $\lambda^* q.let\ [x, y] = q\ in\ [a^\sigma_f.x, y]$

- given a $M_\sigma \in RTm(\Gamma, S)$, $M_\sigma : \forall\gamma \in U(\Gamma_\sigma).S_\sigma(\gamma)$ with realizers a^σ_m, let $\overline{\rho M}_\sigma := \lambda\gamma.(\gamma, M_\sigma(\gamma))$ realized by $\lambda^* q.let\ [x, y] = q\ in\ [x, F_\rho.(!_\rho a^\sigma_m).y]$

- given a $M_\sigma \in Tm(U\Gamma, S) = M_\sigma : \forall\gamma \in U\Gamma_\sigma.S_\sigma(\gamma)$, let $\overline{M}_\sigma := \lambda\gamma.(\gamma, M_\sigma(\check{\gamma}))$ realized by the K combinator.

From now on, we refer to the constructed model as $\mathbb{M}$.

3 Type formers

Definition 5 (Dependent function types a QCwF)**.** A QCwF supports dependent function types with usage information, if

- the underlying CwF $\mathcal{C}$ supports them, namely, if for all $\Gamma \in Ob(\mathcal{C}), S \in Ty(\Gamma), T \in Ty(\Gamma.S)$, $\pi \in R$, there exist type $\Pi\pi ST \in Ty(\Gamma)$ and a bijection

$$\Lambda : Tm(\Gamma.S, T) \cong Tm(\Gamma, \Pi\pi ST),$$

  natural in Γ.

- for $\Gamma \in Ob(\mathcal{L}), S \in Ty(U\Gamma), T \in Ty(U\Gamma.S), \pi \in R$, there exists a bijection

$$\Lambda_{\mathcal{L}} : RTm(\Gamma.\pi S, T) \cong RTm(\Gamma, \Pi\pi ST),$$

  natural in Γ such that $U \circ \Lambda_{\mathcal{L}} = \Lambda \circ U$ and $U \circ \Lambda_{\mathcal{L}}^{-1} = \Lambda^{-1} \circ U$.

To show that our model supports Π types, fix some $\pi \in R$, suppose Γ is a r.g in $Ob(\mathcal{C})$, $S \in Ty(\Gamma)$, $T \in Ty(\Gamma.S)$. Define the semantic type $\Pi\pi ST$ as the family of assemblies over Γ, consisting of:

- $(\Pi\pi ST)_O(\gamma) := (X, \vDash_X)$ for $\gamma \in \Gamma_O$, where

$$\begin{aligned} X := \{(f_O, f_R) \quad | \quad & \\ & f_O : \forall s \in |S_O(\gamma)|.T_O(\gamma, s), \\ & f_R : \forall r \in S_R(\Gamma_{refl}(\gamma)).T_R(\Gamma_{refl}(\gamma), r), \\ & \forall s \in S_O(\gamma).T_{refl}(\gamma, s)(f_O(s)) = f_R(S_{refl}(\gamma)(s)), \\ & \forall r \in S_R(\Gamma_{refl}(\gamma)).T_{src}(\Gamma_{refl}(\gamma), r)(f_R(r)) = f_O(S_{src}(\Gamma_{refl}(\gamma))(r)), \\ & \forall r \in S_R(\Gamma_{refl}(\gamma)).T_{tgt}(\Gamma_{refl}(\gamma), r)(f_R(r)) = f_O(S_{tgt}(\Gamma_{refl}(\gamma))(r))\} \end{aligned}$$

$$a \vDash_X (f_O, f_R) \text{ iff } \forall s \in |S_O(\gamma)|, a_s \in \mathscr{A}.a_s \vDash_{S_O(\gamma)} s \implies a \cdot !_\pi a_s \vDash_{T_O(\gamma,s)} f_O(s).$$

  Note that f_R does not contribute any realisability information to $\vDash_X$.

- $(\Pi\pi ST)_R(\gamma) :=$

$$\begin{aligned} \{((f_O^{src}, f_R^{src}), (f_O^{tgt}, f_R^{tgt}), r) \quad | \quad & \\ & (f_O^{src}, f_R^{src}) \in (\Pi\pi ST)_O(\Gamma_{src}(\gamma)), \\ & (f_O^{tgt}, f_R^{tgt}) \in (\Pi\pi ST)_O(\Gamma_{tgt}(\gamma)), \\ & r : \forall s \in S_R(\gamma).T_R(\gamma, s), \\ & \forall s \in S_R(\gamma).T_{src}(\gamma, s)(r(s)) = f_O^{src}(S_{src}(\gamma)(s)), \\ & \forall s \in S_R(\gamma).T_{tgt}(\gamma, s)(r(s)) = f_O^{tgt}(S_{tgt}(\gamma)(s))\} \end{aligned}$$

- $(\Pi\pi ST)_{refl}(\gamma) := \lambda(f_O, f_R).((f_O, f_R), (f_O, f_R), f_R)$ for $\gamma \in \Gamma_O$.
- $(\Pi\pi ST)_{src}(\gamma) := \lambda(f^{src}, f^{tgt}, r).f^{src}$ for $\gamma \in \Gamma_R$.
- $(\Pi\pi ST)_{tgt}(\gamma) := \lambda(f^{src}, f^{tgt}, r), f^{tgt}$ for $\gamma \in \Gamma_R$.

Unwinding Definition 5, we get

$$\{(M_O, M_R) : \forall(\gamma, s) \in (\Gamma.\pi S).T(\gamma, s)\} \cong \{(N_O, N_R) : \forall\gamma \in \Gamma.(\Pi\pi ST)(\gamma)\}$$

where the terms have the following elaborated types:

$$\begin{aligned}
&M_O : \forall(\gamma, s) \in (\Gamma.\pi S)_O.T_O(\gamma, s) \\
&M_R : \forall(\gamma, r) \in (\Gamma.\pi S)_R.T_R(\gamma, s) \\
&N_O : \forall\gamma \in \Gamma_O. \\
&\quad \{(f_O, f_R) \quad | \quad f_O : \Pi S(\gamma)_O.T(\gamma)_O \\
&\qquad\qquad\qquad\quad f_R : \Pi S_R(\Gamma_{refl}(\gamma).T(\Gamma_{refl}(\gamma)))\} \\
&N_R : \forall\gamma \in \Gamma_R. \\
&\quad \{(f^{src}, f^{tgt}, r) \quad | \quad f^{src} \in (\Pi\pi ST)_O(\Gamma_{src}(\gamma)) \\
&\qquad\qquad\qquad\qquad f^{tgt} \in (\Pi\pi ST)_O(\Gamma_{tgt}(\gamma)) \\
&\qquad\qquad\qquad\qquad r : \Pi S_R(\gamma).T_R(\gamma)\}
\end{aligned}$$

Thus, we can define Λ as $\Lambda(M_O, M_R) = (N_O, N_R)$, where

$$N_O := \lambda\gamma_o.(\lambda s.M_O(\gamma_o, s), \lambda s_r.M_R(\Gamma_{refl}(\gamma_o), s_r))$$

$$N_R := \lambda\gamma_r.(N_O(\Gamma_{src}(\gamma)), N_O(\Gamma_{tgt}(\gamma)), \lambda s_r.M_R(\gamma, s_r)))$$

For $\Lambda_{\mathcal{L}}$, a realizer a_m of M
(*that is* $\forall(\gamma, s), \forall(a_\gamma, a_s), [a_\gamma, a_s] \vDash_{\Gamma.\pi S} (\gamma, s) \implies a_m{\cdot}[a_\gamma, a_s] \vDash_{T(\gamma,s)} M(\gamma, s)$)
can be transformed to a realizer a_n of N by:

$$a_n := \lambda^* y.(\lambda^* s.(a_m{\cdot}[y, s]))$$

The conditions $U \circ \Lambda_{\mathcal{L}} = \Lambda \circ U$ and $U \circ \Lambda_{\mathcal{L}}^{-1} = \Lambda^{-1} \circ U$ follow trivially.

Universe of small types A plausible candidate for the universe U is given by the general construction:
$U_O :=$ the set of small reflexive graphs
$U_R := \{(A, B, R, R_{src}, R_{tgt}) : R \in \mathcal{U}, R_{src} : R \to A_O, R_{tgt} : R \to A_O, A, B \text{ are small r.g.}\}^{\cdot}$

However, this universe turns out to be "too big' - types do not carry enough structure to model parametricity accurately. To remedy the definition, we impose restrictions laid out by the following characterization:

A family of reflexive graphs of assemblies S over Γ is

- small - if for all $\gamma_\sigma \in \Gamma_\sigma$, $S_R(\gamma_R) \in \mathcal{U}$ and $|S_O(\gamma_O)| \in \mathcal{U}$.
- discrete - if for every $\gamma \in \Gamma_O$, there exists $X \in \mathcal{A}sm^\dagger(\mathscr{A})$, s.t.

$$\begin{tikzcd}
S_O(\gamma) \arrow[d, "S_{refl}(\gamma)"] & & X \arrow[d, "id"] \\
S_R(\Gamma_{refl}(\gamma)) \arrow[u, bend left=60, "S_{src}(\Gamma_{refl}(\gamma))"] \arrow[u, bend right=60, "S_{src}(\Gamma_{refl}(\gamma))"'] & \cong & {|X|} \arrow[u, bend left, "id"] \arrow[u, bend right, "id"']
\end{tikzcd}$$

- proof-irrelevant - if for all $\gamma \in \Gamma_R$, the function $\langle S_{src}(\gamma), S_{tgt}(\gamma)\rangle : S_R(\gamma) \to |S_O(\Gamma_{src}(\gamma))| \times |S_O(\Gamma_{tgt}(\gamma))|$ is injective.

For any reflexive graph Γ , define the small, discrete, proof-irrelevant universe $U \in Ty(\Gamma)$ and the type decoder $T \in Ty(\Gamma.U)$ as:

- $|U_O(\gamma)| :=$ the set of small, discrete r.g. of assemblies $a \vDash_{U_O(\gamma)} S$ for $a \in \mathscr{A}$, $a = I$ and $S \in |U_O(\gamma)|$.
- $U_R(\gamma_R) :=$
  $$\begin{aligned} \{(S, T, R, R_{src}, R_{tgt}) \quad | \quad & R \in \mathcal{U} \\ & S, T \text{ are small discrete r.g. of assemblies} \\ & \langle R_{src}, R_{tgt}\rangle : R \to |S_O| \times |T_O| \textit{ is injective}\} \end{aligned}$$
- $U_{refl}(\gamma_R)(S) := (S, |S|, S_{refl}, S_{src}, S_{tgt})$
- $U_{src}(\gamma_R)(S, T, R, R_{src}, R_{tgt}) = S$
- $U_{src}(\gamma_R)(S, T, R, R_{src}, R_{tgt}) = T$

and $T \in Ty(\Gamma.U)$ as:

- $T_O(\gamma_O, S) := S_O$
- $T_R(\gamma_R, (S, T, R, R_{src}, R_{tgt})) := R$
- $T_{refl}(\gamma_O, S) := S_{refl}$
- $T_{src}(\gamma_R, (S, T, R, R_{src}, R_{tgt})) := R_{src}$
- $T_{tgt}(\gamma_R, (S, T, R, R_{src}, R_{tgt})) := R_{tgt}$.

Claim. *U is closed under Π types.*

Given some r.g Γ and $S \in Ty(\Gamma)$, $T \in Ty(\Gamma.S)$, it suffices to show that $\Pi\pi TS \in Ty(\Gamma)$ is a small, discrete and proof-irrelevant family of r.g. of assemblies† [3] For brevity, let $V := \Pi\pi ST$.

Smallness follows by the closure under Π-types in the ambient set-theoretical universe $\mathcal{U}$.
For proof-irrelevance, take some $\gamma_R \in \Gamma$ and $f, g \in V_R(\gamma_R)$, s.t $\langle V_{src}(\gamma_R), V_{tgt}(\gamma_R)\rangle(f) = \langle V_{src}(\gamma_R), V_{tgt}(\gamma_R)\rangle(g)$. WTP $f = g$, by def. we get immediately that $f^{src} = g^{src}$ and $f^{tgt} = g^{tgt}$. Given that $(f^{src}, f^{tgt}, r_f), (f^{src}, f^{tgt}, r_g) \in V_R(\gamma_R)$, note that ,

$$\forall s \in S_R(\gamma_R).\langle T_{src}(\gamma_R, s), T_{src}(\gamma_R, s)\rangle(r_f(s)) = (f_O^{src}(S_{src}(\gamma_R)(s)), f_O^{tgt}(S_{tgt}(\gamma_R)(s)))$$
$$\forall s \in S_R(\gamma_R).\langle T_{src}(\gamma_R, s), T_{src}(\gamma_R, s)\rangle(r_g(s)) = (f_O^{src}(S_{src}(\gamma_R)(s)), f_O^{tgt}(S_{tgt}(\gamma_R)(s)))$$

Since T is proof-irrelevant, it follows directly that $r_f = r_g$ and thus $f = g$.

Reduced Π types - $\Pi^R\pi ST$ There is plethora of problems with the definition of Π types. In order to satisfy the bijections posited in $Definition$ 5, we had to include too many inhabitants of $(\Pi\pi ST)_O$. This leads to an internal conflict w.r.t realisability - some of the elements might not be realised at all, while others might share the same realiser. To see the latter, consider pairs $(f_O, f_R), (f'_O, f'_R) \in |(\Pi\pi ST)_O(\gamma)|$, such that f_O and f'_O have identical behaviour on $\lfloor S_O(\gamma)\rfloor$, but there exists some $s \in |S_O(\gamma)$, s.t $f_O(\gamma, s) \neq f'_O(\gamma, s)$.

4 Some (free) theorems

Definition 6 (No universal realizer)**.** Given a context Γ and a type T, s.t $\Gamma \vdash T$, the model $\mathcal{C}$ constructed so far has no universal realizer iff $\bigcap_{\gamma\in[\![\Gamma]\!]_O} \{a \in \mathscr{A}$: for every x $\in |[\![T]\!]_O(\gamma)|, a \vDash_{[\![T]\!]_O(\gamma_O)} x\}$ is empty.

Theorem 7. *Let Γ be a context and $T := \Pi a \overset{0}{:} \mathbf{U}.\Pi_{\_} \overset{0}{:} \mathbf{T}a.\mathbf{T}a$ - a type. Assume the model has no universal realizers. There is no resourced term M of that type - i.e. $\Gamma \vdash M \overset{1}{:} T$ does not hold in $\mathbb{M}$.*

Assume such term M, $\Gamma \vdash M \overset{1}{:} T$ exists. Fix some $\gamma \in \Gamma_O$ and consider the uncurried term M', s.t $\Gamma, a \overset{O}{:} \mathbf{U}, \_ \overset{0}{:} \mathbf{T}a \vdash M' \overset{1}{:} \mathbf{T}a$ and

$$M'_O(\gamma_O, a_O, \_) := \textit{ let } ((f''_O, f''_R), f'_R) = M_O(\gamma_O) \textit{ in } f''_O(\gamma_O, a_O, \_)$$
$$M'_R(\gamma_R, a_R, \_) := \textit{ let } (f'^{src}, f'^{tgt}, (f''^{src}, f''^{tgt}, r)) = M_R(\gamma_R) \textit{ in } r(\gamma_R, a_R, \_)$$

Fix some $\gamma_O \in \Gamma_O$. Spelling out explicitly the type of M'_R, "instantiated" at $\Gamma_{refl}(\gamma_O)$ (or equivalently, of $r(\Gamma_{refl}(\gamma_O), -, -)$ and suppressing the realizability information, we get that:

$$M'_R(\Gamma_{refl}(\gamma_O), -, -) : \forall a_R \in \mathbf{U}_R(\Gamma_{refl}(\gamma_O)).\mathbf{T}_R(\Gamma_{refl}(\gamma_O), a_R) \to \mathbf{T}_R(\Gamma_{refl}(\gamma_O), a_R)$$

[3] it actually does not suffice, we implicitly assume a lemma that gives an isomorphism between terms of type U and small, discrete, p.i. r.g of assemblies†

Unpacking the definition of $\mathbf{U}_R(\Gamma_{refl}(\gamma_O))$, we get (by conditions in Definition 4):

$$\forall S, T \in \mathbf{U}_O(\gamma_O), \forall R \in \mathcal{U}, R_{src} : R \to S_O, R_{tgt} : R \to T_O :$$

$$R_{src}(M'_R(\Gamma_{refl}(\gamma_O)), (S, T, R, R_{src}, R_{tgt}), (s, t))) = M'_O(\gamma_O, S, s)$$
$$R_{tgt}(M'_R(\Gamma_{refl}(\gamma_O)), (S, T, R, R_{src}, R_{tgt}), (s, t))) = M'_O(\gamma_O, T, t)$$

Thus, we conclude that

$$\forall S, T \in \mathbf{U}_O(\gamma_O), \forall R \subseteq S \times T, \forall (s, t) \in R.\ (M'_O(\gamma_O, S, s), M'_O(\gamma_O, T, t)) \in R \tag{1}$$

Let X be some type, s.t $\Gamma \vdash X : \mathbf{U}$. Consider the term M instantiated at γ_O and X and $R^X := \{(x, x) | x : X\}$. Substituting X for S and T, R^X for R in (1) and applying currying, we get that for each $x : X$, $(M(\gamma_O, X(\gamma_O), x), M(\gamma_O, X(\gamma_O), x)) \in R$ holds. Hence $M(\gamma_O, X(\gamma_O), x) = x$.

Now since $M \in RTm(\Gamma, T)$, M is realizable - in particular, there exists an $a \in \mathscr{A}$ that tracks $M(\gamma_O X(\gamma_O), -)$. By def. we get that $\forall x \in X_O, \forall b \in \mathscr{A}.b \vDash_{\Gamma.0X} (\gamma_O, x) \implies a.!_0 b \vDash_{X(\gamma_O)} M(\gamma_O, X(\gamma_O), x)$ - a is a realizer for every element x in $X_O(\gamma_O)$. But that is a contradiction, as no universal realizer exists for X by assumption. Therefore no resourced term $M \overset{1}{:} T$ exists.

Definition 8 (Separability of realizers)**.** Let Γ be a context and $S \in Ty(\Gamma)$. S has separable realizers if for every $\gamma \in \Gamma_O$ and $x, y \in |S_O(\gamma)|$ and $a \in \mathscr{A}$, if $a \vDash_{S_O(\gamma)} x$ and $a \vDash_{S_O(\gamma)} y$, then $x = y$.

Theorem 9. *Let $\Gamma \vdash A : \mathbf{U}$ and $\Gamma, a \overset{\sigma}{:} A \vdash B : \mathbf{U}$. If A has a realisable inhabitant and B has separable realisers, then $[\![B]\!] \cong [\![\Pi x \overset{0}{:} A.\, B]\!]$ holds in $\mathbb{M}$.*

Let a^* be the realisable inhabitant of A, $a^* \in RTm(\Gamma, A)$, $a^*_o : \forall \gamma_o \in \Gamma_O.\lfloor A_O(\gamma_o) \rfloor$,
$a^*_r : \forall \gamma_r \in \Gamma_R.A_R(\gamma_r)$.
Define the morphisms $g : \Pi x \overset{0}{:} A.\ B \to B$ and $h : B \to \Pi x \overset{0}{:} A.\ B$ as [4]:

$$g_O(\gamma_o) := \lambda(f_O, f_R).f_O(a^*_o(\gamma)) \qquad\qquad h_O(\gamma_o) := \lambda b_O.(\lambda a_o.b_O, \lambda a_r.B_{refl}(b_O))$$

Let $V := \Pi x \overset{0}{:} A.\ B$, observe that if $(f_O, f'_R), (f_O, f''_R) \in |V_O|$, then $f'_R = f''_R$ using the proof-irrelevance of B. Hence, to show that two inhabitants of $|V_O|$ are equal, it suffices to prove it for the first components only (1).

Fix some $(f_O, f_R) \in \lfloor V_O \rfloor$. To find a realiser of $g_O(\gamma_o)$, consider the realiser $a_f \vDash_{V_O(\gamma_o)} (f_O, f_R)$. By def. we have that $\forall x \in |A_O(\gamma_O)|, a \in \mathscr{A}.a \vDash_{A_O(\gamma_o)} x \implies a_f.!_0 a \vDash_{B_O(\gamma_o)} f_O(x)$. As B has

[4]for the time being, focus only on the object part

separable realisers, it must be the case that f_O is a constant function (2). Define the realiser a_g of $g_O(\gamma_o)$ as $a_g := \lambda^* x.(x.I)$.
As for $h_O(\gamma_o)$, we must first ensure that h_O outputs well-defined inhabitants of $V_O(\gamma_o)$. Notice that both $f' := \lambda a_o.b_O$ and $f'' := \lambda a_r.B_{refl}(b_O)$ are well-typed and we can easily verify they satisfy the conditions in Definition 5 by direct substitution (where the extension of f' is the natural extension over $|A_O|$). To construct the realiser of f', let $a_b \vDash_{B_O(\gamma_o)} b_O$ (such a_b exists due to $b_O \in \lfloor B_O(\gamma_o) \rfloor$), then $K.a_b$ realises f'.

Let $b^* \in \lfloor B_O(\gamma_o) \rfloor$, by def. we have that $(g_O \circ h_O)(\gamma_o, b^*) = b^*$. As for the converse direction, let $(h_O \circ g_O)(\gamma_o)(f_O, f_R) = (f'_O, f'_R)$ for some $(f_O, f_R) \in V_O(\gamma_o)$. By (1), we show only that $f_O = f'_O$. As f_O is a constant function by (2), expand the definition of h_o to obtain immediately $f\emptyset = f'_O$.

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