Relational realisability model for QTT

Our aim is to build a concrete realisability model for QTT which allows for metareasoning with results derived from parametricity. To that end, a construction of a quantitative category with families (QCwF) is presented [**Atkey2018**]. We follow the relational approach to types introduced by Reynolds for typed lambda calculus [**Ma1992**] and later refined for dependent types theories [**Atkey2014a**].
Once and for all fix a usage semiring R and an R-linear combinatory algebra $\mathscr{A}$[1].

Taking stocks

Definition 1 (Assembly†)**.** An assembly† Γ is a pair $(|\Gamma|, e)$ where $|\Gamma|$ is a carrier set and e is a function $|\Gamma| \to \mathcal{P}(\mathscr{A})$.

e encodes realisability information - given some $\gamma \in |\Gamma|$, $e(\gamma)$ is interpreted as the set of witnesses for the existence of γ. To emphasize on that aspect, we write $a \vDash_\Gamma \gamma$ to denote $a \in e(\gamma)$.
A morphism between two assemblies† $(|\Gamma|, e_\Gamma)$ and $(|\Delta|, e_\Delta)$ is a function $f : |\Gamma| \to |\Delta|$ that is realizable when acting on elements with realizers - there exists $a_f \in \mathscr{A}$, s.t for every $\gamma \in |\Gamma|$ and $a_\gamma \in \mathscr{A}$, the following holds:

$$a_\gamma \vDash_\Gamma \gamma \implies a_f . a_\gamma \vDash_\Delta f(\gamma)$$

a_f is said to track f. Also note that multiple realizers for the same function f do not induce multiple morphisms.
Using these notions we can construct a category $\mathcal{A}sm^\dagger(\mathscr{A})$.

Definition 2 (Reflexive graph)**.** A reflexive graph (r.g.) G is a tuple $(G_O, G_R, G_{refl}, G_{src}, G_{tgt})$, where G_O and G_R are sets, G_{src} and G_{tgt} are functions $G_R \to G_O$ and G_{refl} is a function $G_O \to G_R$, s.t $G_{src} \circ G_{refl} = id_{G_O} = G_{tgt} \circ G_{refl}$.

G_O and G_R stand for a set of objects and a set of relations, respectively. In general, reflexive graphs are less structured than categories as relations need not compose.
A morphism between reflexive graphs G and H is a pair of functions $(f_o : G_O \to H_O, f_r : G_R \to H_R)$, s.t. all of the depicted squares commute:

$$\begin{tikzcd}
G_O \arrow[rr, "f_o"] \arrow[d, "G_{refl}" description] & & H_O \arrow[d, "H_{refl}" description] \\
G_R \arrow[u, bend left=40, "G_{src}"] \arrow[u, bend right=40, "G_{tgt}"'] \arrow[rr, "f_r"] & & H_R \arrow[u, bend left=40, "H_{src}"] \arrow[u, bend right=40, "H_{tgt}"']
\end{tikzcd}$$

Reflexive graphs equipped with r.g. morphisms form a category $\mathcal{R}Gph(\mathcal{S}et)$.
We use reflexive graphs to give a dyadic interpretation of types in the spirit of [**Ma1992**].

[1] In case some non-trivial properties of $\mathscr{A}$ are required, we will tacitly assume that $\mathscr{A}$ is a graph model(see [**fill**]) - **also to fix**

Reflexive graphs of assemblies†

One could easily generalize reflexive graphs by considering object and relation components from arbitrary categories instead of $\mathcal{S}et$. As our purpose is to build a realisability model, we replace the set of objects with an assembly† $\mathcal{A}sm^\dagger(\mathscr{A})$ identify two appropriate notions of reflexive graph of assemblies and a family of reflexive graphs of assemblies.

Definition 3 (Reflexive graph of assemblies). **FNF:** It's not easy, but we should think of a good name for this — "reflexive graph of X" usually means "reflexive graph in the category of Xs", i.e. where G_O and G_R are Xs, and the maps are X morphisms. A reflexive graph of assemblies G is a tuple $(G_O, G_R, G_{refl}, G_{src}, G_{tgt})$, where G_O is an assembly, G_R - a set and the functions $G_{refl} : |G_O| \to G_R$, $G_{src} : |G_R| \to G_O$, $G_{tgt} : |G_R| \to G_O$) are such that the identities in Definition 2 are satisfied.

With these components, we obtain a category $\mathcal{R}Gph(\mathcal{A}sm^\dagger(\mathscr{A}))$. By considering r.g. of assemblies of shape $(X, |X|, id_X, id_X, id_X)$, we identify isomorphic copy of $\mathcal{A}sm^\dagger(\mathscr{A})$ inside $\mathcal{R}Gph(\mathcal{A}sm^\dagger(\mathscr{A}))$.
A terminal object $\mathbf{1}_{\mathcal{R}Gph(\mathcal{A}sm^\dagger(\mathscr{A}))}$ in $\mathcal{R}Gph(\mathcal{A}sm^\dagger(\mathscr{A}))$ is a tuple $(\mathbf{1}_{\mathcal{A}sm^\dagger(\mathscr{A})}, \{*\}, id, id, id)$, where $\mathbf{1}_{\mathcal{A}sm^\dagger(\mathscr{A})}$ is the terminal assembly† $(\{\star\}, f)$, with f defined as $\star \mapsto \{I\}$.

Definition 4 (Family of reflexive graphs of assemblies). **FNF:** ditto here, namingwise Let $\mathcal{C}$ be a category with a terminal object. Given a reflexive graph $\Gamma \in Ob(\mathcal{C})$, a family of internal r.g. over Γ is a tuple $S := (S_O, S_R, S_{refl}, S_{src}, S_{tgt})$, where:

- $S_O : \Gamma_O \to \mathcal{A}sm^\dagger(\mathscr{A})$
- $S_R : \Gamma_R \to \mathcal{S}et$
- a Γ-indexed collection of functions $S_{refl} := \{f_\gamma : |S_O(\gamma)| \to S_R(\Gamma_{refl}(\gamma))\}_{\gamma \in \Gamma_O}$
- $S_{src} := \{f_\gamma : S_R(\gamma) \to |S_O(\Gamma_{src}(\gamma))|\}_{\gamma \in \Gamma_R}$
- $S_{tgt} := \{f_\gamma : S_R(\gamma) \to |S_O(\Gamma_{tgt}(\gamma))|\}_{\gamma \in \Gamma_R}$

such that

- each identity in the following collection is satisfied:

$$S_\sigma(\Gamma_{refl}(\gamma)) \circ S_{refl}(\gamma) = id \text{ for every } \gamma \in \Gamma_O, \sigma \in \{src, tgt\}$$

We are only interested in cases when $\mathcal{C} = \mathcal{S}et$ or $\mathcal{C} = \mathcal{A}sm^\dagger(\mathscr{A})$.
A morphism M between two families S and T of internal r.g. over Γ is a pair (M_O, M_R) of Γ-indexed collections of functions:

- $M_O := \{f_\gamma : |S_O(\gamma)| \to T_O(\gamma)\}_{\gamma \in \Gamma_O}$ **FNF:** should be tracked?
- $M_R := \{f_\gamma : |S_R(\gamma)| \to T_R(\gamma)\}_{\gamma \in \Gamma_O}$

such that:

- $T_{refl}(M_O(\gamma)(x)) = M_R(\Gamma_{refl}(\gamma))(S_{refl}(\gamma)(x))$ for every $\gamma \in \Gamma_O$, $x \in S_O(\gamma)$
- $T_{src}(M_R(\gamma)(r)) = M_O(\Gamma_{src}(\gamma))(S_{src}(\gamma)(r))$ for every $\gamma \in \Gamma_R$, $r \in S_R(\gamma)$
- $T_{tgt}(M_R(\gamma)(r)) = M_O(\Gamma_{tgt}(\gamma))(S_{tgt}(\gamma)(r))$ for every $\gamma \in \Gamma_R$, $r \in S_R(\gamma)$

A terminal family of r.g. over Γ, 1_Γ, consists of two constant functions, mapping $\gamma \in \Gamma$ to a terminal assembly† 1, and three Γ-indexed collections with a sole element id_1.

A CwF from families of reflexive graphs of assemblies

As a first step toward obtaining a relationally parametric realisability model of QTT, we construct a concrete CwF using families of internal graphs of assemblies.
Consider the category $\mathcal{R}Gph$ with terminal object $1 := (\{\star\}, \{\star\}, id, id, id)$.
Let $\Gamma, \Delta \in Ob(\mathcal{R}Gph)$, define:

- the collection of semantic types $Ty(\Gamma)$ as the collection of families of reflexive graphs of assemblies Γ.
- given a type $S \in Ty(\Gamma)$, an element $M \in Tm(\Gamma, S)$ is a pair of functions ($M_O : \forall \gamma \in \Gamma_O.|S_O(\gamma)|$, $M_R : \forall \gamma \in \Gamma_R.S_R(\gamma)$), such that

$$\forall \gamma \in \Gamma_O.S_{refl}(M_O(\gamma)) = M_R(\Gamma_{refl}(\gamma))$$
$$\forall \gamma \in \Gamma_R.S_{src}(M_R(\gamma)) = M_O(\Gamma_{src}(\gamma))$$
$$\forall \gamma \in \Gamma_R.S_{tgt}(M_R(\gamma)) = M_O(\Gamma_{tgt}(\gamma))$$

- given $f : \Gamma \to \Delta$, substitutions in types and terms is a precomposition with f on the object and relation components of types and terms respectively. Clearly, these operations are compatible with identity and composition in $\mathcal{R}Gph$.
- context extension: Suppose $S \in Ty(\Gamma)$, construct a r.g. $\Gamma.S$ as :

$$\begin{aligned}(\Gamma.S)_O &= \{(\gamma, x) : \gamma \in \Gamma_O, x \in |S_O(\gamma)|\} \\ (\Gamma.S)_R &= \{(\gamma, r) : \gamma \in \Gamma_R, r \in S_R(\gamma)\} \\ (\Gamma.S)_{refl}(\gamma, x) &= (\Gamma_{refl}(\gamma), S_{refl}(\gamma)(x)) \\ (\Gamma.S)_\sigma(\gamma, r) &= (\Gamma_\sigma(\gamma), S_\sigma(\gamma)(r)), \quad \sigma \in \{src, tgt\}\end{aligned}$$

Claim. *$Hom_{\mathcal{R}Gph}(\Delta, \Gamma.S) \cong \{(f, M) : f : \Delta \to \Gamma, M \in Tm(\Delta, S\{f\}\}$, natural in Δ.*

Upgrading to a QCwF

Recall the definition of a QCwF from [**Atkey2018**]. Given a usage semiring R, a R-QCwF consists of:

1. A CwF $(\mathcal{C}, 1, Ty, Tm, -.-, \langle -.- \rangle)$
2. A category $\mathcal{L}$ with a faithful functor $U : \mathcal{L} \to \mathcal{C}$

3. A functor $(+) : \mathcal{L} \times_{\mathcal{C}} \mathcal{L} \to \mathcal{L}$, s.t $U(\Gamma_1 + \Gamma_2) = U(\Gamma_1) = U(\Gamma_2)$[2]. $\mathcal{L} \times_{\mathcal{C}} \mathcal{L}$ denotes the pullback $\mathcal{L} \xrightarrow{U} \mathcal{C} \xleftarrow{U} \mathcal{L}$.
   Additionally, there exists an object $\lozenge \in \mathcal{L}$, s.t. $U\lozenge = 1$.
4. A functor $\rho(-) : \mathcal{L} \to \mathcal{L}$ for each $\rho \in R$, s.t $U(\rho(-)) = U(-)$.
5. A collection $RTm(\Gamma, S)$ for each $\Gamma \in \mathcal{L}$ and $S \in Ty(U\Gamma)$, equipped with an injective function $U_{\Gamma,S} : RTm(\Gamma, S) \to Tm(U\Gamma, S)$.
   For an $\mathcal{L}$ morphisms $f : \Gamma \to \Delta$ and types $S \in Ty(U\Gamma)$, a function $-\{f\} : RTm(\Delta, S) \to RTm(\Gamma, S\{f\})$, s. t. $U(-\{f\}) = (U(-))\{Uf\}$.
6. Given $\Gamma \in \mathcal{L}$, $\rho \in R$ and $S \in Ty(U\Gamma)$, an object $\Gamma.\rho S$, s.t $U(\Gamma.\rho S) = U\Gamma.S$.
   Additionally, there exist the following natural transformations:
   $emp_\pi : \lozenge \to \pi\lozenge$, s.t. $U(emp_\pi) = id_1$ **FNF:** since 1 terminal, this is automatic?
   $emp_+ : \lozenge \to \lozenge + \lozenge$, s.t. $U(emp_+) = id_1$ **FNF:** ditto
   $ext_\pi : \pi\Gamma.(\pi\rho S) \to \pi(\Gamma.\rho S)$, s.t. $U(ext_\pi) = id$
   $ext_+ : (\Gamma_1 + \Gamma_2).(\rho_1 + \rho_2)S \to \Gamma_1.\rho_1 S + \Gamma_2.\rho_2 S$, s.t. $U(ext_+) = id$
7. Given $\Gamma \in \mathcal{L}$, $S \in Ty(U\Gamma)$, there exists:
   a morphism $p_{\Gamma.S} : \Gamma.0S \to \Gamma$, s.t. $U(p_{\Gamma.S}) = p_{U\Gamma.S}$
   an element $v_{\Gamma.S} \in RTm(0\Gamma.1S, S\{p_{U\Gamma.S}\})$, s.t. $U(v_{\Gamma.S}) = v_{U\Gamma.S}$
   a morphism $wk(f, \rho S') : \Gamma.\rho S'\{Uf\} \to \Delta.\rho S'$ for each $f : \Gamma \to \Delta$, $S' \in Ty(U\Gamma, \Delta)$ s.t. $U(wk(f, \rho S')) = wk(Uf, S')$
   let $\Gamma_1, \Gamma_2 \in \mathcal{L}$, s.t $U\Gamma_1 = U\Gamma_2$ and $M \in RTm(\Gamma_2, S)$. There is a morphism $\overline{\rho M} : \Gamma_1 + \rho\Gamma_2 \to \Gamma_1.\rho S$, s.t $U(\overline{\rho M}) = \overline{UM}$
   a morphism $\overline{M} : \Gamma \to \Gamma.0S$ for $M \in Tm(U\Gamma, S)$, s.t. $U(\overline{M}) = \overline{M}$.

Now to construct the concrete QCwF, consider the CwF from previous section as the underlying category.
Take $\mathcal{L} := \mathcal{R}Gph(\mathscr{A})$ **FNF:** "half-assemblied" r.g.s (yes, we also need notation for them) and let U be the functor $\mathcal{R}Gph(\mathscr{A}) \to \mathcal{R}Gph$, sending an assembly† to its underlying set, forgetting the realisability function.
For the addition structure, let Γ', Γ'' be r.g. of assemblies, s.t $|\Gamma'_O| = |\Gamma''_O|$ and $|\Gamma'_R| = |\Gamma''_R|$ **FNF:** no $|\cdot|$ for $_R$ part. Construct the r.g. of assemblies $\Gamma := \Gamma' + \Gamma''$, where:

- $\Gamma_O := (|\Gamma'_O|, \vDash_\Gamma)$ with $a \vDash_\Gamma \gamma$ iff there exist $x, y \in \mathscr{A}$, s.t. $a = [x, y]$ and $x \vDash_{\Gamma'} \gamma$ and $y \vDash_{\Gamma''} \gamma$.
- define Γ_R similarly as Γ_O. **FNF:** this is just $\Gamma_R = \Gamma'_R (= \Gamma''_R)$
- Γ_{refl}, Γ_{src}, Γ_{tgt} are the same as their Γ' counterparts (or Γ'').

Define $\lozenge$ as the terminal object $\mathbf{1}_{\mathcal{R}Gph(\mathcal{A}sm^\dagger(\mathscr{A}))}$
Consider the scaling structure and let $\Gamma := \rho(\Gamma')$:

- $\Gamma_\sigma = (|\Gamma'_\sigma|, \vDash_{\Gamma_\sigma})$ with $a \vDash_{\Gamma_\sigma} \gamma$ iff there is $x \in \mathscr{A}$, s.t $a =!_\rho x$ and $x \vDash_{\Gamma'_\sigma} \gamma$ for $\sigma \in \{O, R\}$

[2] the second equality being trivially satisfied

- again, scaling leaves unmodified Γ_σ for $\sigma \in \{src, tgt, rfl\}$.

Let $RTm(\Gamma, S)$ be the collection of assembly† morphisms from the terminal object to S (note any set-theoretic function from the terminal object is realizable). Spelling this out, $RTm(\Gamma, S)$ consists of tuples (M_O, M_R), s.t. the conditions from Definition 4 are satisfied **FNF:** would be good for clarity to spell this out (up to isomorphism, removing the unit type element). $U_{\Gamma,S}$ just forgets the realisability information and is trivially injective. Substitution in terms is given by precomposition with $f : \Gamma \to \Delta$ - $-\{f_O\} := \lambda M_O.\forall\gamma \in \Gamma.M_O(f(\gamma))$ and similarly, $-\{f_R\} := \lambda M_R.\forall\gamma \in \Gamma.M_R(f(\gamma))$. The functor U interacts nicely with the so-defined $-\{f\}$ as essentially the substitution in terms in the underlying CwF is defined in the same way.

Let $\Gamma.\rho S := (\Gamma'_O, \Gamma'_R, \Gamma'_\sigma)$, where

- $|\Gamma'_O| := \{(\gamma, x) : \gamma \in \Gamma_O, x \in S_O(\gamma)\}$ and $a \vDash_{\Gamma.\rho S} (\gamma, x)$ iff there exists $b, c \in \mathscr{A}$, s.t $a = [b, !_\rho c]$, $b \vDash_\Gamma \gamma$ and $c \vDash_{S(\gamma)} \pi_1(\check{(\gamma, x)})$, where $\check{(-)} : \Gamma.S \to U(\Gamma.S)$, $\check{(-)} := id$ as the set-theoretic part of the extensions in the CwF and $\mathcal{L}$ is the same by definition.
- $|\Gamma'_R|$ is defined analogously. **FNF:** no realizability anymore
- Each Γ'_σ is defined pointwise.

$emp_\pi : \Diamond \to \pi\Diamond$ is given by identity function on both the object and relational part. It is realized by $K.!_\rho I$. Similarly, emp_+ is realized by $K.[I, I]$,
ext_π - by $\lambda^*q.let\ [x, y] = q\ in\ F_\pi.(F_\pi.(!_\pi\lambda^*stu.ust).x).\delta_{\pi\rho}y$ and
ext_+ - by $\lambda^*q.let\ [[x, y], z] = q\ in\ W_{\pi\rho}.(\lambda^*ab.[[x, a], [y, b]]).z$

The underlying function part of the above-defined natural transformations is given by identity, hence naturality follows trivially.
Finally, we construct the morphisms, listed in 7:

- $p_{\Gamma.S} : \Gamma.0S \to \Gamma$ is the first projection of $(\Gamma.0S)_\sigma = \{(\gamma, s) : \gamma \in \Gamma_\sigma.s \in S(\gamma)\}$, $(\sigma \in \{_{O,R}\})$ and is realized by $\lambda^*t.(t.K)$.
The equality $U(p_{\Gamma.S}) = p_{U\Gamma.S}$ holds trivially due to the identical structure of context extension in the underlying CwF and $\mathcal{L}$.
- define $v_{\Gamma.S} \in RTm(0\Gamma.1S, S\{p_{U\Gamma.S}\})$ as the second projection. $v_{\Gamma.S}$ is realized by $\lambda^*t.B.t.K.D$.
- Let a^σ_f realizes f_σ, then $wk(f, \rho S') := \lambda(\gamma, s).(f(\gamma), s)$ is realized by $\lambda^*q.let\ [x, y] = q\ in\ [a^\sigma_f.x, y]$
- given a $M_\sigma \in RTm(\Gamma, S) = M_\sigma : \forall\gamma \in U\Gamma_\sigma.S_\sigma(\gamma)$ with realizers a^σ_m, let $\overline{\rho M}_\sigma := \lambda\gamma.(\gamma, M_\sigma(\gamma))$ realized by $\lambda^*q.let\ [x, y] = q\ in\ [x, F_\rho.(!_\rho a^\sigma_m).y]$
- given a $M_\sigma \in Tm(U\Gamma, S) = M_\sigma : \forall\gamma \in U\Gamma_\sigma.S_\sigma(\gamma)$, let $\overline{M}_\sigma := \lambda\gamma.(\gamma, M_\sigma(\check{\gamma}))$ realized by the K combinator.

Type formers

Definition 5 (Dependent function types a QCwF)**.** A QCwF supports dependent function types with usage information, if

- the underlying CwF $\mathcal{C}$ supports them, namely, if for all $\Gamma \in Ob(\mathcal{C}), S \in Ty(\Gamma), T \in Ty(\Gamma.S)$, $\pi \in R$, there exist type $\Pi\pi ST \in Ty(\Gamma)$ and a bijection

  $$\Lambda : Tm(\Gamma.S, T) \cong Tm(\Gamma, \Pi\pi ST),$$

  natural in Γ.

- for $\Gamma \in Ob(\mathcal{L}), S \in Ty(U\Gamma), T \in Ty(U\Gamma.S), \pi \in R$, there exists a bijection

  $$\Lambda_{\mathcal{L}} : RTm(\Gamma.\pi S, T) \cong RTm(\Gamma, \Pi\pi ST),$$

  natural in Γ such that $U \circ \Lambda_{\mathcal{L}} = \Lambda \circ U$ and $U \circ \Lambda_{\mathcal{L}}^{-1} = \Lambda^{-1} \circ U$.

To show that our model supports Π types, fix some $\pi \in R$, suppose Γ is a r.g in $Ob(\mathcal{C})$, $S \in Ty(\Gamma)$, $T \in Ty(\Gamma.S)$. Define the semantic type $\Pi\pi ST$ as the family of assemblies over Γ, consisting of:

- $(\Pi\pi ST)_O(\gamma) := (X, \vDash_X)$ for $\gamma \in \Gamma_O$, where

  $$\begin{aligned} X := \{(f_O, f_R) \quad | \quad & \\ & f_O : \forall s \in S_O(\gamma).T_O(\gamma, s), \\ & f_R : \forall r \in S_R(\Gamma_{refl}(\gamma)).T_R(\Gamma_{refl}(\gamma), r), \\ & \forall s \in S_O(\gamma).T_{refl}(\gamma, s)(f_O(s)) = f_R(S_{refl}(\gamma)(s)), \\ & \forall r \in S_R(\Gamma_{refl}(\gamma)).T_{src}(\Gamma_{refl}(\gamma), r)(f_R(r)) = f_O(S_{src}(\Gamma_{refl}(\gamma))(r)), \\ & \forall r \in S_R(\Gamma_{refl}(\gamma)).T_{tgt}(\Gamma_{refl}(\gamma), r)(f_R(r)) = f_O(S_{tgt}(\Gamma_{refl}(\gamma))(r))\} \end{aligned}$$

  $a \vDash_X (f_O, f_R)$ iff

  $$\forall s \in |S_O(\gamma)|, b \in \mathscr{A}.b \vDash_{S_O(\gamma)} s \implies a.!_\rho b \vDash_{T_O(\gamma,s)} f_O(s)$$

  Note that f_R does not contribute any realisability information to $\vDash_X$.

- $$\begin{aligned} (\Pi\pi ST)_R(\gamma) := \{((f_O^{src}, f_R^{src}), (f_O^{tgt}, f_R^{tgt}), r) \quad | \quad & \\ & (f_O^{src}, f_R^{src}) \in (\Pi\pi ST)_O(\Gamma_{src}(\gamma)), \\ & (f_O^{tgt}, f_R^{tgt}) \in (\Pi\pi ST)_O(\Gamma_{tgt}(\gamma)), \\ & r : \forall s \in S_R(\gamma).T_R(\gamma, s), \\ & \forall s \in S_R(\gamma).T_{src}(\gamma, s)(r(s)) = f_O^{src}(S_{src}(\gamma)(s)), \\ & \forall s \in S_R(\gamma).T_{tgt}(\gamma, s)(r(s)) = f_O^{tgt}(S_{tgt}(\gamma)(s))\} \end{aligned}$$

- $(\Pi\pi ST)_{refl}(\gamma) := \lambda(f_O, f_R).((f_O, f_R), (f_O, f_R), f_R)$ for $\gamma \in \Gamma_O$.
- $(\Pi\pi ST)_{src}(\gamma) := \lambda(f^{src}, f^{tgt}, r).f^{src}$ for $\gamma \in \Gamma_R$.
- $(\Pi\pi ST)_{tgt}(\gamma) := \lambda(f^{src}, f^{tgt}, r), f^{tgt}$ for $\gamma \in \Gamma_R$.

Unwinding Definition 5, we get

$$\{(M_O, M_R) : \forall(\gamma, s) \in (\Gamma.\pi S).T(\gamma, s)\} \cong \{(N_O, N_R) : \forall\gamma \in \Gamma.(\Pi\pi ST)(\gamma)\}$$

where the terms are of the following type structure:

$$\begin{aligned}
&M_O : \forall(\gamma, s) \in (\Gamma.\pi S)_O.T_O(\gamma, s)\\
&M_R : \forall(\gamma, r) \in (\Gamma.\pi S)_R.T_R(\gamma, s)\\
&N_O : \forall\gamma \in \Gamma_O.\\
&\quad \{(f_O, f_R) \quad | \quad f_O : \Pi S(\gamma)_O.T(\gamma)_O\\
&\qquad\qquad\qquad\quad f_R : \Pi S_R(\Gamma_{refl}(\gamma).T(\Gamma_{refl}(\gamma)))\}\\
&N_R : \forall\gamma \in \Gamma_R.\\
&\quad \{(f^{src}, f^{tgt}, r) \quad | \quad f^{src} \in (\Pi\pi ST)_O(\Gamma_{src}(\gamma))\\
&\qquad\qquad\qquad\qquad f^{tgt} \in (\Pi\pi ST)_O(\Gamma_{tgt}(\gamma))\\
&\qquad\qquad\qquad\qquad r : \Pi S_R(\gamma).T_R(\gamma)\}
\end{aligned}$$

Thus, we can define Λ as $\Lambda(M_O, M_R) = (N_O, N_R)$, where

$$N_O := \lambda\gamma_o.(\lambda s.M_O(\gamma_o, s), \lambda s_r.M_R(\Gamma_{refl}(\gamma_o), s_r))$$

$$N_R := \lambda\gamma_r.(N_O(\Gamma_{src}(\gamma)), N_O(\Gamma_{tgt}(\gamma)), \lambda s_r.M_R(\gamma, s_r)))$$

For $\Lambda_{\mathcal{L}}$, a realizer a_m of M
(*that is* $\forall(\gamma, s), \forall(a_\gamma, a_s), [a_\gamma, a_s] \vDash_{\Gamma.\pi S} (\gamma, s) \implies a_m.[a_\gamma, a_s] \vDash_{T(\gamma,s)} M(\gamma, s)$)
can be transformed to a realizer a_n of N by:

$$a_n := \lambda^* y.(\lambda^* s.(a_m.[y, s]))$$

The conditions $U \circ \Lambda_{\mathcal{L}} = \Lambda \circ U$ and $U \circ \Lambda_{\mathcal{L}}^{-1} = \Lambda^{-1} \circ U$ follow trivially.

Universe of small types A plausible candidate for the universe U is the general definition in [**fill**]:
$U_O :=$ the set of small r.g.
$U_R := \{(A, B, R, R_{src}, R_{tgt}) : R \in \mathcal{U}, R_{src} : R \to A_O, R_{tgt} : R \to A_O, A, B \text{ are small r.g.}\}^{\cdot}$

However, this universe turns out to be "too big' - types do not carry enough structure to model parametricity accurately. To remedy the definition, we impose restrictions laid out by the following characterization:

A family of reflexive graphs of assemblies S over Γ is

- small - if for all $\gamma_\sigma \in \Gamma_\sigma$, $S_R(\gamma_R) \in \mathcal{U}$ and $|S_O(\gamma_O)| \in \mathcal{U}$.

- discrete - if for every $\gamma \in \Gamma_O$, there exists $X \in \mathcal{A}sm^{\dagger}(\mathscr{A})$, s.t.

$$\begin{tikzcd} S_O(\gamma) \arrow[d, "S_{refl}(\gamma)"] \\ S_R(\Gamma_{refl}(\gamma)) \arrow[u, bend left=60, "S_{src}(\Gamma_{refl}(\gamma))"] \arrow[u, bend right=60, "S_{src}(\Gamma_{refl}(\gamma))"'] \end{tikzcd} \cong \begin{tikzcd} X \arrow[d, "id"] \\ |X| \arrow[u, bend left=60, "id"] \arrow[u, bend right=60, "id"'] \end{tikzcd}$$

- proof-irrelevant - if for all $\gamma \in \Gamma_R$, the function $\langle S_{src}(\gamma), S_{tgt}(\gamma)\rangle : S_R(\gamma) \to |S_O(\Gamma_{src}(\gamma))| \times |S_O(\Gamma_{tgt}(\gamma))|$ is injective.

For any reflexive graph Γ , define the small, discrete, proof-irrelevant universe $U \in Ty(\Gamma)$ and the type decoder $T \in Ty(\Gamma.U)$ as:

- $|U_O(\gamma)| :=$ the set of small, discrete r.g. of assemblies
  $a \vDash_{U_O(\gamma)} S$ for any $a \in \mathscr{A}$, $S \in |U_O(\gamma)|$. **FNF:** Again maybe $a \vdash S$ iff $a = I$ is better for linearity reasons?

- $$\begin{aligned} U_R(\gamma_O) := \{(S, T, R, R_{src}, R_{tgt}) \quad | \quad & R \in \mathcal{U} \\ & S, T \text{ are small discrete r.g. of assemblies} \\ & \langle R_{src}, R_{tgt}\rangle : R \to |S_O| \times |T_O| \textit{ is injective}\} \end{aligned}$$

- $U_{refl}(\gamma_R)(S) := (S, |S|, S_{refl}, S_{src}, S_{tgt})$

- $U_{src}(\gamma_R)(S, T, R, R_{src}, R_{tgt}) = S$

- $U_{src}(\gamma_R)(S, T, R, R_{src}, R_{tgt}) = T$

and $T \in Ty(\Gamma.U)$ as:

- $T_O(\gamma_O, S) := S_O$

- $T_R(\gamma_R, (S, T, R, R_{src}, R_{tgt})) := R$

- $T_{refl}(\gamma_O, S) := S_{refl}$

- $T_{src}(\gamma_R, (S, T, R, R_{src}, R_{tgt})) := R_{src}$

- $T_{tgt}(\gamma_R, (S, T, R, R_{src}, R_{tgt})) := R_{tgt}$.

Claim. *U is closed under Π types.*

Given some r.g Γ and $S \in Ty(\Gamma)$, $T \in Ty(\Gamma.S)$, it suffices to show that $\Pi\pi TS \in Ty(\Gamma)$ is a small, discrete and proof-irrelevant family of r.g. of assemblies† [3] For brevity, let $V := \Pi\pi ST$.

Smallness follows by the closure under Π-types in the ambient set-theoretical universe $\mathcal{U}$.

To show discreteness,
For proof-irrelevance, take some $\gamma_R \in \Gamma$ and $f, g \in V_R(\gamma_R)$, s.t $\langle V_{src}(\gamma_R), V_{tgt}(\gamma_R)\rangle(f) = \langle V_{src}(\gamma_R), V_{tgt}(\gamma_R)\rangle(g)$. WTP $f = g$, by def. we get immediately that $f^{src} = g^{src}$ and $f^{tgt} = g^{tgt}$. Given that $(f^{src}, f^{tgt}, r_f), (f^{src}, f^{tgt}, r_g) \in V_R(\gamma_R)$, note that ,

$$\forall s \in S_R(\gamma_R).\langle T_{src}(\gamma_R, s), T_{src}(\gamma_R, s)\rangle(r_f(s)) = (f_O^{src}(S_{src}(\gamma_R)(s)), f_O^{tgt}(S_{tgt}(\gamma_R)(s)))$$
$$\forall s \in S_R(\gamma_R).\langle T_{src}(\gamma_R, s), T_{src}(\gamma_R, s)\rangle(r_g(s)) = (f_O^{src}(S_{src}(\gamma_R)(s)), f_O^{tgt}(S_{tgt}(\gamma_R)(s)))$$

Since T is proof-irrelevant, it follows directly that $r_f = r_g$ and thus $f = g$.

A free theorem

Theorem 6. *If* $\Gamma \vdash M : \Pi a \overset{0}{:} \mathbf{U}.\Pi\_ \overset{0}{:} \mathbf{T}a.\mathbf{T}a$*, then for any* $\Gamma \vdash X : \mathbf{U}$*,* $\Gamma \vdash MX = \mathbf{0} : \mathbf{T}X \to \mathbf{T}X$ *is sound when interpreted in the model constructed so far.*

Sketch Instantiate M_R with $\Gamma_{refl}(\gamma_O)$ and uncurry once to get:
$M_R(\Gamma_{refl}(\gamma_O), -) : \forall R \in \mathbf{U}_R(\Gamma_{refl}(\gamma_O)).\Pi 0(T_R(\Gamma_{refl}(\gamma_O)), R).(T_R(\Gamma_{refl}(\gamma_O)), R)$ (1)
By the def. of terms in our model, we have:
$U_{src}(\Gamma_{refl}(\gamma_O)).M_R(\Gamma_{refl}(\gamma_O)) = M_O(\gamma_O)$
$U_{tgt}(\Gamma_{refl}(\gamma_O)).M_R(\Gamma_{refl}(\gamma_O)) = M_O(\gamma_O)$
Expanding the def. in (1), we get:

$$\forall A, B \in U_O(\gamma_O), \forall R \subseteq A_O \times B_O,$$
$$(M_O(\gamma_O)(A_O), M_O(\gamma_O)(B_O)) \in (\Pi 0 R.R)_R$$

Instantiate B with $\mathbf{0}$.

There is no $\Gamma \vdash M \overset{1}{:} \Pi a \overset{0}{:} \mathbf{U}.\Pi\_ \overset{0}{:} \mathbf{T}a.\mathbf{T}a$

[3] it actually does not suffice, we implicitly assume a lemma that gives an isomorphism between terms of type U and small, discrete, p.i. r.g of assemblies†