# A Brief Introduction to Synthetic Differential Geometry

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### Motivation





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SDG attempts to work with the classical constructs of DG from an axiomatic standpoint. All of the backdrop of coordinate frames is done away with, and we start directly with geometric considerations.





### **Outline**

- 1 Introductory theory
  - Single Variable Calculus
  - A Primer on Toposes
- 2 The generalized axioms
  - Weil Algebras
  - The General KL Axiom
- 3 Microlinear Objects
  - The Tangent Bundle
  - Vector Fields and Flows





### The Derivative

"The rate of change, for very small changes"

$$f(x+h) = f(x) + f'(x)h + \mathcal{O}(h^2)$$

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#### **Axiom**

R is an algebra over  $\mathbb{Q}$ .



# Recreating the Derivative Axiomatically The Infinitesimals

Let D be the set

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#### **KL** Axiom

For all functions

$$f: D \rightarrow R$$
,

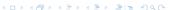
there exist unique  $a, b \in R$  such that

$$f(d) = a + bd$$

for all  $d \in D$ .

■ The coefficient a is easily seen to be f(0).





- The derivative can now be defined.
- Let  $f: R \to R$  be a function. Fix x, define  $g: D \to R$  by

$$g(d)=f(x+d).$$



■ By the KL axiom, there exist unique *a*, *b* with

$$g(d) = f(x+d) = a + bd$$

■ The first coefficient is g(0) = f(x).

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#### Definition

The derivative of f, written f'(x), is the unique coefficient b.



Thus we have

$$f(x+d) = f(x) + f'(x)d \quad \forall d \in D$$

# An Example

#### Example

Consider  $f(x) = x^n$ . Then

$$f(x+d) = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}d + K_2d^2 + \dots + K_nd^n$$
$$= x^n + nx^{n-1}d$$

Therefore  $f'(x) = nx^{n-1}$ .





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- We have to weaken logic to only constructive logic (no LEM).
- Toposes are the correct setting for SDG.
- The objects in a topos behave like sets.

[...] all notions, constructions, and proofs involved are presented as if the base category were the category of sets; in particular all constructions on the objects involved are described in terms of "elements" of them. However, it is necessary and possible to be able to understand this naive writing as referring to Cartesian closed categories. It is necessary because the basic axioms of synthetic differential geometry have no models in the category of sets (cf. 1 S. 1); and it is possible [...]

### Weil Algebras

A Weil algebra is something of the form

$$R[X_1,\ldots,X_n]/(P_1,\cdots,P_s)$$

(with some more conditions)

### Weil Algebras Spectra

#### Definition

Let  $W = R[X_1, \dots, X_n]/(P_1, \dots, P_s)$  be a Weil algebra. The spectrum of W in R is

$$\mathsf{Spec}_{\mathsf{R}} \; W = \{(a_1, \dots, a_n) \in R^n \; | \; P_j(a_1, \dots, a_n) = 0, \; j = 1, \dots, s\}$$

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#### Example

Consider 
$$W = R[X]/(X^2)$$
. In that case

Spec<sub>R</sub> 
$$W = \{a \in R \mid a^2 = 0\} = D.$$



### The General KL Axiom

• 
$$W=R[X_1,\ldots,X_n]/(P_1,\cdots,P_s)$$
. Define  $lpha:W o R^{\operatorname{Spec}_R W}$  by sending  $g(X_1,\ldots,X_n)$  to the map  $(a_1,\ldots,a_n)\mapsto g(a_1,\ldots,a_n)$ 



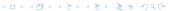
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#### **KL** Axiom

The map  $\alpha$  is an R-algebra isomorphism.





# The General KL Axiom Examples

- In the case  $W = R[X]/(X^2)$ , this is the first axiom we gave.
- Another case is  $W = R[X]/(X^{k+1})$ , which gives us the Taylor expansion of order k.



### Microlinear Objects

Define "2-infinitesimals":

$$D(2) = \{(d_1, d_2) \in D \times D \mid d_1 d_2 = 0\}$$

### Microlinear Objects

■ In particular, if *M* is microlinear then two maps

$$f,g:D\to M$$

define a unique map

$$h: D(2) \rightarrow M$$

such that

$$h(d,0)=f(d)$$

$$h(0,d)=g(d)$$





### Microlinear Objects

- To be thought of as "manifolds".
- We can define tangent vectors (and they will form a "vector space").





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#### Definition

Let *M* be microlinear. A tangent vector to *M* at *p* is a map

$$t:D\to M$$

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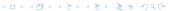
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■ With that, the tangent space at p,  $T_pM$ , is the set of all tangent vectors at p.





- The tangent space  $T_pM$  is an R-module.
  - $\bullet$  0 = the constant map p.
  - $\lambda t(d) := t(\lambda d)$
- For addition we need to use microlinearity.



# Addition of Tangent Vectors

■ Microlinearity guarantees that for  $t_1$ ,  $t_2$  tangent vectors, there exists a unique

$$s: D(2) \rightarrow M$$

such that

$$s(d,0) = t_1(d)$$
  
 $s(0,d) = t_2(d)$ 

■ Define  $(t_1 + t_2)(d) = s(d, d)$ 





### Addition of Tangent Vectors

The Neutral Element

- Recall we designated  $0 \equiv p$  (ct.)
- Let t be a t.v. The map  $s: D(2) \rightarrow M$  must satisfy

$$s(d,0) = t(d,0)$$
  
 $s(0,d) = 0(d) = p$ 

so s must be

$$s(d_1,d_2)=t(d_1),$$

and 
$$(t+0)(d) = s(d,d) = t(d)$$

# The Tangent Bundle

- It is simply the collection of all tangent spaces,  $M^D$ . We write TM.
- We have a projection  $\pi: TM \to M$  given by

$$t \mapsto t(0)$$



### **Vector Fields**

- They are sections of TM; maps  $X : M \to TM$  such that  $(\pi \circ X)(p) = p$ .
- The set of vector fields is denoted  $\mathfrak{X}(M)$ .





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  - Each  $T_pM$  is an R-module, so by fiberwise operations so is  $\mathfrak{X}(M)$ .
  - $\mathbf{X}(M)$  is also an  $M^R$ -module, by

$$(fX)(p) = f(p)X(p)$$





### **Flows**

 $\blacksquare$  Exponentials  $A^X$  have the property that

$$B \rightarrow A^X$$

is equivalent to

$$X \times B \rightarrow A$$

$$\left(\tilde{f}(x,b)=f(b)(x)\right)$$





### **Flows**

■ Vector fields are maps  $X : M \rightarrow M^D$ , therefore the same as maps

$$X: D \times M \rightarrow M$$
.

We write the same *X* because it's comfortable.

#### **Proposition**

Flows are additive.

#### Proof.

Let f, g : D(2) be defined by

$$f(d_1, d_2) = X(d_1 + d_2, p)$$
  

$$g(d_1, d_2) = X(X(d_1, p), d_2)$$

we have

$$f(d,0) = g(d,0)$$
  
 $g(0,d) = f(0,d)$ 

since M is microlinear, f = g.







### Summary

- In SDG we define geometric objects directly.
- This requires working axiomatically.
- The right context for these needs is that of topos theory.
- It's possible to recover the "same" constructs from classical geometry.

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- This requires working axiomatically.
- The right context for these needs is that of topos theory.
- It's possible to recover the "same" constructs from classical geometry.
- Of course, we need to construct a category which satisfies the axioms.
- These are models of SDG.



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