

A Brief Introduction to Synthetic Differential Geometry

Gaspar Pérez Scornik

Supervised by Xavier Gràcia Sabaté

Universitat Politècnica de Catalunya
Faculty of Mathematics and Statistics

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Motivation

SDG attempts to work with the classical constructs of DG from an axiomatic standpoint. All of the backdrop of coordinate frames is done away with, and we start directly with geometric considerations.

- 1 **Introductory theory**
 - Single Variable Calculus
 - A Primer on Toposes
- 2 **The generalized axioms**
 - Weil Algebras
 - The General KL Axiom
- 3 **Microlinear Objects**
 - The Tangent Bundle

The Derivative

- “The rate of change, for very small changes”

$$f(x + h) = f(x) + f'(x)h + \mathcal{O}(h^2)$$

Recreating the Derivative Axiomatically

- We begin with an object R . Naturally, this is the “synthetic version” of \mathbb{R} .

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Axiom

R is an algebra over \mathbb{Q} .

Recreating the Derivative Axiomatically

The Infinitesimals

- Let D be the set

$$D = \{d \in R \mid d^2 = 0\}.$$

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KL Axiom

For all functions

$$f : D \rightarrow R,$$

there exist unique $a, b \in R$ such that

$$f(d) = a + bd$$

for all $d \in D$.

- *The coefficient a is easily seen to be $f(0)$.*

Recreating the Derivative Axiomatically

- The derivative can now be defined.
- Let $f : R \rightarrow R$ be a function. Fix x , define $g : D \rightarrow R$ by

$$g(d) = f(x + d).$$

Recreating the Derivative Axiomatically

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Definition

The **derivative** of f , written $f'(x)$, is the unique coefficient b .

Recreating the Derivative Axiomatically

- Thus we have

$$f(x + d) = f(x) + f'(x)d \quad \forall d \in D$$

An Example

Example

Consider $f(x) = x^n$. Then

$$\begin{aligned} f(x+d) &= \binom{n}{0}x^n + \binom{n}{1}x^{n-1}d + K_2d^2 + \cdots + K_nd^n \\ &= x^n + nx^{n-1}d \end{aligned}$$

Therefore $f'(x) = nx^{n-1}$.

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- We have to weaken logic to only constructive logic (no LEM).
- Toposes are the correct setting for SDG.
- The objects in a topos behave like sets.

How is this possible?

[...] all notions, constructions, and proofs involved are presented as if the base category were the category of sets; in particular all constructions on the objects involved are described in terms of “elements” of them. However, it is necessary and possible to be able to understand this naive writing as referring to Cartesian closed categories. It is necessary because the basic axioms of synthetic differential geometry have no models in the category of sets (cf. I S. 1); and it is possible [...]

- A Weil algebra is something of the form

$$R[X_1, \dots, X_n]/(P_1, \dots, P_s)$$

(with some more conditions)

Definition

Let $W = R[X_1, \dots, X_n]/(P_1, \dots, P_s)$ be a Weil algebra. The **spectrum** of W in R is

$$\operatorname{Spec}_R W = \{(a_1, \dots, a_n) \in R^n \mid P_j(a_1, \dots, a_n) = 0, j = 1, \dots, s\}$$

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Example

Consider $W = R[X]/(X^2)$. In that case

$$\operatorname{Spec}_R W = \{a \in R \mid a^2 = 0\} = D.$$

The General KL Axiom

- $W = R[X_1, \dots, X_n]/(P_1, \dots, P_s)$. Define

$$\alpha : W \rightarrow R^{\text{Spec } R} W$$

by sending $g(X_1, \dots, X_n)$ to the map

$$(a_1, \dots, a_n) \mapsto g(a_1, \dots, a_n)$$

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KL Axiom

The map α is an R -algebra isomorphism.

The General KL Axiom

Examples

- In the case $W = R[X]/(X^2)$, this is the first axiom we gave.
- Another case is $W = R[X]/(X^{k+1})$, which gives us the Taylor expansion of order k .

- Define “2-infinitesimals”:

$$D(2) = \{(d_1, d_2) \in D \times D \mid d_1 d_2 = 0\}$$

Microlinear Objects

- In particular, if M is microlinear then two maps

$$f, g : D \rightarrow M$$

define a unique map

$$h : D(2) \rightarrow M$$

such that

$$h(d, 0) = f(d)$$

$$h(0, d) = g(d)$$

Microlinear Objects

- To be thought of as “manifolds”.
- We can define tangent vectors (and they will form a “vector space”).

Tangent Vectors

- Classically, they are equivalence classes of paths.

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Let M be microlinear. A **tangent vector** to M at p is a map

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- With that, the **tangent space** at p , $T_p M$, is the set of all tangent vectors at p .

Tangent Vectors

- The tangent space $T_p M$ is an R -module.
 - 0 = the constant map p .
 - $\lambda t(d) := t(\lambda d)$
- For addition we need to use microlinearity.

Addition of Tangent Vectors

- Microlinearity guarantees that for t_1, t_2 tangent vectors, there exists a unique

$$s : D(2) \rightarrow M$$

such that

$$s(d, 0) = t_1(d)$$

$$s(0, d) = t_2(d)$$

- Define $(t_1 + t_2)(d) = s(d, d)$

Addition of Tangent Vectors

The Neutral Element

- Recall we designated $0 \equiv p$ (ct.)
- Let t be a t.v. The map $s : D(2) \rightarrow M$ must satisfy

$$s(d, 0) = t(d, 0)$$

$$s(0, d) = 0(d) = p$$

so s must be

$$s(d_1, d_2) = t(d_1),$$

$$\text{and } (t + 0)(d) = s(d, d) = t(d)$$

The Tangent Bundle

- It is simply the collection of all tangent spaces, M^D . We write TM .
- We have a projection $\pi : TM \rightarrow M$ given by

$$t \mapsto t(0)$$





Summary

- In SDG we define geometric objects directly.
- This requires working axiomatically.
- The right context for these needs is that of topos theory.
- It's possible to recover the “same” constructs from classical geometry.

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- Of course, we need to construct a category which satisfies the axioms.
 - These are **models** of SDG.

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