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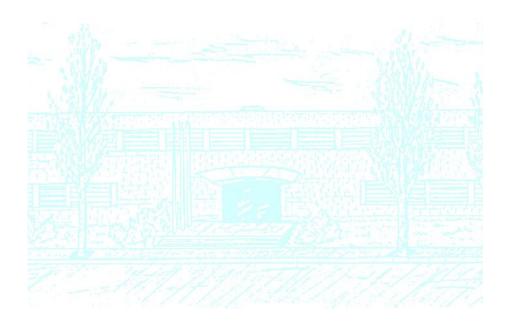
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A Brief Introduction to Synthetic Differential Geometry

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Abstract

ABSTRACT. The goal of this thesis is to explore the basic axiomatic theory of Synthetic Differential Geometry (SDG). This field aims to put the study of smooth manifolds, and geometry therein, in a topos-theoretic framework. Though the full depth of application and consequences of SDG require knowledge of topos theory to comprehend, a large part of the theory can be appreciated with only some notions of basic category theory (as well as with a standard undergraduate mathematics syllabus). In this work we look at this part of SDG, called the "axiomatic" theory because it is indeed developed axiomatically. Specifically, under the axiomatic theory of SDG we look at differential calculus, then "manifolds" (their analogue in SDG), vector bundles (the tangent bundle as a particular case), and vector fields (and Lie algebras thereof).

RESUM. L'objectiu d'aquest treball és el d'explorar la teoria axiomàtica bàsica de l'anomenada *Geometria Diferencial Sintètica* (GDS). La GDS es refereix a un camp que té l'objectiu d'enquadrar l'estudi de les varietats diferenciables i la seva geometria dins el marc de la teoria de topos. Malgrat que l'apreci complet de les aplicacions i conseqüències d'aquesta teoria requereixen conèixer la teoria de topos, una gran part de la teoria es pot tractar amb només unes nocions bàsiques de teoria de categories (juntament amb una educació en matemàtiques a nivell de grau). En aquest treball visitem aquesta part de la teoria, que s'anomena "axiomàtica" perqué efectivament ho és. En particular sota la teoria axiomàtica de la GDS tractem el càlcul diferencial, varietats (el seu anàleg), fibrats vectorials (el fibrat tangent en especial), i finalment camps vectorials (amb la seva estructura d'àlgebra de Lie).

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Contents

Preface

The title of this work may sound strange if one is unfamiliar with the topic. Of course, that is true of anything in life, but my experience in particular when stating the phrase "Synthetic Differential Geometry" to my mathematician colleagues (students and professors) is of being met immediately with a confused look, albeit one of intrigue. If they are a classmate of mine then they also will have likely not seen any category theory (what's that?) as it is not covered in the curriculum. I was in the same position when I did perchance upon the topic, and I was sufficiently intrigued so as to explore it myself and compile my findings into a short document that would hopefully serve as a light introduction for undergraduate students of mathematics.

What is *synthetic* about synthetic differential geometry (SDG)? Quite simply, it means that it is an attempt to study, or at times simply describe the behavior of smooth manifolds (and smooth functions) from an axiomatic framework. To help explain this, we'll make an analogy with modern Euclidean planar geometry ($\mathbb{R}^2$ with a scalar product) vs. "synthetic Euclidean geometry", which is the name we give to the study of geometry as the Greeks did, from axioms concerning the elemental objects of geometry: points and lines, and relations between them. Euclides' five axioms were not completely rigorous by today's standards, but in that case we simply refer to any modern revisiting of axiomatic Euclidean geometry, such as the works of Hilbert or Tarski. The important point is the contrast with doing geometry with an analytic backdrop ($\mathbb{R}^2$), from which we define lines and points set-theoretically. In axiomatic geometry one begins directly with lines and points and describes how they interact.

A highly similar approach is taken with synthetic differential geometry. Naturally, there is a considerable complexity gap between flat 2-space, and general smooth manifolds. In accordance with the jump in complexity between the objects we wish to study there accompanies such a jump in the type of axioms we need. In particular, synthetic differential geometry begins by describing requirements for a certain *category*. If the reader is not familiar, I would highly recommend they read one or both of [7] and [6]. One wouldn't have to read the entirety of either to understand most of what I write here, but I would certainly say that readers will find themselves completely comfortable with anything encountered here, having

read the first five chapters of [**6**]. From here on this knowledge is assumed, to the aim of avoiding highly verbose detours into purely category theoretic details.

After parsing what is meant (in informal terms) by "synthetic" differential geometry as we have done, the immediate reaction is to ask why we'd like to do that (other than for pure interest, which is not a bad answer). One practical answer is that it can give a rigorous foundation to certain ways that geometers would *like* to think, such as the notion of "infinitesimals" (this is one of the first constructions in SDG here and in many sources), or higher order concepts such as the interpretation of vector fields as "infinitesimal flows". I am not a geometer, and my own motivation for studying SDG was in large part out of an interest in *any* modern category-theoretic construction, but we could say that "thinking synthetically" in differential geometry was an idea that held much appeal, whatever I believed it meant at the time. I quote Anders Kock, quoting Sophus Lie in [**3**], should it spark the same intrigue it did in me some time ago:

> What is meant by "synthetic" reasoning? Of course, we do not know exactly what Lie meant, but the following is the way we would describe it: It deals with space forms in terms of their structure, i.e. the basic geometric and conceptual constructions that can be performed on them. Roughly, these constructions are the morphisms which constitute the base category in terms of which we work; the space forms themselves being objects of it.

SECTION 1

Introduction

This principal objective of this thesis was to research the topic of synthetic differential geometry, both for the sake of learning about the discipline itself, as well as to gain practical experience in the realization of mathematical research. With regards to the former, on the outset the goal was to examine what could be meant by "an axiomatic treatment of differential geometry", and to discern in which way this treatment could reproduce what was already known from "standard" geometry. In many ways SDG does accomplish this (without meaning to imply that the original authors of SDG intended for it to exactly replicate classical geometry); we have seen differential calculus, manifolds, tangent spaces and vector fields in their synthetic form. Many other classical constructions are also possible as will be described below.

In second place we'd also note that before this research preceded an independent study of category theory (and to some extent topos theory), as it is the basic language of the theory of SDG. Conversely, in studying a practical application of category theory - SDG - a greater degree of fluency in the basics of category theory was gleaned (and we could say this was a secondary objective).

1.1. Overview

The written work is organized as follows. Section one begins by laying down (a reduced version of) the axiomatic framework in which the rest of the text takes place. As mentioned, we can summarize that framework as to say "there exists a category with these properties". Again, the reader with a basic grasp of category theory will get along fine. Strictly speaking one should be familiar with the notion of a *topos*, but we do not dedicate any large portion to the details, instead taking what "high-level" properties we need for the rest of our discourse. Those abstract (and convenient) properties of toposes will be detailed. The end result is that we end up with an object R, the smooth line, which is more or less the synthetic version of $\mathbb{R}$, the reals. Particularly, there is a subset of R, called D throughout, which are to be the embodiment of "first order infinitesimals". In analysis we may sometimes like to think of "infinitesimal increments", but of course there exists no such a thing. We have suggestive notation such as $f(x + h) = f(x) + f'(x)h + \mathcal{O}(h^2)$, and we may even think of the last term as "negligible" for small enough h. Perhaps not in

analysis proper anyway, but it is known as a useful technique for rough estimations in, say, numerics. With our object D this is made a reality. Elements of d are "so small" that their square is actually 0. Using D and a strong axiom about their behavior it is possible to revisit single and multivariable calculus, and such is done. By introducing further axioms one can speak of integration as well, but this is not covered here, and instead we refer the reader to [**5**] for a discussion on this.

In the second section further (and final) axiomatics are given. In this way we mimic the style of many sources of SDG, which introduce successively stronger axioms until reaching the desired general statement. It is "general" in the sense that every axiom up till that point becomes a particular case. The reasons for going about it this way are twofold. One, because as an introductory text it's appropriate to ease the reader in. Though mentioned that category theory is required, the first axiom and the entire first section are very much palpable for any student of mathematics (this may sound confusing or contradictory but it becomes clear upon reading). Secondly, the nature of studying SDG itself often takes the same approach. Axioms are introduced as they are needed, and attention is given to which theories are possible within one set of axioms or another. In fact, this is indicated as the method for studying SDG in [**1**]. In any case, the second section of this text arrives at the general axiomatics and employs them henceforth.

Until the mentioned point the main object of study is R. As classical geometry moves from $\mathbb{R}$ (or $\mathbb{R}^n$) to smooth manifolds, so do we explore "manifolds" in the third section. Instead of manifolds they are called *microlinear objects*, because that is what they are. That is, "microlinearity" is a property of certain objects which essentially demands that they behave "similar to R", in a way that is made precise. And, as it will result, microlinearity is a sufficient condition for effectively carrying out the constructions we'd desire in differential geometry, namely the tangent bundle and vector fields. Furthermore, microlinear objects are sufficiently well-behaved under typical constructions such as the Cartesian product, yet they are necessarily more general than classical manifolds, for one because "function sets" N^M where M, N are microlinear, are too microlinear, something which cannot be said if we replaced "microlinear" with "smooth manifold".

Having said that microlinear objects are "more general" than manifolds, section four deals with ways in which they are very similar. Taking advantage of the properties of microlinear objects we can define tangent spaces in an eminently geometric way. In classical geometry tangent vectors can be defined as equivalence classes of curves passing through a point, parametrized by some interval. Here a tangent vector is literally a single infinitesimal curve, a map $t : D \to M$, from the infinitesimal interval to the space in question. The collection of such curves (through a fixed point) exhibits a vector space structure, promptly named the tangent space at a

point. The tangent bundle follows, and so do vector fields as sections of it. Finally, the set of vector fields on a space will be found to form a Lie algebra.

The last section dedicates just a few paragraphs to outlining the subject of *models* of SDG. As with any list of axioms, if we eventually wish to relate the resulting theory to the rest of existing mathematics, then we must produce a model (the term has many technical definitions - in the mentioned section it will be clearly defined) which verifies those axioms. Concerning SDG in particular, we are interested in models that allow us to compare results in SDG to those in classical geometry. These are called well-adapted models and are also commented on briefly.

1.2. Concluding remarks and further study

The focus of this thesis was foremostly on taking a first look at the theory of SDG, as well as gaining practice with the application of category theory in modern mathematics (particularly as relates to geometric fields). We were able to find that one can start from purely axiomatic foundations, and arrive at the "same" familiar world of manifolds and smooth maps from classical geometry.

Additionally, what is written here is only a small portion. In the literature one will find still more topics from classical geometry in their synthetic form. To name a few, these are differential forms - the exterior differential, integration of forms, leading up to De Rham theory [5]; connections - covariant differentiation, torsion, curvature [5], metrics - Riemannian and pseudo-Riemannian metrics, the Levi-Civita connection [4]; and Hamiltonian mechanics [5, 8]. The approach taken for those topics is of varying complexity, but altogether in line with the types of reasoning employed in this thesis and are excellent candidates for us to study in the future.

Right alongside, or inseparable from the depth of geometric material that's possible to explore from here, are the subjects of category theory, categorical logic, sheaf theory, topos theory, etc. To say that those are worthy of multiple theses in their own right would be a criminal understatement. However, in researching the matter of this thesis, a solid ground has been laid on which to continue building. For any interested readers, a very recent publication in SDG can be found in [1], in which topos theory and categorical logic, in contrast to the present text, are at the forefront of discourse.

1.3. Bibliographical remarks

Much of the main matter in this thesis is drawn in combination from [3] and [5], and between those two the former plays a bigger part, at least in direct citations or use of arguments. The two other main sources are [1] and [4], which are cited only

a few times but were useful as resources, seeing as the four books cover much of the same material in different ways, thus leading to a more complete understanding.

There are two books on category theory in the bibliography, [**6**] and [**7**]. Since the material is common to any introductory category theory text, and the style of this text is to take it as common knowledge, they are not cited anywhere directly. They have a place in the bibliography simply because they were the main foundation of an extensive previous study of category theory, and are recommended for any readers who would like to do the same.

In the short section on constructive logic, [**2**] was consulted in order to complete a particular argument. Lastly, [**8**], though consulted extensively at an earlier phase of research, is mentioned only in the front matter as a reference for further study, in particular that of Hamiltonian mechanics.

1.4. Historical note

The majority of the literature points to SDG originating with William Lawvere, whose doctoral advisor was Samuel Eilenberg, co-founder of the theory of categories. In particular, the subject is first introduced in Chicago 1967, in his seminar *Categorical Dynamics*. That is to say, *dynamics*, from the point of view of category theory (as opposed to "the dynamics of categories"). From there the subject gained collaborators, and the first book on SDG was published by Anders Kock in 1981 (here we consult the 2006 reprint of that book, [**3**]). During or after that time many other works were published, by authors such as Belair, L.; Bunge, M.; Dubuc, E.J.; Koszul, J.-L.; Lavendhomme, R.; Moerdijk, I.; Penon, J.; Reyes, G.E.; and Wraith, G.E., among others. If one consults those sources they would find that, according to them, they are picking up on the works left by mathematicians before them in similar fields, notably Henri Cartan, Charles Ehresmann, Alexander Grothendieck, and André Weil.

SECTION 2

The basic theory

The axiomatic theory of Synthetic Differential Geometry begins by assuming a certain topos $\mathcal{E}$ exists. A topos is a category with certain properties which allow us to to abstract the way in which the category of sets behaves. In fact, most naive set theoretic and logical notions have meaningful interpretation in terms of objects and morphisms of a topos. For example, comprehension by formulas such as

$$\{x \in A \mid f(x) = 0\}$$

is possible in a topos (at least for polynomial f, which is all that we need here), and it behaves in the way we would expect it to. Some authors prefer to use different notation for those types of constructions, to distinguish "internal" constructions in the topos, from regular set theory. For example the above might be written as

$$[[x \in A \mid x^2 = 0]]$$

We will not make such distinctions here, but do mention it now so as to make clear that the set-theoretic and logical constructions that appear throughout are those internal to $\mathcal{E}$. We will also use the terms set and object interchangeably, though both refer to "objects of $\mathcal{E}$".

Another fundamental (defining) property of toposes is that of the existence of exponential objects. Again, this is completely analogous to a familiar property of sets. Given sets A, X, we can always form the exponential:

$$A^X = \{f \mid f : X \to A\}$$

and it satisfies the universal property that for any set B, giving a function

$$B \to A^X$$

is equivalent to giving a function

$$X \times B \to A$$

In a topos we can always form exponential objects, and they too are characterized by the mentioned universal property.

Thus, from this point on we will mainly omit the fact that we are working in a topos, and use standard set theoretic notation, giving only brief reminders now and then. There are of course limitations - not every formula in set theory can

be meaningfully interpreted as an internal formula of a topos, particularly when it comes to logic. Notably, we do not have access to nonconstructive logic. However, this will also be demanded by our geometric theory, as will be seen very shortly.

It is duly noted that a vast oversimplification has been made of the details of the internal language of toposes. However there exists ample literature on the subject; the reader is urged to consult [**1**] or [**3**] for topos theory preliminaries. There are of course more primary sources, the cited works are simply those that discuss the preliminaries in the context of SDG.

2.1. Axiomatics

We begin with an object R. We can think of R as an extension of the notion of the standard continuum, $\mathbb{R}$. Thus R should be a commutative ring with unit. However, due to an axiom that we will shortly introduce, we cannot ask that R be a field. Later on, we will need to strengthen this requirement to, for example "2:=1+1 is invertible". But later on we may need that "3 is invertible". Instead of introducing these small axioms successively, we will take care of them succinctly with our:

AXIOM 2.1. *R is a $\mathbb{Q}$-algebra.*

With that out of the way, we proceed to introduce a central object of study, the "infinitesimals". These come in the form of nilpotent elements of R. To be precise, let $D = \{d \in R \mid d^2 = 0\}$. The (first version of the) defining axiom of SDG concerns D, and as in all the literature we refer to it as the Kock-Lawvere axiom.

KL AXIOM 1. *For all $f : D \to R$ there exist unique $a, b \in R$ such that $f(d) = a + bd \, \forall d \in D$*

Note that, in particular, $a = f(0)$. Of course, this is a strong requirement. So strong in fact, that we will have to weaken our logic in order for the theory to be remotely interesting. This is because of the following proposition.

PROPOSITION 2.1. *$R = \{0\}$*

PROOF. Let $f : D \to R$ be defined by

$$f(d) = \begin{cases} 1, & d \neq 0 \\ 0, & d = 0 \end{cases}$$

By the KL axiom 1, there exist unique $a, b \in R$ such that $f(d) = a + bd$ for all d. Since $a = f(0) = 0$, there exists $b \in R$ such that $f(d) = bd$ for all d. Thus for non-zero d we have $1 = bd$. Squaring both sides yields $1 = 0$, and since R is a ring this concludes the proof. $\square$

However, the above proof relies on the "fact" that, for any $x \in R$ we have either $x = 0$ or $x \neq 0$. Indeed the well-definedness of the function f depends on this assertion. This is true in classical logic, as it is a simple application of the **Law of the Excluded Middle** (LEM):

$$\forall p \, p \vee \neg p$$

To carry on with our theory, we see that we have to reject LEM. The above proof then fails, but of course just because of that we cannot conclude that the theory functions properly. To do so would by all means require exhibiting the category $\mathcal{E}$ explicitly. For now we will continue ignoring this, and develop the naive theory, knowing that we have to abstain from assuming LEM.

Without LEM, proposition 2.1 of course still contradicts (because we do want R to be a $\mathbb{Q}$-algebra) the statement

(2.1) $$\forall d \in D \, (d = 0) \vee \neg(d = 0)$$

In other words, its negation is true. One may be inclined to "simplify" the negation statement, leading to the apparent absurdity that

$$\exists d \in D \, \neg(d = 0) \wedge \neg\neg(d = 0)$$

Note that we haven't eliminated the double negation, since $\neg\neg p \to p$ is equivalent to LEM. Still, the above would be a contradiction, since $p \wedge \neg p$ is still false in intuitionistic logic. But, the first equivalence we used on the universally quantified statement (2.1) also fails to be valid in intuitionistic logic (see [**2**, p. 248] for example).

So what are the elements of D? For one they are not all zero, for if they were then every function $d \mapsto ad, a \in R$ would be equal to the constant 0, but since each coefficient a is unique, we again reach the contradiction that every $a \in R$ is the same (equal to zero). The conclusion is the simple statement that

$$\exists d \in D \, \neg\neg(d = 0)$$

The infinitesimals D can thus be neither zero nor nonzero. This is a contradictory statement in classical logic, but it is precisely of the type of phrases that we should expect to appear if we weaken our logic to intuitionistic logic.

2.2. Elementary Calculus

With our basic axiom scheme in place we can begin revisiting the classical notion of a derivative in this new context. We begin, as expected, with a function $f : R \to R$. Fix $x \in R$ and define a new function $g : D \to R$ by

$$d \mapsto f(x + d)$$

By the KL axiom 1 there are unique $a, b \in R$ such that

$$\forall d \in D \, f(x + d) = f(x) + bd$$

(since $g(0) = f(x)$). This naturally leads to the following

DEFINITION 2.2. Let $f : R \to R$. The *derivative* of f at x, denoted $f'(x)$, is the unique $b \in R$ such that $f(x + d) = f(x) + bd \, \forall d \in D$.

Clearly this also allows us to define the derivative *function* $f' : R \to R$. For instance, let us calculate the derivative of a simple polynomial function. Let $f : x \mapsto x^2 + x$ then

$$f(x + d) = x^2 + 2xd + d^2 + x + d$$
$$= x^2 + x + (2x + 1)d$$

by which $f'(x) = 2x + 1$. So far our "formal" calculus agrees with standard calculus. In fact, we have the following proposition:

PROPOSITION 2.3. *Let $f, g : R \to R$ and let $\alpha \in R$. The following hold:*
(1) $(f + g)' = f' + g'$
(2) $(\alpha f)' = \alpha f'$
(3) $(fg)' = f'g + fg'$

PROOF. The proof is as simple as one could suspect, here we only prove the third identity as an example. We have:

$$f(x + d)g(x + d) = (f(x) + f'(x)d)(g(x) + g'(x)d)$$
$$= f(x)g(x) + (f'(x)g(x) + f(x)g'(x))d + f'(x)g'(x)d^2$$
$$= f(x)g(x) + (f'(x)g(x) + f(x)g'(x))d$$

At a given $x \in R$, this clearly holds for all $d \in D$, giving us the result. $\square$

Furthermore we have the Chain Rule:

PROPOSITION 2.4. *Let $f, g : R \to R$ be two functions. Then the derivative of the composite $g \circ f$ is*

$$(f \circ g) = (f' \circ g) \cdot g'$$

PROOF. Let $x \in R$. Then

$$(f \circ g)(x + d) = f(g(x)) + (f \circ g)'(x)d$$

by definition. Meanwhile, if we expand $g(x + d)$ first:

$$(f \circ g)(x + d) = f(g(x + d))$$
$$= f(g(x) + g'(x)d)$$

Observe that $g'(x)d \in D$, since $(g'(x)d)^2 = g'(x)^2d^2 = 0$. Now, $f'(g(x))$ is the unique coefficient satisfying

$$f(g(x) + g'(x)d) = f(g(x)) + f'(g(x))g'(x)d$$

$\square$

The reader may notice that these statements seem lazily worded. Normally, we would include in the hypotheses something like "*Let $f, g : R \to R$ be differentiable functions. Then* (some other function) *is differentiable, and ... etc.*" We are not in fact forgetting hypotheses. The Kock-Lawvere Axiom 1 actually implies that *every* function from R to R is differentiable. This is easy to verify by noting that in the definition of the derivative (2.2), no additional assumptions are made other than f being a function. As a direct consequence, every function from R to R is infinitely differentiable.

This leads us to ask if we also have Taylor expansions of functions. In classical calculus, a Taylor expansion of second order involves a second degree polynomial, and "terms of order 3". In our context, these are simply nilcube elements, i.e. $\delta \in R : \delta^3 = 0$. An example of such a nilcube is any $d_1 + d_2$ where d_1, d_2 are both in D. In effect,

$$(d_1 + d_2)^3 = d_1^3 + d_2^3 + 3d_1^2 d_2 + 3d_1 d_2^2 = 0$$

Furthermore, if $f : R \to R$ is a function,

$$f(x + d_1 + d_2) = f(x + d_1) + f'(x + d_1)d_2$$
$$= f(x) + f'(x)d_1 + f'(x)d_2 + f''(x)d_1 d_2$$

and meanwhile,

$$(d_1 + d_2)^2 = 2d_1 d_2$$

Joining the two together yields:

$$f(x + d_1 + d_2) = f(x) + f'(x)(d_1 + d_2) + \frac{f''(x)}{2}(d_1 + d_2)^2$$

What we have just proven is the following proposition.

PROPOSITION 2.5. *If $f : R \to R$ is a function and $d_1, d_2 \in D$, then $\delta = d_1 + d_2$ satisfies*

$$f(x + \delta) = f(x) + f'(x)\delta + \frac{f''(x)}{2}\delta^2$$

This justifies the earlier heuristic of considering nilcubes as candidates for second order Taylor expansions. Unfortunately, there is no way to assert as of now that *any* nilcube must be of the form $d_1 + d_2$ with $d_1, d_2 \in D$. It is a partial result. In section 2.2 we will introduce Taylor expansions for every nilpotent element by way

of an axiom that generalizes the KL axiom 1. Still, it serves as a nice exercise to prove the following, more general, Taylor formula.

PROPOSITION 2.6. *Let $f : R \to R$ be a function. Then for all $n \in \mathbb{N}$ and $\delta = d_1 + \cdots + d_n$*

$$f(x + \delta) = f(x) + f'(x)\delta + \frac{f''(x)}{2!}\delta^2 + \cdots + \frac{f^{(n)}}{n!}\delta^n$$

where $x \in R$, $d_1, \ldots, d_n \in D$

PROOF. To begin with, let us look at the powers of δ, that is, the usual multinomial formula:

$$\delta^k = (d_1 + \cdots + d_n)^k = \sum_{i_1 + \cdots + i_n = k} \binom{k}{i_1, \ldots, i_n} \prod_{r=1}^{n} d_r^{i_r}$$

Recall that the d_r are in D, so only the only non-zero summands will be those with each d_r raised to either 1 or 0. That is, choosing a subset of the i_r to be 1, and the rest 0. In that case the multinomial coefficients are simply $k!$, giving us

$$\delta^k = \sum_{\substack{I \subset \{1, \ldots, n\} \\ |I| = k}} k! \prod_{t \in I} d_t$$

in other words, the *k-th* elementary symmetric polynomial of n variables (multiplied by $k!$). Denote this by $e_k(X_1, \ldots, X_n)$, and convene $e_0 \equiv 1$ and $e_k(X_1, \ldots, X_n) = 0$ if $k > n$. We proceed inductively:

$$f(x + \delta) = f(x + d_1 + \cdots + d_n)$$

$$= f(x + d_1 + \cdots + d_{n-1}) + f'(x + d_1 + \cdots + d_{n-1})d_n$$

$$= \sum_{i=0}^{n-1} \frac{f^{(i)}(x)}{i!} i! e_i(d_1, \ldots, d_{n-1})$$

(2.2)

$$+ d_n \sum_{i=0}^{n-1} \frac{f^{(i+1)}(x)}{i!} i! e_i(d_1, \ldots, d_{n-1})$$

$$= \sum_{i=0}^{n-1} f^{(i)}(x) \left(e_i(d_1, \ldots, d_{n-1}) + d_k e_{i-1}(d_1, \ldots, d_{n-1})\right)$$

$$+ d_k f^{(n)}(x) e_{k-1}(d_1, \ldots, d_{k-1})$$

It's now useful to use the following recursion that elementary symmetric polynomials satisfy:

$$e_k(X_1, \ldots, X_n) = e_k(X_1, \ldots, X_{n-1}) + X_n e_{k-1}(X_1, \ldots, X_{n-1})$$

Finally, observe that $e_{k-1}(d_1, \ldots, d_{k-1})$ is just $d_1 d_2 \cdots d_{k-1}$, meaning that

$$d_k e_{k-1}(d_1, \ldots, d_{k-1}) = e_k(d_1, \ldots, d_{k-1}, d_k)$$

With these two identities, 2.2 becomes:

$$\sum_{i=0}^{n} f^{(i)}(x) e_i(d_1, \ldots, d_k) = \sum_{i=0}^{n} \frac{f^{(i)}(x)}{i!} \delta^j$$

$$\square$$

We'd do well to note that, for $n \geq k+1$, δ^n is 0. The above Taylor formula is of course still valid for any k and n, but the terms are all zero after $k+1$. We again find a unique characterization of all functions on a certain set of nilpotents. This property will be stated in section 3 as a general axiom.

2.3. Calculus of several variables

We now take the usual course in calculus, which is to generalize the previous study to functions of more than one variable, i.e. $R^n \to R$, and further on functions $E \to V$ where E and V are "vector spaces" in a sense that will be made precise.

2.3.0.1. *Scalar functions.* Let $f : R^n \to R$ be a function, and $r = (r_1, \ldots, r_n) \in R^n$. As one would expect, we calculate the partial derivatives by adding an infinitesimal increment along a single coordinate direction. That is, let $g : D \to R$ be defined by

$$d \mapsto f(r_1 + d, \ldots, r_n)$$

By the KL axiom 1, there exists a unique $b \in R$ such that

$$f(r_1 + d, \ldots, r_n) = f(r) + bd$$

We define $\frac{\partial f}{\partial x_1}(r_1, \ldots, r_n)$ as b, which simultaneously defines a function $\frac{\partial f}{\partial x_1} : R^n \to R$ Similarly, we define $\frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_n}$. By iterating the process we obtain higher partial derivatives, denoted

$$\frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}$$

A nice result is a parallel of Schwarz' theorem of the interchangeability of partial derivatives.

PROPOSITION 2.7. *Let $f : R^n \to R$ be a function. Then for any $1 \leq i, j \leq n$*

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

The previous result follows directly from the following lemma which generalizes the second-order Taylor expansion to two dimensions.

LEMMA 2.8. *Let $f : R^n \to R$ be a function. Then for all $(d_1, d_2) \in D \times D$*

$$f(r_1, \ldots, r_i + d_1, \ldots, r_j + d_2, \ldots, r_n) = f(x_1, \ldots, x_i, \ldots, x_j + d_2, \ldots, x_n) +$$

$$+ \frac{\partial f}{\partial x_i}(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) d_1 + \frac{\partial^2 f}{\partial x_i \partial x_j}(x_1, \ldots, x_n) d_2 d_1$$

PROOF. Apply the definition of the partial derivative in x_i, and successively in x_j

$$f(r_1, \ldots, r_i + d_1, \ldots, r_j + d_2, \ldots, r_n) =$$

$$= f(x_1, \ldots, x_i, \ldots, x_j + d_2, \ldots, x_n)$$

$$+ \frac{\partial f}{\partial x_i}(x_1, \ldots, x_i, \ldots, x_j + d_2, \ldots, x_n) d_1$$

$$= f(x_1, \ldots, x_n) + \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n) d_1 + \frac{\partial^2 f}{\partial x_i \partial x_j}(x_1, \ldots, x_n) d_2 d_1$$

and $f(x_1, \ldots, x_n), \frac{\partial f}{\partial x_i}(x_1, \ldots, x_n), \frac{\partial f}{\partial x_j}(x_1, \ldots, x_n), \frac{\partial^2 f}{\partial x_i \partial x_j}(x_1, \ldots, x_n)$ are the unique coefficients that satisfy this for universally quantified $(d_1, d_2) \in D$. □

On the other hand, exchanging d_1 and d_2 (and commuting products and sums) in the above equations yields the same conclusion for $\frac{\partial^2 f}{\partial x_j \partial x_i}$. Since they are unique they must equal each other, proving (2.7). By induction we also obtain that the order in which we differentiate higher partial derivatives does not matter.

2.4. Vector functions

We will now examine functions between R-modules. However, for a general R-module V, it does not follow from the KL axiom (1) that a version of it holds for functions $D \to V$. In this case what we do is simply restrict our attention to those R-modules where this is the case. This leads to the following definition:

DEFINITION 2.9. An R-module V is said to be a *Euclidean R-module* if the vector form of the KL axiom holds. That is, for any function $f : D \to V$,

$$\exists!\, \underline{a}, \underline{b} \in V \text{ such that } f(d) = \underline{a} + d\underline{b}\, \forall d \in D$$

(again $\underline{a}$ is evidently $f(0)$)

This may seem like a very ad-hoc definition, but we will encounter many Euclidean R-modules naturally. For example we have the following lemma.

LEMMA 2.10.
  (1) R^n *is a Euclidean R-module.*
  (2) *For any object X, and Euclidean R-module V, the exponential V^X is a Euclidean R-module.*

PROOF.
  (1) This is evident since a function $f : D \to R^n$ is uniquely determined by each of its component functions. We let $f_i = \pi_i \circ f$ where $\pi_i : R^n \to R$ is the *i-th* coordinate projection. Then $\underline{a} = (a_1, \ldots, a_n), \underline{b} = (b_1, \ldots, b_n)$ are the unique coefficients in the claim. Here each a_i, b_i is obtained by applying the KL axiom to f_i.
  (2) First of all, V^X is an R-module by the pointwise sum of maps. Let $f : D \to V^X$ be a function. This defines, for each $x \in X$, a function $g(x) : D \to V$ by
$$d \mapsto f(d)(x)$$
  since V is Euclidean, there exist unique $a(x), b(x)$ such that
$$f(d)(x) = \underline{a}(x) + d\underline{b}(x)\, \forall d \in D$$
  Evidently, varying x defines functions $\underline{a}, \underline{b} : X \to V$, which are unique such that $f(d) = \underline{a} + d\underline{b}\, \forall d \in D$

$\square$

Additionally, it's clear that any V which is isomorphic to R^n as an R-module is too Euclidean. In [**4**] for example, it begins directly with this notion, calling them "finite dimensional vector spaces". Another fundamental example of a Euclidean R-module that we'll study are tangent spaces, in section 5.

For now let us go back to doing calculus. The Euclidean structure allows one to define directional derivatives, as is done in [**5**], for example.

DEFINITION 2.11. Let V, E be Euclidean R-modules, $f : V \to E$ a function and $\underline{u}, \underline{a} \in V$. The derivative of f at $\underline{a}$ in direction $\underline{u}$ is the unique $\underline{b} \in E$ such that

$$\forall d \in D,\ f(\underline{a} + d\underline{u}) = f(\underline{a}) + d\underline{b}$$

We denote this by $\partial_{\underline{u}} f(a)$.

We note that $\partial_{(-)}f(\underline{a})$ is a linear function. In other words, for all $\lambda, \mu \in R$,

$$\partial_{\lambda\underline{u}+\mu\underline{v}}f(\underline{a}) = \lambda\partial_{\underline{u}}f(\underline{a}) + \mu\partial_{\underline{v}}f(\underline{a})$$

The proof is a straightforward exercise (one may consult [**5**, p. 13]).

If V is finitely generated, i.e. $V = \text{span}\{e_1, \ldots, e_n\}$, then one can define

$$\frac{\partial f}{\partial x_i}(\underline{a}) = \partial_{e_1}f(\underline{a})$$

and if $\underline{u} = \sum_{i=1}^{n} u_i e_i$ then by linearity

$$\partial_{\underline{u}}f(\underline{a}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}u_i$$

Finally, the familiar total differential makes an appearance here too.

DEFINITION 2.12. Let $f : V \to E$ be a function. The *differential* of f at $\underline{a} \in V$ is the function $df(\underline{a}) : V \to E$ defined by

$$\underline{u} \mapsto \partial_{\underline{u}}f(\underline{a})$$

As noted before, $df(\underline{a})$ defines a linear map $V \to E$, for each $a \in V$. It will also be useful to prove that, if f is linear, it is equal to its own differential. First we need a

LEMMA 2.13. *Let V be an R-module, and E a Euclidean R-module. Let $f : V \to E$ be such that*

$$f(\lambda\underline{a}) = \lambda f(\underline{a})$$

for all $\underline{a} \in V$ and λ in R (f is homogeneous). Then $\forall \underline{a} \in V$, $df(\underline{a}) = df(\underline{0})$.

PROOF. Let $\underline{a} \in V$. Since E is Euclidean, we have

$$f(\underline{x} + d\underline{u}) = f(\underline{x}) + d\partial_{\underline{u}}f(\underline{x})$$

for all $d \in D$ and $\underline{x}, u \in V$. Letting $x = \lambda a$ and $d = \lambda\bar{d}$ for some $\bar{d} \in D$, we find that

$$f(\lambda\underline{a} + \lambda d\underline{u}) = f(\lambda\underline{a}) + \lambda d\partial_{\underline{u}}(\lambda\underline{a})$$

At the same time, since f is homogeneous,

$$f(\lambda\underline{a} + \lambda d\underline{u}) = \lambda f(\underline{a} + d\underline{u})$$
$$= \lambda(f(\underline{a}) + d\partial_{\underline{u}}f(\underline{a}))$$

Putting together the two equalities, we see that for all $d \in D$,

$$\lambda d\partial_{\underline{u}}f(\lambda\underline{a}) = \lambda d\partial_{\underline{u}}f(\underline{a}) \Rightarrow$$
$$\Rightarrow \lambda\partial_u f(\lambda\underline{a}) = \lambda\partial_{\underline{u}}(\underline{a})$$

Finally, we differentiate both sides with respect to λ, which results in

$$\partial_{\underline{u}} f(\lambda \underline{a}) + \lambda \frac{\partial}{\partial \lambda}(\partial_{\underline{u}} f(\lambda \underline{a})) = \partial_{\underline{u}} f(\underline{a})$$

Letting $\lambda = 0$ concludes the proof.   □

With that, we can quickly prove the following

PROPOSITION 2.14. *Let V be an R-module, and E a Euclidean R-module. Let $f : V \to E$ be homogeneous. Then f is linear and $f = df(\underline{0})$.*

PROOF. Since E is Euclidean, for all $\underline{u} \in V$ and $d \in D$,

$$\begin{aligned}
d\partial_{\underline{u}} f(\underline{u}) &= f(\underline{u} + d\underline{u}) - f(\underline{u}) \\
&= (1 + d)f(\underline{u}) - f(\underline{u}) \\
&= d \cdot f(\underline{u})
\end{aligned}$$

by which

$$f(\underline{u}) = \partial_{\underline{u}} f(\underline{u}) = \partial_{\underline{u}}(\underline{0}) = df(\underline{0})(\underline{u})$$

□

The general KL axiom

3.1. A closer look at the first axiom

Let $(a, b) \in R \times R$. To this pair we can associate the function

$$f : D \to R$$
$$d \mapsto a + bd$$

In this manner we obtain a map

$$\alpha : R \times R \to R^D$$

Axiom 1 can then be succinctly stated by demanding that α be bijective. Note that the R-module structure is also preserved, so that α is an R-module isomorphism. Furthermore, R^D has a natural R-algebra structure given by the pointwise product of maps:

(3.1) $$(a_1 + b_1 d)(a_2 + b_2 d) = a_1 a_2 + (a_1 b_2 + b_1 a_2)d + b_1 b_2 d^2$$
(3.2) $$= a_1 a_2 + (a_1 b_2 + b_1 a_2)d$$

As such, $R \times R$ does not possess an R-algebra structure, but if we define a product by the above formula, that is

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2, a_1 b_2 + b_1 a_2)$$

then α is an R-algebra isomorphism.

It will be helpful for motivating the forthcoming definitions to state this in yet another way. The previous structure on $R \times R$ is nothing more than the natural R-algebra on

$$W = R[X]/(X^2)$$

thus there is an isomorphism of R-algebras

$$W \overset{\alpha}{\cong} R^D$$

In the following section we will generalize this idea.

3.2. Weil Algebras

The infinitesimals D are one among many other "small objects" in SDG, called *Weil algebras*, due to André Weil. We will use the general definition as per [**1**].

DEFINITION 3.1. A *Weil algebra* W is a $\mathbb{Q}$-algebra together with a morphism

$$\pi : W \to \mathbb{Q}$$

such that W is a local ring with maximal ideal $I = \pi^{-1}(0)$ with I nilpotent and such that W is finite dimensional as a $\mathbb{Q}$ vector space. A homomorphism of Weil algebras $f : W_1 \to W_2$ is thus an algebra homomorphism sending the maximal ideal of W_1 to that of W_2.

It will do to explore some simple consequences of this definition.

LEMMA 3.2. *Any Weil algebra W may be finitely presented, that is, W is isomorphic to*

$$\mathbb{Q}[X_1, \ldots, X_n]/(P_1, \ldots, P_s)$$

where $P_1, \ldots P_s$ are polynomials of $\mathbb{Q}[X_1, \ldots, X_n]$.

PROOF. Let $e_1, \ldots, e_n$ be a $\mathbb{Q}$-basis of W. The algebra structure is then determined by the constants γ_{ij}^k given by

$$e_i \cdot e_j = \sum_{k=1}^{n} \gamma_{ij}^k e_k$$

in other words, the kernel of the surjection

$$E : \mathbb{Q}[X_1, \ldots, X_n] \to W$$

given by $X_k \mapsto e_k$ is the ideal generated by the polynomials

$$X_i X_j - \sum_{k=1}^{n} \gamma_{ij}^k X_k$$

With $1 \leq i, j \leq n$. Thus W is isomorphic to $\mathbb{Q}[X_1, \ldots, X_n]/\ker E$ with $\ker E$ finitely generated. $\square$

Naturally, there is no reason for $X_1, \ldots, X_n$ to be a minimal set of generators. In [**5**], this minimal number of generators is referred to as the *breadth* of a Weil algebra, and the smallest power that annihilates the maximal ideal as the *height*.

That a Weil algebra W is finite dimensional over $\mathbb{Q}$ means that it is isomorphic to $\mathbb{Q}^n$ with a certain product given by γ_{ij}^k, for some n. If A is any other $\mathbb{Q}$-algebra, then so can A^n be endowed with a product by letting $e_i e_j = \gamma_{ij}^k e_k$ where $e_l = (0, \overset{(l)}{\ldots}, 1, \ldots, 0)$, making A^n an A-algebra. This structure is manifestly dependent on the e_l. However, it does not depend on the presentation of W. This algebra structure

on A^n is denoted $A[W]$. For instance, the R-algebra on $R \times R$ from (3.2) arises in this way as $R[W]$, where
$$W = \mathbb{Q}[X]/(X^2)$$
If we repeat the steps that we took to obtain a presentation of a Weil algebra $W \cong \mathbb{Q}[X_1, \ldots, X_n]/(P_1, \ldots, P_s)$ then it's clear that for any other $\mathbb{Q}$-algebra A, the algebra $A[W]$ is also presented as $A[X_1, \ldots, X_n](P_1, \ldots, P_s)$.

It's worth mentioning that in [**5**], the author defines Weil algebras directly as objects which according to our definition arise as $R[W]$ with W a Weil algebra as defined above. The distinction is that $R[W]$ need not be a local ring (for instance, (3.2) has the ideal $((0, a))$ with $a \in R$ which is not maximal since any element $(d, 0)$ with $d \in D$ is in its complement, but it is not a unit. However this only changes the phrasing involved for the general KL axiom (the goal of this section), and not the content.

3.3. Spectra of Weil Algebras

The final piece involved in the statement of the general KL axiom is that of the *spectrum* of a Weil algebra.

DEFINITION 3.3. Let W be a Weil algebra, and C be an R-algebra. If W is presented as
$$\mathbb{Q}[X_1, \ldots, X_n]/(P_1 \ldots, P_s)$$
the *spectrum* of W in C is
$$\mathrm{Spec}_C W = \{(a_1, \ldots, a_n) \in C^n \mid P_j(a_1, \ldots, a_n) = 0, j = 1, \ldots, s\}$$

It is not difficult but it is prudent to check that this is well defined. In fact we shall prove that $\mathrm{Spec}_C(-)$ is functorial. Note that elements of $\mathrm{Spec}_C(W)$ can be "evaluated" at by classes in W, since two representatives differ by an element of $I = (P_1, \ldots, P_j)$, but every polynomial in I is zero on $\mathrm{Spec}_C(W)$ by definition. We shorten such an evaluation, $(P + I)(a)$, simply to $P(a)$.

Let W_1, W_2 be Weil algebras, with
$$W_1 \cong \mathbb{Q}[X_1, \ldots, X_n]/I$$
$$W_2 \cong \mathbb{Q}[Y_1, \ldots, Y_n]/J$$
Now let $f : W_1 \to W_2$ be a homomorphism of Weil algebras. Define
$$\mathrm{Spec}_C(f) : \mathrm{Spec}_C(W_2) \to \mathrm{Spec}_C(W_1)$$
by
$$\mathrm{Spec}_C(f)(b_1, \ldots, b_m) = (a_1, \ldots, a_n)$$
where
$$a_i = f(X_i)(b_1, \ldots, b_m)$$

We have

(1) $\mathrm{Spec}_C(-)$ preserves identities. In effect, the identity on a Weil algebra id_W sends X_i to X_i, and since $X_i(b_1, \ldots, b_m) = b_i$ so is $\mathrm{Spec}_C(\mathrm{id}_W)$ the identity on $\mathrm{Spec}_C(W)$.

(2) $\mathrm{Spec}_C(g \circ f) = \mathrm{Spec}_C(f) \circ \mathrm{Spec}_C(g)$, where defined. The proof is a simple verification.

By virtue of the (contravariant) functoriality we get as a corollary that isomorphic Weil algebras induce a bijection on their spectra. With these definitions in place we may state the complete version of the KL axiom.

3.4. The KL Axiom

Let $W \cong \mathbb{Q}[X_1, \ldots, X_n]/I$ be a Weil algebra. Earlier we observed that we can evaluate (classes of) polynomials in W at elements of $\mathrm{Spec}_R(W)$ (just replace C with R). In other words each element of W defines a mapping $\mathrm{Spec}_R(W) \to R$

$$W \to R^{\mathrm{Spec}_R(W)}$$

In the same manner we also obtain a mapping

$$\alpha : R[W] \to R^{\mathrm{Spec}_R(W)}$$

again given by sending $(p + I) \in R[W]$ to the map $a \mapsto p(a)$. It is of course an algebra homomorphism. The KL Axiom asserts that it is bijective.

AXIOM 3.1 (KL). *Let $W \cong \mathbb{Q}[X_1, \ldots, X_n]/I$ be a Weil algebra. Then the map*

$$\alpha : R[W] \to R^{\mathrm{Spec}_R(W)}$$

given by $p + I \mapsto (a \mapsto p(a))$ is an R-algebra isomorphism.

By asserting this axiom we obtain many of the propositions in the first part of this monograph in a stronger version, and further generalizations. But before looking at those examples we'll note that the first KL axiom given is too a consequence of axiom 3.1. This can be seen simply by applying the axiom to the Weil algebra $\mathbb{Q}[X]/(X^2)$. The result is the equivalent version of axiom 1 that we had already observed in section 3.1.

What the general KL axiom provides is essentially, given a particular set of "small" elements of R, a correspondence between functions from that set to R with a particular finite dimensional R-algebra. These "small" elements are the objects $\mathrm{Spec}_R(W)$ for some Weil algebra W. We refer to them as "small" because they are meant to be a rigorization of the historically nebulous concept of "infinitesimals". We first introduced nilsquares $(\mathrm{Spec}_R(Q[X]/(X^2)))$, on which all functions are affine, giving us first order derivatives. The Taylor formulas that came afterwards were restricted, however, to sums of elements of D and not any nilpotent element of R.

Since other nilpotents are too spectra of certain Weil algebra, we now have Taylor formulae for these, given by axiom 3.1. We'll look at a few such spectra, study the KL axiom in regards to them and at the same time introduce their common nomenclature.

- $D_k := \mathrm{Spec}_R(W_{k+1})$ with $W_{k+1} = \mathbb{Q}[X]/(X^{k+1})$. These are elements of R whose $(k+1)$-th power is zero (in particular $D_2 = D$). One can check that W_{k+1} is isomorphic to the the set $\{a_0 + a_1\epsilon + \cdots + a_k\epsilon^k \mid a_i \in R\}$ with ϵ denoting the equivalence class of X in W_{k+1}. By axiom 3.1 the set of functions $D_k \to R$ are isomorphic as an R-algebra to the algebra W_{k+1}. Another way of saying this is that:

    PROPOSITION 3.4. *For all functions* $g : D_k \to R$ *there exist unique coefficients* $a_0, \ldots, a_k \in R$ *such that*
    $$g(\delta) = a_0 + a_1\delta + \cdots + a_k\delta^k \quad \forall \delta \in D_k$$

    And as an immediate consequence we have that, for any $f : R \to R$ there exist unique coefficients $a_0, \ldots, a_k \in R$ such that
    $$f(x + \delta) = a_0 + a_1\delta + \cdots + a_k\delta^k \quad \forall \delta \in R : \delta^{k+1} = 0$$

    That is, we have Taylor formulas of all orders (and one can check that the a_i are precisely the Taylor coefficients).

- $(D_k)^n := \mathrm{Spec}_R((W_{k+1})^n)$ where $(W_{k+1})^n$ just means the Cartesian product with itself n times (and so does taking the spectrum of the product of W_k coincide with the product of D_k). This is a Weil algebra, and it is presented by $\mathbb{Q}[X_1, \ldots, X_n](X_1^{k+1}, \ldots, X_n^{k+1})$. The general KL axiom applied to this particular algebra is what yields higher dimensional Taylor formulae. We state it now without proof (a tedious but simple combinatorial exercise).

    PROPOSITION 3.5. *For any function* $f : R^n \to R$,
    $$f(\underline{x} + \underline{\delta}) = \sum_{\alpha \leq k} \frac{\underline{\delta}}{\alpha!} \cdot \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(\underline{x}) \quad \forall \delta : \delta^{k+1} = 0$$

    where we're using the usual multi-index notation - if $\alpha = (\alpha_1, \ldots, \alpha_n)$ then $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \cdot \cdots \cdot \alpha_n!$ and $\frac{\partial^{|\alpha|} f}{\partial x^\alpha} = \frac{\partial^{|\alpha|} f}{\partial x^{\alpha_1} \ldots \partial x^{\alpha_n}}$. The partial derivatives are defined in the same way as before.

- $D(n) := \mathrm{Spec}_R(W(n))$ where
    $$W(n) = \mathbb{Q}[X_1, \ldots, X_n]/(\{X_i X_j\}_{1 \leq i,j \leq n})$$

    A simpler way of writing this is

    $$D(n) = \{(d_1, \ldots, d_n) \subset R^n \mid \text{the product of any two } d_i, d_j \text{ is zero}\}$$

- $D_k(n) := \mathrm{Spec}_R(W_k(n))$ where
$$W_k(n) = \mathbb{Q}[X_1, \ldots, X_n]/(\{X_{i_1} \cdots X_{i_k}\}_{1 \le i_1, \ldots, i_k \le n})$$

  Again these are the elements of R^n such that the product of any k of their components is equal to zero.

There are some notable inclusions:

$$D_k(n) \subset D_l(n) \text{ iff } k \le l$$
$$D_k(n) \subset (D_k)^n$$
$$(D_k)^n \subset D_{n \cdot k}(n)$$

Microlinearity

4.1. A simple example

We begin by observing a few specific cases. Let

$$D \times D \to D$$

be the multiplication map, i.e. $(d_1, d_2) \mapsto d_1 d_2$. Nothing in our theory suggests that this map should be surjective, but for many purposes it suffices to have the following property:

PROPOSITION 4.1. *If $f_1, f_2 : D \to R$ verify that, for each $d_1, d_2 \in D$, $f_1(d_1 d_2) = f_2(d_1 d_2)$ then $f_1 = f_2$.*

That is, it suffices to know any function $f : D \to R$ on products of elements of D. This is referred to in [**3**] as "*R thinks that the multiplication : $D \times D \to D$ is surjective*".

PROOF. Define $g = f_1 - f_2$. By our hypothesis g is identically zero on all elements $d_1 d_2$ with $d_1, d_2 \in D$. By the KL axiom (since the previous section this refers to axiom 3.1, but in this case axiom 1 also suffices), there exist unique $a, b \in R$ such that

$$g(d) = a + bd \quad \forall d \in D$$

On the one hand $a = g(0) = g(0 \cdot 0) = 0$, so that $g(d) = bd$. Thus, for all $d_1, d_2 \in D$ we have $0 = g(d_1 d_2) = b d_1 d_2$, or in other words that $b d_1 d_2 = 0$ for all $d_1, d_2 \in D$, which means that $b = 0$. $\qquad\square$

4.2. A slightly more interesting example

Consider two functions $f, g : D \to X$ such that $f(0) = g(0)$, where X is some set (object of $\mathcal{E}$). Here there is no reason to assume that this will define a map $D(2) \to X$. It will be the case, however, if X is R.

PROPOSITION 4.2. *Let $f, g : D \to R$ be functions such that $f(0) = g(0)$. Then there exists a unique function $h : D(2) \to R$ such that $h(d, 0) = f(d)$ and $h(0, d) = g(d)$ for all $d \in D$.*

PROOF. Let $a_0 = g(0) = f(0)$. Then we have that for all $d \in D$,

$$f(d) = a_0 + a_1 d$$
$$g(d) = a_0 + b_1 d$$

Thus, we simply define h by

$$h(d_1, d_2) = a_0 + a_1 d_1 + b_1 d_2$$

This clearly satisfies the existence requirements. The Uniqueness is a consequence of the KL axiom. Let k be another function that satisfies the hypotheses. By the KL axiom applied to $D(2)$,

$$k(d_1, d_2) = \alpha_0 + \alpha_1 d_1 + \alpha_2 d_2$$

for unique α_i. By $k(d, 0) = f(d)$ we obtain $\alpha_0 = a_0$ and $\alpha_1 = a_1$. Similarly $\alpha_2 = b_1$. Since the α_i are unique and determine k, we get $k = h$ (if preferred we could have concluded at the very definition of h, since the KL axiom asserts that functions on $D(2)$ are defined as such - the argument has simply been made explicit a bit further here).

$\square$

4.3. Perceived colimits

The previous examples can be stated in diagrammatic form, which will then lend itself to a generalization in terms of (co)limits. We'll focus on the second example. Consider the following commutative diagram:

(4.1)
$$
\begin{array}{ccc}
\{0\} & \xrightarrow{\ 0\ } & D \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle i_2} \\
D & \xrightarrow[\ i_1\ ]{} & D(2)
\end{array}
$$

where i_1 and i_2 are the inclusion maps $i_1(d) = (d, 0)$, $i_2(d) = (0, d)$ and 0 is the constant zero map. The previous observation that two maps $f, g : D \to X$ do not have to define a map $D(2) \to X$ is equivalent to saying that diagram 4.1 is not a pushout. But, if we apply the $R^{(-)}$ functor to it, the result is:

$$R^{\{0\}} \xleftarrow{\quad R^0 \quad} R^D$$

$$R^0 \uparrow \qquad\qquad \uparrow R^{i_2}$$

$$R^D \xleftarrow[\quad R^{i_2} \quad]{} R^{D(2)}$$

PROPOSITION 4.3. *Diagram 4.3 is a pullback.*

PROOF. First we will make the diagram a little neater. Firstly, $R^{\{0\}}$ is isomorphic to R. The map R^0 Takes $f : D \to R$ and sends it to $f \circ 0$, which we simply identify with $f(0)$ the same way we identified $R^{\{0\}}$ with R. Likewise, the map R^{i_1} sends $g : D(2) \to R$ to $g \circ i_1$. Thus, for 4.3 to be a pullback, we'd need that the set $R^D(2)$ be in bijection with the set of pairs $f, g \in R^D$ such that $f(0) = g(0)$. But this is precisely what we proved in proposition 4.2. $\qquad\square$

How we refer to this phenomenon (or how it is often referred to) is *"R perceives diagram 4.1 as a pushout.* This is of course generalizes, as we will see shortly. To do so we will give a few definitions first.

DEFINITION 4.4. Let $\mathbf{W}$ be the category of Weil algebras (a subcategory of the category of $\mathbb{Q}$-algebras $\mathbb{Q}$-$\mathbf{Alg}$). A *good limit* of Weil algebras is a limit $\left(L \xrightarrow{f_i} D_i \right)_{i \in I}$ in $\mathbb{Q}$-$\mathbf{Alg}$ which is also a limit in $\mathbf{W}$. That is, the morphisms in the diagram defining L must be Weil algebra homomorphisms, and so must the projections f_i.

DEFINITION 4.5. A diagram in $\mathcal{E}$ is said to be a (finite) *quasi colimit* if it is the image under the functor

$$\mathrm{Spec}_R(-) : \mathbf{W} \to \mathcal{E}$$

of a good finite colimit in $\mathbf{W}$.

DEFINITION 4.6. An object X is said to perceive a quasi colimit as a colimit if the functor

$$X^{(-)} : \mathcal{E} \to \mathcal{E}$$

takes said quasi colimit into a limit (of $\mathcal{E}$).

EXAMPLE 4.7. The following diagram is a good colimit of Weil algebras (it is a pushout in $\mathbb{Q}$-**Alg**, and the morphisms are Weil algebra homomorphisms).

(4.2)
$$
\begin{array}{ccc}
\mathbb{Q} & \xleftarrow{\quad 0 \quad} & \mathbb{Q}[X]/(X^2) \\
\uparrow{\scriptstyle 0} & & \uparrow{\scriptstyle p_2} \\
\mathbb{Q}[X]/(X^2) & \xleftarrow{\quad p_1 \quad} & \mathbb{Q}[X,Y]/(X^2,Y^2,XY)
\end{array}
$$

where $p_1(X) = X, p_1(Y) = 0, p_2(X) = 0$, and $p_2(Y) = X$. Via the $\mathrm{Spec}_R(-)$ functor this is taken to

$$
\begin{array}{ccc}
\{0\} & \xrightarrow{\quad 0 \quad} & D \\
\downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle i_2} \\
D & \xrightarrow{\quad i_1 \quad} & D(2)
\end{array}
$$

which is the same as diagram 4.1. Thus, since diagram 4.2 is an example of a good finite limit, diagram 4.1 is a finite quasi colimit. Furthermore, R perceives 4.1 as a limit, as observed in proposition 4.3.

For plenty more examples, one should look at [**5**]. The objective of these definitions are to state the upcoming proposition, which generalizes the previous observation that *"R perceives. . . etc"*. The result will be of use in section 5 as a smoothness condition.

PROPOSITION 4.8. *R perceives finite quasi colimits as colimits. Conversely, if R perceives a diagram as a colimit, it is a quasi colimit.*

We'll need a simple result previously, in the form of the following

LEMMA 4.9. *Let*
$$
\left(W_i \xleftarrow{\;p_i\;} W_L \right)_{i \in I}
$$
be any limit of Weil algebras. Then the diagram
$$
\left(R[W_i] \xleftarrow{\;p_i\;} R[W_L] \right)_{i \in I}
$$
is a limit of R-algebras.

With that, we can prove the proposition.

PROOF OF PROPOSITION 4.8. Let $\left(L \xleftarrow{f_i} D_i \right)_{i \in I}$ be a finite quasi colimit. That is, we may write the diagram as

(4.3) $$\left(\mathrm{Spec}_R(W_L) \xleftarrow{\mathrm{Spec}_R(p_i)} \mathrm{Spec}_R(W_i) \right)_{i \in I}$$

where

$$\left(W_i \xleftarrow{p_i} W_L \right)_{i \in I}$$

is some good finite limit of Weil algebras. We need to prove that diagram 4.3 is taken to a limit by the functor $R^{(-)}$. Under this functor, the resulting diagram is

(4.4) $$\left(R^{\mathrm{Spec}_R(W_L)} \xrightarrow{R^{\mathrm{Spec}_R(p_i)}} R^{\mathrm{Spec}_R(W_i)} \right)_{i \in I}$$

Now we invoke the general KL axiom, which says that this diagram is isomorphic to

$$\left(R[W_i] \xleftarrow{p_i} R[W_L] \right)_{i \in I}$$

By lemma 4.9, this is a limit, since by hypothesis

$$\left(W_i \xleftarrow{p_i} W_L \right)_{i \in I}$$

is a limit of Weil algebras. $\qquad\square$

4.4. Microlinear objects

The behavior of R studied above will be a fundamental requirement as we move onto studying other objects of synthetic differential geometry. We call these objects *microlinear*, and it can be said that they are a sort of generalized "manifold" in the context of SDG. Let us state this precisely:

DEFINITION 4.10. An object M is said to be *microlinear* if it perceives finite quasi colimits as limits.

Thus, obviously R is microlinear. We'll make two observations.

PROPOSITION 4.11. *If M is a microlinear object and X is any other object, then M^X is also microlinear.*

PROPOSITION 4.12. *If*

$$\left(M \xrightarrow{f_\alpha} M_\alpha \right)_{\alpha \in A}$$

is a limit, with M_α microlinear for every $\alpha \in A$, then M is microlinear.

SECTION 5

Tangency

At this point we have enough of the basic theory in place to begin doing actual *geometry* on "manifolds" (microlinear objects).

5.1. The tangent bundle

Let M be a microlinear object and $p \in M$. We make the following

DEFINITION 5.1. A tangent vector to M at p is a mapping

$$t : D \to M$$

such that $t(0) = p$.

We call the collection of all such tangent vectors T_pM, the tangent space at p of M. In resemblance to classical geometry, we of course expect each T_pM to have a vector space structure (and to justify calling them *vectors* in the first place). It will not be so, since R is not a field in the classical sense, but we will prove that each T_pM is a Euclidean R-module. Let us begin by defining scalar multiplication:

$$R \times T_pM \to T_pM$$
$$(\lambda, t) \mapsto \lambda t$$

where the map $\lambda t : D \to M$ is defined by

$$(\lambda t)(d) = t(\lambda d)$$

As for addition, consider two tangent vectors at p, $t_1, t_2 : D \to M$. We now use that M is microlinear. Recall from earlier that the diagram below is a quasi pushout:

(5.1)

$$\begin{array}{ccc} \{0\} & \xrightarrow{\ 0\ } & D \\ {\scriptstyle 0}\big\downarrow & & \big\downarrow{\scriptstyle i_2} \\ D & \xrightarrow[\ i_1\ ]{} & D(2) \end{array}$$

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Since M is microlinear, it perceives 5.1 as a pushout. As we studied in the case of R, this means that the maps $D(2) \to M$ are in bijection with pairs of maps $D \to M$ that are equal at 0. This is the case with t_1 and t_2, since $t_1(0) = p = t_2(0)$. Thus, there exists a unique map, which we call

$$s_{t_1,t_2} : D(2) \to M$$

allowing us to define

$$t_1 + t_2 : d \mapsto s_{t_1,t_2}(d,d)$$

Finally, let us define $0 \in T_pM$ as the constant map $0(d) = p \, \forall d \in D$, and the additive opposite of a vector t as $-t$, given by $(-t)(d) = t(-d)$.

PROPOSITION 5.2. *Let $p \in M$. With the operations defined above, T_pM is an R-module.*

PROOF.   (1) Addition is commutative. Let $t_1, t_2 \in T_pM$. Again, this defines

$$s_{t_1,t_2} : D(2) \to M$$

as the unique map satisfying $s_{t_1,t_2}(d_1, 0) = t_1(d_1)$ and $s_{t_1,t_2}(0, d_2) = t_2(d_2)$. On the other hand, t_1 and t_2 also define the map

$$s_{t_2,t_1} : D(2) \to M$$

unique among maps satisfying $s_{t_2,t_1}(d_2, 0) = t_2(d_2)$ and $s_{t_2,t_1}(0, d_1) = t_1(d_1)$. Therefore, we have that for all $(d_1, d_2) \in D(2)$

$$s_{t_1,t_2}(d_1, d_2) = s_{t_2,t_1}(d_2, d_1)$$

In particular, s_{t_1,t_2} and s_{t_2,t_1} are equal on the diagonal,

$$\{(d,d) \mid d \in D\} \subset D(2)$$

So, by definition of $t_1 + t_2$ and $t_2 + t_1$, these two are equal.
        (2) Addition is associative. This requires a generalization of proposition 4.3, that is, that R perceives the diagram

(5.2)

$$\begin{array}{ccc} \{0\} & \xrightarrow{\ 0\ } & D(q) \\ {\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle i_2} \\ D(p) & \xrightarrow[\ i_1\ ]{} & D(p+q) \end{array}$$

as a pushout, for any $p, q \geq 1$. Here the maps i_1, i_2 are

$$i_1(d_1, \ldots, d_p) = (d_1, \ldots, d_p, 0, \ldots, 0)$$
$$i_2(d_1, \ldots, d_q) = (0, \ldots, 0, d_1, \ldots, d_q)$$

The proof is nearly identical to the case we have examined ($p = q = 1$), one can refer to [**5**, p. 48] for details. That R perceives 5.2 as a pushout means that M does as well, M being microlinear. In turn, this implies that the diagram

$$
\begin{array}{ccc}
M & \xleftarrow{\ M^0\ } & M^{D(2)} \\
{\scriptstyle M^0}\big\uparrow & & \big\uparrow{\scriptstyle M^{i_2}} \\
M^D & \xleftarrow[\ M^{i_1}\ ]{} & M^{D(3)}
\end{array}
$$

is a pullback. That is, any function

$$
g : D(3) \to M
$$
$$
(d_1, d_2, d_3) \mapsto g(d_1, d_2, d_3)
$$

is uniquely determined by the functions

$$
g_1 : D \to M
$$
$$
d_1 \mapsto g(d_1, 0, 0)
$$

and

$$
g_{23} : D(2) \to M
$$
$$
(d_2, d_3) \mapsto g(0, d_2, d_3)
$$

Now, by applying the characterization of maps $D(2) \to M$ on g_{23}, we obtain that g is the unique map satisfying $g(d_1, 0, 0) = g_1(d_1)$, $g(0, d_2, 0) = g_{23}(d_2, 0)$, and $g(0, 0, d) = g_{23}(0, d_3)$ for all $(d_1, d_2, d_3) \in D(3)$. In other words, any map

$$
g : D(3) \to M
$$

is uniquely determined by three maps $g_1, g_2, g_3 : D \to M$ with $g_1(0) = g_2(0) = g_3(0)$, and such that

$$
g_1(d_1) = g(d_1, 0, 0)
$$
$$
g_2(d_2) = g(0, d_2, 0)
$$
$$
g_3(d_3) = g(0, 0, d_3)
$$

With that said, let $t_1, t_2, t_3 \in T_p M$ be three tangent vectors. As before, t_1 and t_2 determine a unique map

$$
s_{t_1, t_2} : D(2) \to M
$$

So that, together with t_3, they determine a map $g_l : D(3) \to M$ given by

$$g_l(d_1, 0, 0) = s_{t_1, t_2}(d_1, 0)$$
$$g_l(0, d_2, 0) = s_{t_1, t_2}(0, d_2)$$
$$g_l(0, 0, d_3) = t_3(d_3)$$

Likewise, we obtain a map g_r by

$$g_r(d_1, 0, 0) = t_1(d_1)$$
$$g_r(0, d_2, 0) = s_{t_2, t_3}(d_2, 0)$$
$$g_r(0, 0, d_3) = s_{t_2, t_3}(0, d_3)$$

But these are the same map, since they both satisfy

$$g_l(d_1, 0, 0) = g_r(d_1, 0, 0) = t_1(d_1)$$
$$g_l(0, d_2, 0) = g_r(0, d_2, 0) = t_2(d_2)$$
$$g_l(0, 0, d_3) = g_r(0, 0, d_3) = t_3(d_3)$$

Now note that

$$((t_1 + t_2) + t_3)(d) = s_{t_1+t_2, t_3}(d, d) = g_l(d, d, d)$$
$$(t_1 + (t_2 + t_3))(d) = s_{t_1, t_2+t_3}(d, d) = g_r(d, d, d)$$

by which we conclude that $((t_1 + t_2) + t_3) = (t_1 + (t_2 + t_3))$.

(3) The map 0 is the additive identity. Let $t : D \to M$ be a tangent vector at p, and $0 : D \to M$ be the constant p. The identities

$$s_{t,0}(d, 0) = t(d)$$
$$s_{t,0}(0, d) = 0(d) = p$$

uniquely determine $s_{t,0} : D(2) \to M$. But the function $t \circ \pi_1 : D(2) \to M$, where $\pi_2 : D(2) \to D$ is the projection onto the second coordinate, also satisfies those equations, making $s_{t,0} = t \circ \pi_2$. Therefore,

$$(t + 0)(d) = s_{t,0}(d, d) = (t \circ \pi_2)(d, d) = t(d)$$

(4) The opposite of a vector, as defined, satisfies $t + (-t) = 0$. The function $s_{t,-t} : D(2) \to M$ is unique such that $s_{t,-t}(d, 0) = t(d)$ and $s_{t,-t}(0, d) = (-t)(d) = t(-d)$. Again, we explicitly exhibit the function

$$f : D(2) \to M$$
$$(d_1, d_2) \mapsto t(d_1 - d_2)$$

and f satisfies the same equations, making it equal to $s_{t,-t}$. Therefore, for any $d \in D$

$$(t + (-t))(d) = s_{t,-t}(d, d) = f(d, d) = t(d - d) = t(0) = p$$

(5) We have the identities
  (a) $(\alpha + \beta)t = \alpha t + \beta t$
  (b) $\alpha(t_1 + t_2) = \alpha t_1 + \alpha t_2$
  (c) $\alpha(\beta t) = (\alpha \beta)t$
  (d) $1 \cdot t = t$

With $\alpha, \beta \in R$ and $t, t_1, t_2 \in T_p M$. The tangent vector $\alpha t + \beta t$ is the unique map $s_{\alpha t, \beta t} : D(2) \to M$ satisfying

$$s_{\alpha t, \beta t}(d_1, 0) = (\alpha t)(d_1) = t(\alpha d_1)$$
$$s_{\alpha t, \beta t}(0, d_2) = (\beta t)(d_2) = t(\beta d_2)$$

On the other hand, the map c, defined by

$$c(d_1, d_2) = t(\alpha d_1 + \beta d_2)$$

satisfies the equations, by which $s_{\alpha t, \beta t} = c$. So,

$$(\alpha t + \beta t)(d) = c(d, d) = t(\alpha d + \beta d) = t((\alpha + \beta)d) = (\alpha + \beta)t(d)$$

by definition. That proves 5a. To prove 5b we make an analogous argument, this time considering

$$c(d_1, d_2) = s_{t_1, t_2}(\alpha d_1, \alpha d_2)$$

leaving 5c and 5d, which are immediate.

This concludes the proof that $T_p M$ is an R-module. We will now prove that it is Euclidean. Recall that this means that, for any map $\varphi : D \to T_p M$, there should be a unique vector $t \in T_p M$ such that, for all $d \in D$

$$\varphi(d) = \varphi(0) + d \cdot t$$

This will again be a consequence of microlinearity. Define

$$\tau : D \times D \to M$$
$$(d_1, d_2) \mapsto (\varphi(d_1) - \varphi(0))(d_2)$$

Note that $\varphi(d_1) - \varphi(0)$ is a tangent vector at p, so that

$$\tau(d_1, 0) = \tau(0, d_2) = \tau(0, 0) = p$$

for all $d_1, d_2 \in D$. We'll now need another lemma concerning a quasi colimit.

LEMMA 5.3. *M perceives the diagram*

(5.3)
$$D \underset{\underset{0}{\xrightarrow{\hspace{1cm}}}}{\overset{\overset{i_1}{\xrightarrow{\hspace{1cm}}}}{\underset{i_2}{\xrightarrow{\hspace{1cm}}}}} D \times D \xrightarrow{\mu} D$$

as a colimit (a "triple coequalizer", if one wishes). Here the maps i_1, i_2 are defined by $i_1(d) = (d, 0), i_2(d) = (0, d)$, 0 is the zero map, and μ is the multiplication of elements of R, restricted to elements of D.

In other words, the diagram

$$M^D \xleftarrow[\substack{M^{i_2} \\ M^{i_1}}]{M^0} M^{D \times D} \xleftarrow{M^\mu} M^D$$

is a limit (a triple equalizer). That is, if $g \in M^{D \times D}$ is such that

(5.4) $$g(d_1, 0) = g(0, d_2) = g(0, 0) \quad \forall d_1, d_2 \in D$$

then there exists a unique mapping

$$t : D \to M$$

such that

$$g(d_1, d_2) = (t \circ \mu)(d_1, d_2) = t(d_1 d_2)$$

This is all we need, since the map τ verifies the equations 5.4. Therefore there exists a unique $t : D \to M$ with

$$\tau(d_1, d_2) = t(d_1 d_2) \quad \forall d_1, d_2 \in D$$

In other words for all d_1, d_2,

$$(\varphi(d_1) - \varphi(0))(d_2) = t(d_1 d_2) = (d_1 t)(d_2)$$

where the last equality is by definition of scalar multiplication of tangent vectors. The above holds in particular for all $d_2 \in D$, so that there is an equality of maps

$$\varphi(d_1) = \varphi(0) + d_1 t$$

$\square$

.

The proof of lemma 5.3 is fairly straightforward. Since M is microlinear, it amounts to proving that diagram 5.3 is a quasi colimit.

PROOF OF LEMMA 5.3. The diagram of Weil algebras

(5.5) $$\mathbb{Q}[X]/(X^2) \xleftarrow[\substack{f_2 \\ f_1}]{f_3} \mathbb{Q}[X, Y]/(X^2, Y^2) \xleftarrow{m} \mathbb{Q}[X]/(X^2)$$

is a limit. Here the functions m, f_1, f_2, f_3 are defined by

$$f_1 : \begin{array}{c} X \mapsto X \\ Y \mapsto 0 \end{array}$$

$$f_2 : \begin{array}{c} X \mapsto 0 \\ Y \mapsto X \end{array}$$

$$f_3 : \begin{array}{c} X \mapsto 0 \\ Y \mapsto 0 \end{array}$$

$$m : X \mapsto XY$$

To begin with, the diagram obviously commutes. Now, let A be any object, and $g : A \to \mathbb{Q}[X,Y]/(X^2, Y^2)$ a map such that $f_i \circ g = f_j \circ g$ for any i, j (g makes a similar diagram commute). Let $a \in A$. Its image under g is an element of $\mathbb{Q}[X,Y]/(X^2, Y^2)$, and such can be written, modulo (X^2, Y^2) as

$$g(a) = c_{00} + c_{10}X + c_{01}Y + c_{11}XY$$

The condition that g commutes with the f_i force $c_{10} = c_{01} = 0$. Define

$$h(a) = c_{00} + c_{11}X$$

By varying a this defines a map $h : A \to \mathbb{Q}[X]/(X^2)$. Since c_{00}, c_{11} are unique modulo (X^2, Y^2), the map h is the unique map such that the following diagram commutes:

This concludes that diagram 5.5 is a limit. It is left to the reader to check that diagram 5.3 is the result of applying the Spec_R functor to 5.5, making it a quasi colimit. Since M was microlinear, the lemma is proven.   □

The natural next step is to define the differential (or derivative, or tangent map, etc.) of a mapping between two microlinear objects. Let M, N be microlinear, and

$f : M \to N$ a map. Let $p \in M$. We define:

$$df_p : T_pM \to T_{f(p)}N$$

by

$$df_p(t) = f \circ t$$

As we should expect, this map is linear. Since T_pM is Euclidean, by proposition 2.14 it suffices to see that it is homogeneous. For $d \in D$, we have

$$df_p(\alpha t)(d) = (f \circ (\alpha t))(d) = f(t(\alpha d)) = (f \circ t)(\alpha d) = (\alpha \cdot df_p(t))(d)$$

Another desirable property is that if V is a Euclidean R-module, it is canonically isomorphic to T_pV at every $p \in V$. This is easy to see by presenting the isomorphism explicitly. Define

$$\lambda_p : V \to T_pV$$

by sending $v \in V$ to the map

$$d \mapsto p + dv$$

This is a bijection since, by virtue of V being Euclidean, every map $t : D \to V$ is characterized by a unique $b \in V$ such that

$$d \mapsto t(0) + db$$

In T_pV, each $t(0)$ is equal to p, so the inverse of λ_p is $\lambda_p^{-1}(t) = b$. The map λ_p is obviously homogeneous, so again by proposition 2.14 it is linear. As a short exercise, one can prove that, under this identification, df_p corresponds to $df(p)$ (definition 2.12) for maps of Euclidean R-modules $f : V \to E$.

5.2. Vector bundles; the tangent bundle

DEFINITION 5.4. Let $\pi : E \to M$ be a mapping of microlinear objects. We say that π is a *(resp. Euclidean) vector bundle* if each fiber $\pi_p = \pi^{-1}(\{p\})$ of π is a (resp. Euclidean) R-module.

As a prominent example, we have

DEFINITION 5.5. Let M be a microlinear object. The *tangent bundle* on M is given by

$$\pi : M^D \to M$$
$$t \mapsto t(0)$$

The previous map indeed defines a Euclidean vector bundle. First, M^D is microlinear by proposition 4.11, and second each fiber is just T_pM, which we have just seen to be a Euclidean R-module.

We have defined vector bundles, so we should define what it means to be a morphism of vector bundles. If

$$\pi_1 : E_1 \to M_1$$
$$\pi_2 : E_2 \to M_2$$

are two vector bundles, then a pair (φ, f) is said to be a morphism of the vector bundles π_1, π_2 if the following diagram commutes:

$$
\begin{array}{ccc}
E_1 & \xrightarrow{\varphi} & E_2 \\
\downarrow{\pi_1} & & \downarrow{\pi_2} \\
M_1 & \xrightarrow{f} & M_2
\end{array}
$$

Equivalently, φ takes the fiber at $p \in M_1$ to the fiber at $f(p) \in M_2$.

Again, an important example is provided by the tangent bundle to a microlinear object. If M, N are microlinear objects, and $f : M \to N$ is a map, then (f^D, f) is a morphism of the tangent bundle at M to the tangent bundle at N, as evidenced by the commutative diagram

$$
\begin{array}{ccc}
M^D & \xrightarrow{f^D} & N^D \\
\downarrow{\pi_M} & & \downarrow{\pi_N} \\
M & \xrightarrow{f} & N
\end{array}
$$

Where π_M, π_N are the projections defining the tangent bundles of M, N, respectively. If we define **Mic** to be the category of microlinear objects (with morphisms regular maps), and **Fib** the category of vector bundles (with morphisms of vector bundles as previously defined), then the association of each microlinear object to its tangent bundle defines a functor

$$T : \mathbf{Mic} \to \mathbf{Fib}$$

with

$$TM = M^D$$
$$Tf = (f^D, f)$$

which we call the *tangent functor*. Note that fiberwise, this is exactly the association of each object to its tangent space at a point p, and each map f to its differential df_p.

5.3. Vector Fields

As in classical differential geometry, we define a *section* of a vector bundle to be a right inverse to the projection map of the bundle. That is, if $\pi : E \to M$ is a vector bundle, then a section of π is a map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$. We will write $\Gamma(\pi)$ for the set of sections of π.

Let $\pi : E \to M$ be a vector bundle. It's immediate that $\Gamma(\pi)$ comes equipped with an R-module structure, by the R-module structure on each fiber: define $(s_1 + s_2)(p) = s_1(p) + s_2(p)$, and so on (where s_1, s_2 are sections). Furthermore we have the following proposition.

PROPOSITION 5.6. *The set $\Gamma(\pi)$ is microlinear, and if π is a Euclidean vector bundle, $\Gamma(\pi)$ is a Euclidean R-module.*

PROOF. The objects E and M are microlinear by definition of a vector bundle, and by proposition 4.11 so are E^M, M^M microlinear. Now observe that the diagram

$$\Gamma(\pi) \rightarrowtail E^M \underset{I}{\overset{\varphi}{\rightrightarrows}} M^M$$

is an equalizer, where the first arrow is just the inclusion of $\Gamma(\pi)$ into E^M, $\varphi(s) = \pi \circ s$, and $I(s) = \mathrm{id}_M$. Thus $\Gamma(\pi)$ is a limit of microlinear objects, meaning it is microlinear by proposition 4.12. Now suppose that π is a Euclidean vector bundle, and let $f : D \to \Gamma(\pi)$ be a map. For fixed $p \in M$, the Euclidean R-module structure on π_p gives that there exists a unique $b(p) \in \pi_p$ (we make explicit the dependence on p) such that for all $d \in D$

$$f(d)(p) = f(0)(p) + d \cdot b(p)$$

Varying p thus gives a section $b : M \to E$ (it is a section, since $b(p) \in \pi_p = \pi^{-1}(p)$) which is unique such that

$$f(d) = f(0) + d \cdot b$$

and so $\Gamma(\pi)$ is Euclidean. $\square$

We should also make the observation that $\Gamma(\pi)$ is a module over R^M, as well. For $f : M \to R$ and $s \in \Gamma(\pi)$ define

$$(f \cdot s)(p) = f(p) \cdot s(p)$$

We can now define what a vector field is, in a much expected way.

DEFINITION 5.7. Let M be a microlinear object. A *vector field* on M is a section of the tangent bundle on M. We denote the set of vector fields by $\mathfrak{X}(M)$.

Now since the tangent bundle is an exponential object, we can can automatically establish a correspondence between three familiar conceptions of a vector field. On the first hand a vector field is a map

$$X : M \to M^D$$

such that $X(p)(0) = p$ for every $p \in M$. But then X is equivalent to giving a map

$$X : M \times D \to M$$

such that $X(p, 0) = p$. Thus, the notion of infinitesimal flow is recovered. Furthermore we have the principal of superposition.

PROPOSITION 5.8. *Let $X : M \times D \to M$ be a vector field. Then for every $p \in M$ and $(d_1, d_2) \in D(2)$*

$$X(p, d_1 + d_2) = X(X(p, d_1), d_2)$$

PROOF. By fixing p we obtain two maps $f, g : D(2) \to M$, which are defined by

$$f(d_1, d_2) = X(p, d_1 + d_2)$$
$$g(d_1, d_2) = X(X(p, d_1), d_2)$$

Since X is a vector field we have the equalities

$$f(d, 0) = g(d, 0)$$
$$f(0, d) = g(0, d)$$

hence, since M is microlinear, f and g are the same map. $\square$

Lastly, we also have the notion of a vector field as an "infinitesimal transformation of the identity". That is, from a map

$$X : D \times M \to M$$

there corresponds a map (which we'll call by the same name)

(5.6) $$X : D \to M^M$$

and it is such that $X(0) = \mathrm{id}_M$.

From here on we will use the three ways of giving a vector field interchangeably. As in [5] we refer to the image of an element $d \in D$ by a map such as 5.6, as $X_d : M \to M$. For $p \in M$, we should visualize $X_d(p)$ as the result of letting p flow

along the vector field for d units of "time". Whereas in classical differential geometry the flow corresponding to a vector field need not define a group of transformations (but it does a semi-group), it will be the case here.

PROPOSITION 5.9. *Let $X : M \to M$ be a vector field. Then for all $d \in D$, X_d is a bijection and $X_d^{-1} = X_{-d}$.*

PROOF. This is a simple consequence of proposition 5.8, since

$$X(X(p, d), -d) = X(p, 0) = p$$

for all $p \in M$.   □

The following proposition also establishes a strong parallel with classical Lie algebras of vector fields (although the "Lie" part will be seen later).

PROPOSITION 5.10. *Let M be microlinear. Denote the set of invertible maps from M to itself as $\mathrm{Aut}(M)$. This is a microlinear object, and $\mathfrak{X}(M)$ is R-module isomorphic to $T_{\mathrm{id}_M} \mathrm{Aut}(M)$, the tangent space at the identity.*

PROOF. To prove that $\mathrm{Aut}(M)$ is microlinear, we proceed as usual to prove that it is a limit of microlinear objects. To do this, note that $\mathrm{Aut}(M)$ is equal to the set

$$\{(f, g) \in M^M \times M^M \mid g \circ f = f \circ g = \mathrm{id}_M\}$$

In other words, we have a diagram

$$\mathrm{Aut}(M) \xrightarrow{\ \ i\ \ } M^M \times M^M \; \substack{\xrightarrow{c_1} \\ \xrightarrow{c_2} \\ \xrightarrow{\ I\ }} \; M^M$$

where i is the mapping $f \mapsto (f, f^{-1})$, c_1 is the mapping $(f, g) \mapsto f \circ g$, c_2 is $(f, g) \mapsto g \circ f$, and I is $(f, g) \mapsto \mathrm{id}_M$. The diagram commutes, and any object mapping into $M^M \times M^M$ in such a way will clearly be in bijection with (a subset of) $\mathrm{Aut}(M)$. And, of course, $M^M \times M^M$ along with M^M are microlinear, as per proposition 4.11. Now, any element t of $T_{\mathrm{id}_M} \mathrm{Aut}(M)$ is clearly associated with an element of $T_{\mathrm{id}_M}(M^M)$, namely the "same" map, extending the codomain. Thus, it is an element of $\mathfrak{X}(M)$, thanks to the last interpretation of vector fields we observed above. Conversely, by proposition 5.9, any element of $\mathfrak{X}(M)$ maps D to invertible maps in M^M, in other words elements of $\mathrm{Aut}(M)$. In summary, the map $\mathfrak{X}(M) \to T_{\mathrm{id}_M} \mathrm{Aut}(M)$ given by

$$(t : D \to M^M) \mapsto (\bar{t} : D \to \mathrm{Aut}(M))$$

is a bijection. It is trivially homogeneous ($\lambda t \mapsto \lambda \bar{t}$), which means it is linear, thanks to proposition 2.14.   □

5.4. The Lie algebra of vector fields

In classical differential geometry, the Lie bracket of two vector fields is often thought of a commutator of the "flows" induced by the vector fields (and in the case of matrix Lie groups the Lie bracket is indeed represented by matrix commutators). The "problem", as noted earlier, is that classical flows may not correspond to permutations of the manifold (again, in general only a semi-group of transformations). That isn't to say that differential geometers suffer greatly from this, but we again take the opportunity to show how synthetic differential geometry actually behaves how we would like to *think* about geometry behaving. That is, the Lie bracket will be directly associated to the commutator of the bijections defined by their flows.

Let X, Y be vector fields on M. They each define bijections $X_d, Y_d : M \to M$ for all $d \in D$. Let $d_1, d_2 \in D$, and consider the commutator of X_{d_1}, Y_{d_2} as a map $\tau : D \times D \to M^M$:

$$\tau(d_1, d_2) = Y_{-d_2} \circ X_{-d_1} \circ Y_{d_2} \circ X_{d_1}$$

Let us recall lemma 5.3, which, in summary, states that for any $g : D \times D \to M$ such that

$$g(d_1, 0) = g(0, d_2) = g(0, 0) \quad \forall d_1, d_2 \in D$$

there exists a unique map $t : D \to M$ with the property

$$g(d_1, d_2) = t(d_1 d_2)$$

The same conclusion applies when instead we have $g : D \times D \to M^M$, as the only property of M used in the proof is that it is microlinear. It will be the case that τ satisfies those conditions:

$$
\begin{aligned}
\tau(d_1, 0) &= Y_0 \circ X_{-d_1} \circ Y_0 \circ X_{d_1} \\
&= X_{-d_1} \circ X_{d_1} \\
&= \mathrm{id}_M = \tau(0, 0) \\
\tau(0, d_2) &= Y_{-d_2} \circ X_0 \circ Y_{d_2} \circ X_0 \\
&= Y_{-d_2} \circ Y_{d_2} \\
&= \mathrm{id}_M = \tau(0, 0)
\end{aligned}
$$

Therefore there exists a unique $t : D \to M$ with $\tau(d_1, d_2) = t(d_1 d_2)$ for all d_1, d_2. This allows us make the following

DEFINITION 5.11. Let X, Y be vector fields on M (a microlinear object). The *Lie bracket* of X and Y is the unique vector field, denoted $[X, Y]$, such that

$$[X, Y](d_1, d_2) = Y_{-d_2} \circ X_{-d_1} \circ Y_{d_2} \circ X_{d_1}$$

What we'll be pleased to find is that this bracket is bilinear, antisymmetric, and it satisfies the Jacobi identity. In short:

PROPOSITION 5.12. *The pair $(\mathfrak{X}(M), [-, -])$ is a Lie algebra.*

PROOF.   • The Lie bracket is antisymmetric. We only need to verify that $[X, Y](d) = -[Y, X](d)$ for $d = d_1 d_2$, for some $d_1, d_2 \in D$, by virtue of the uniqueness property of $[-, -]$.

$$\begin{aligned}
[X, Y]_{d_1 d_2} &= [X, Y]_{-(-d_1 d_2)} \\
&= ([X, Y]_{-d_1 d_2})^{-1} \\
&= (Y_{-d_2} \circ X_{d_1} \circ Y_{d_2} \circ X_{-d_1})^{-1} \\
&= X_{d_1} \circ X_{-d_2} \circ X_{-d_1} \circ Y_{d_2} \\
&= [Y, X]_{d_2(-d_1)} \\
&= (-[Y, X])_{d_1 d_2}
\end{aligned}$$

• Again we utilize that $\mathfrak{X}(M)$ is Euclidean to prove linearity, by only proving homogeneity (prop. 2.14). Let $\lambda \in R$. Then

$$\begin{aligned}
\lambda[X, Y]_{d_1 d_2} &= [X, Y]_{\lambda d_1 d_2} \\
&= Y_{-d_2} \circ X_{-\lambda d_1} \circ Y_{d_2} \circ X_{\lambda d_1} \\
&= Y_{-d_2} \circ (\lambda X)_{-d_1} \circ Y_{d_2} \circ (\lambda X)_{d_1} \\
&= [\lambda X, Y]_{d_1 d_2}
\end{aligned}$$

• It remains to prove the Jacobi identity:

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

for all vector fields X, Y, Z. We will not prove it here, but the idea is the same as in the previous proofs. First note that it suffices to prove it on elements $d_1 d_2 d_3$. You then expand the left hand side to

$$\begin{aligned}
& Z_{-d_3} \circ X_{-d_1} \circ Y_{-d_2} \circ X_{d_1} \circ Y_{d_2} \circ Z_{d_3} \circ Y_{-d_2} \circ X_{-d_1} \circ Y_{d_2} \circ X_{d_1} \\
& \circ X_{-d_1} \circ Y_{-d_2} \circ Z_{-d_3} \circ Y_{d_2} \circ Z_{d_3} \circ X_{d_1} \circ Z_{-d_3} \circ Y_{-d_2} \circ Z_{d_3} \circ Y_{d_2} \\
& \circ Y_{-d_2} \circ Z_{-d_3} \circ X_{-d_1} \circ Z_{d_3} \circ X_{d_1} \circ Y_{d_2} \circ X_{-d_1} \circ Z_{-d_3} \circ X_{d_1} \circ Z_{d_3}
\end{aligned}$$

and make a series of judicious manipulations to reach $0(d_1 d_2 d_3 = \mathrm{id}_M$. The complete proof can be found in [**5**].

$\square$

5.5. Derivations

The final construction we'll look at is that of the Lie derivative associated with a vector field. Its definition is much the same as the classical one; we evaluate functions $M \to R$ along the flow induced by a vector field. Recall from the previous section that a vector field X is equivalent to giving a function

$$X : M \times D \to M$$

Hence, given a function $f : M \to R$ the mapping

$$f(X(p, -) : D \to R$$

is determined uniquely by $f(X(p, 0)) = f(p)$, and some $b \in R$ with

$$f(X(p, d)) = f(p) + b(p)d$$

Here $p \in M$ is fixed, and b depends on it (and f, and X), as we've made explicit. Varying p gives a function $M \to R$, leading to this

DEFINITION 5.13. Let $f : M \to R$ be a function, and $X \in \mathfrak{X}(M)$. The Lie derivative of f along X is the unique function

$$\mathcal{L}_X f : M \to R$$

such that

$$f(X(p, d)) = f(p) + d \cdot (\mathcal{L}_X f)(p)$$

The Lie derivative is a derivation, as we will see shortly. We should call to memory that a *derivation* of an algebra A is a linear map $D : A \to A$ such that $D(fg) = D(f)g + fD(g)$ for all $f, g \in A$.

PROPOSITION 5.14. *The map given by associating each function $f : M \to R$ to its Lie derivative along X is a derivation of R-algebras.*

PROOF. Homogeneity, and therefore linearity, is a given, since

$$(\lambda \mathcal{L}_X f)(p) = \lambda f(p) + d \cdot \lambda (\mathcal{L}_X f)(p)$$
$$(\mathcal{L}_X (\lambda f))(p) = (\lambda f)(p) + d \cdot (\mathcal{L}_X (\lambda f))(p)$$
$$= \lambda f(p) + d \cdot (\mathcal{L}_X (\lambda f))(p)$$

and by uniqueness $\cdot \lambda (\mathcal{L}_X f)(p) = d \cdot (\mathcal{L}_X (\lambda f))(p)$. The "product rule" will follow from the same proof we gave of the calculus version in section 2. Let $f, g : M \to R$

be functions. The Lie derivative along X is defined by the expression

$$
\begin{aligned}
(fg)(X(p,d)) &= f(X(p,d))g(X(p,d)) \\
&= (f(p) + d \cdot (\mathcal{L}_X f)(p))(g(p) + d \cdot (\mathcal{L}_X g)(p)) \\
&= f(p)g(p) + d((\mathcal{L}_X f)(p)g(p) + f(p)(\mathcal{L}_X g)(p))
\end{aligned}
$$

and the unique coefficient multiplying d is clearly the same as for $\mathcal{L}_X f \cdot g + f \cdot \mathcal{L}_X g$. $\square$

In this manner we've recovered the interpretation of vector fields as derivations. One might suspect that furthermore the Lie structure given by the bracket $[-,-]$ of vector fields corresponds to the commutator of their respective derivations (given D_1, D_2 derivations of R^M, their commutator is $[D_1, D_2](f) = D_1(D_2(f)) - D_2(D_1(f))$). If we define the Lie *operator*

$$
\mathcal{L} : \mathfrak{X}(M) \to \mathrm{Der}(R^M)
$$

then we can phrase that suspicion as

PROPOSITION 5.15. *The Lie operator $\mathcal{L}$ is a homomorphism of R-algebras.*

PROOF. Homogeneity is easily established:

$$
\begin{aligned}
f((\alpha X)(p,d)) &= f(X(p, \alpha d)) \\
&= f(p) + \alpha d \cdot (\mathcal{L}_X f)(p)
\end{aligned}
$$

the unique coefficient multiplying d is, by definition $(\mathcal{L}_{\alpha X} f)(p)$. It remains to be seen that

$$
\mathcal{L}_{[X,Y]} f = \mathcal{L}_X(\mathcal{L}_Y f) - L_Y(L_X f)
$$

By definition of the Lie bracket, $\mathcal{L}_{[X,Y]} f$ is determined by $f([X,Y](p,d))$ for those $d = d_1 d_2$ with $d_1, d_2 \in D$. So, we start with that:

$$
f([X,Y](p, d_1 d_2)) = f(p) + d_1 d_2 \cdot (\mathcal{L}_{[X,Y]} f)(p)
$$

Meanwhile, the left hand side is also equal to

$$
f([X,Y](p, d_1 d_2)) = f((Y_{-d_2} \circ X_{-d_1} \circ Y_{d_2} \circ X_{d_1})(p)))
$$

Starting with the right hand side,

$$\ldots = f((X_{-d_1} \circ Y_{d_2} \circ X_{d_1})(p)) - d_2(\mathcal{L}_Y f)((X_{-d_1} \circ Y_{d_2} \circ X_{d_1})(p))$$

$$= f((Y_{d_2} \circ X_{d_1})(p)) - d_1(\mathcal{L}_X f)((Y_{d_2} \circ X_{d_1})(p))$$

$$\quad - d_2(\mathcal{L}_Y f)((Y_{d_2} \circ X_{d_1})(p))$$

$$\quad + d_1 d_2(\mathcal{L}_X(\mathcal{L}_Y f))((Y_{d_2} \circ X_{d_1})(p))$$

$$= f(X_{d_1}(p)) + d_2(\mathcal{L}_Y f)(X_{d_1}(p)) - d_1(\mathcal{L}_X f)(X_{d_1}(p))$$

$$\quad - d_2(\mathcal{L}_Y f)(X_{d_1}(p)) - d_2^2(\mathcal{L}_Y f)(X_{d_1}(p))$$

$$\quad - d_1 d_2(\mathcal{L}_Y(\mathcal{L}_X f))(X_{d_1}(p)) + d_1 d_2(\mathcal{L}_X(\mathcal{L}_Y f))((Y_{d_2} \circ X_{d_1})(p))$$

$$= f(p) + d_1(\mathcal{L}_X f)(p) + d_2(\mathcal{L}_Y f)(p) + d_1 d_2(\mathcal{L}_X(\mathcal{L}_Y f))(p)$$

$$\quad - d_1(\mathcal{L}_X f)(p) - d_1^2(\mathcal{L}_X(\mathcal{L}_X f))(p)$$

$$\quad - d_2(\mathcal{L}_Y f)(p) - d_2^2(\mathcal{L}_Y f)(X_{d_1}(p))$$

$$\quad - d_1 d_2(\mathcal{L}_X(\mathcal{L}_Y f))(p) - d_1 d_2(\mathcal{L}_Y(\mathcal{L}_X f))(p)$$

$$\quad - d_1^2 d_2(\mathcal{L}_X(\mathcal{L}_Y(\mathcal{L}_X f)))(p) + d_1 d_2(\mathcal{L}_X(\mathcal{L}_Y f))(p)$$

$$\quad - d_1 d_2^2(\mathcal{L}_Y(\mathcal{L}_X(\mathcal{L}_Y f)))(p)$$

Cancelling out terms (recall that d_1, d_2 are nilsquares),

$$f([X,Y](p, d_1 d_2)) = f(p) + d_1 d_2(\mathcal{L}_X(\mathcal{L}_Y f) - \mathcal{L}_Y(\mathcal{L}_X f))(p)$$

which proves the identity. $\square$

Though we've established a homomorphism of Lie algebras

$$\mathcal{L} : \mathfrak{X}(M) \to \mathrm{Der}(R^M)$$

we have not proved that it is an isomorphism, as would be the case for classical (paracompact) manifolds. As it would turn out, it's possible that for that to be true require more of M than microlinearity. In [**5**], the author refers to such objects as *reflexive*, and they are those that satisfy that the morphism

$$\mathrm{ev} : M \to \mathrm{alg}(R^M, R)$$

(where $\mathrm{alg}(R^M, R)$ is the set of R-algebra homomorphisms), given by sending each point in M to the evaluation map $f \mapsto f(p)$, is bijective. In the mentioned book it is shown that the Lie operator is bijective if M is reflexive (the converse is left unattended to).

SECTION 6

A quick overview of models

The exposition of the material so far is, as the introduction states, axiomatic. Consider the analogy with a first course in differential calculus; the real numbers most likely simply "given" as some variation of "a complete Archimedean ordered field" that exists (or nothing is stated, and properties are simply invoked when needed). The task of actually exhibiting such an object is left until a later point in time, presumably when the student has matured enough to grasp the necessary material.

Here we have done the same, albeit in a much higher context. To carry on with the theory we have instated a pair $(\mathcal{E}, R)$, where $\mathcal{E}$ is a topos and R is a $\mathbb{Q}$-algebra of $\mathcal{E}$ which satisfies the Kock-Lawvere axiom. Two concerns arise:

- Does there exist such a pair in the standard mathematical universe?
- If so, how do we associate results in SDG to their classical counterparts?

Naturally, both questions have been answered. For instance, in [**3**, §III] the author constructs models which answer the first question (the answer is "yes"). The nature of these models is outside the scope of the present work, but pose an excellent subject for further study (in effect, it would be quite necessary if one were interested in pursuing SDG at a professional level).

The second question is answered by what are referred to as *well-adapted* models of SDG. More work has to be done to exhibit such a model, but again this has been accomplished and one can find it in the literature, for example in the same [**3**]. Once more, the details make heavy use of category theory at an advanced level, but we can describe some of the attractive properties that well-adapted models possess, in regards to comparing SDG with classical geometry.

Let **Mf** be the category of Hausdorff paracompact $\mathcal{C}^\infty$ manifolds. To give a well-adapted model of SDG is to exhibit a category $\mathcal{E}$, and a full and faithful functor

$$i : \mathbf{Mf} \to \mathcal{E}$$

which satisfies certain properties. We won't go into detail about those properties (see [**3**]), but we can look at some of the consequences of such a model. For instance, $\mathcal{E}$ has an object $i(\mathbb{R})$, and this will be our R. It satisfies the Kock-Lawvere axiom for one, and furthermore there are such comparison results such as the following.

Given a smooth map $f : \mathbb{R} \to \mathbb{R}$, through i we obtain a map $i(f) : R \to R$. It is the case (if $(i, \mathcal{E})$ is a well-adapted model), that

$$(i(f))' = i(f')$$

In other words, derivatives in the standard sense correspond exactly with derivatives in SDG. It's also true that i preserves products, so that $\mathbb{R}^n$ gets taken to R^n, and given a function $f : \mathbb{R}^n \to R$, then

$$i\left(\frac{\partial f}{\partial x_j}\right) = \frac{\partial i(g)}{\partial x_j}$$

Another property of well-adapted models is that they preserve tangent bundles. Formally, if M is a manifold in $\mathbf{Mf}$, then there exists a natural isomorphism

$$i(TM) \overset{\alpha_M}{\to} (i(M))^D = T(i(M))$$

With these properties one can now be assured that SDG is well suited to reasoning and drawing conclusions in "standard" differential geometry. One particularly well detailed example of the procedure for doing so can be found in [1], where the authors give a synthetic proof of the Ambrose-Palais-Singer theorem, along with a synthetic treatment of (parts of) the calculus of variations - in particular the characterization of geodesic curves as critical curves of the action integral.

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