A Brief Introduction to Synthetic Differential Geometry

Gaspar Pérez Scornik

Supervised by Xavier Gràcia Sabaté

Universitat Politécnica de Catalunya
Faculty of Mathematics and Statistics

Bachelor's Thesis, 2017

Outline

The Derivative

- "The rate of change, for very small changes"

$$f(x+h) = f(x) + f'(x)h + \mathcal{O}(h^2)$$

Recreating the Derivative Axiomatically

- We begin with an object R. Naturally, this is the “synthetic version” of $\mathbb{R}$.

Recreating the Derivative Axiomatically

- We begin with an object R. Naturally, this is the “synthetic version” of $\mathbb{R}$.

Axiom

R is an algebra over $\mathbb{Q}$.

- Let D be the set

$$D = \left\{ d \in R \mid d^2 = 0 \right\}.$$

- Let D be the set

$$D = \left\{ d \in R \mid d^2 = 0 \right\}.$$

KL Axiom

For all functions

$$f : D \to R,$$

there exist unique $a, b \in R$ such that

$$f(d) = a + bd$$

for all $d \in D$.

- *The coefficient a is easily seen to be $f(0)$.*

Recreating the Derivative Axiomatically

- The derivative can now be defined.
- Let $f : R \to R$ be a function. Fix x, define $g : D \to R$ by

$$g(d) = f(x + d).$$

Recreating the Derivative Axiomatically

- By the KL axiom, there exist unique a, b with

$$g(d) = f(x + d) = a + bd$$

- The first coefficient is $g(0) = f(x)$.

Recreating the Derivative Axiomatically

- By the KL axiom, there exist unique a, b with

$$g(d) = f(x + d) = a + bd$$

- The first coefficient is $g(0) = f(x)$.

Definition

The derivative of f, written $f'(x)$, is the unique coefficient b.

Recreating the Derivative Axiomatically

- Thus we have

$$f(x+d) = f(x) + f'(x)d \quad \forall d \in D$$

An Example

Example

Consider $f(x) = x^n$. Then

$$f(x+d) = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} d + K_2 d^2 + \cdots + K_n d^n$$

$$= x^n + nx^{n-1} d$$

Therefore $f'(x) = nx^{n-1}$.

How is this possible?

How is this possible?

- We have to leave the category of sets.

How is this possible?

- We have to leave the category of sets.
- We have to weaken logic to only constructive logic (no LEM).

How is this possible?

- We have to leave the category of sets.
- We have to weaken logic to only constructive logic (no LEM).
- Toposes are the correct setting for SDG.

How is this possible?

- We have to leave the category of sets.
- We have to weaken logic to only constructive logic (no LEM).
- Toposes are the correct setting for SDG.
- The objects in a topos behave like sets.

Microlinear Objects

- Define “2-infinitesimals”:

$$D(2) = \{(d_1, d_2) \in D \times D \mid d_1 d_2 = 0\}$$

Microlinear Objects

- In particular, if M is microlinear then two maps

$$f, g : D \to M$$

define a unique map

$$h : D(2) \to M$$

such that

$$h(d, 0) = f(d)$$
$$h(0, d) = g(d)$$

Microlinear Objects

- To be thought of as “manifolds”.
- We can define tangent vectors (and they will form a vector space).

Tangent Vectors

- Classically, they are equivalence classes of paths.

Tangent Vectors

- Classically, they are equivalence classes of paths.

Definition

Let M be microlinear. A tangent vector to M at p is a map

$$t : D \to M$$

with $t(0) = p$.

Tangent Vectors

- Classically, they are equivalence classes of paths.

Definition

Let M be microlinear. A tangent vector to M at p is a map

$$t : D \to M$$

with $t(0) = p$.

- With that, the tangent space at p, T_pM, is the set of all tangent vectors at p.

Tangent Vectors

- The tangent space T_pM is an R-vector space.
  - 0 = the constant map p.
  - $\lambda t(d) := t(\lambda d)$
- For addition we need to use microlinearity.

Addition of Tangent Vectors

- Microlinearity guarantees that for t_1, t_2 tangent vectors, there exists a unique

$$s : D(2) \to M$$

such that

$$s(d, 0) = t_1(d)$$
$$s(0, d) = t_2(d)$$

- Define $(t_1 + t_2)(d) = s(d, d)$

- Recall we designated $0 \equiv p$ (ct.)
- Let t be a t.v. The map $s : D(2) \to M$ must satisfy

$$\begin{aligned} s(d,0) &= t(d,0) \\ s(0,d) &= 0(d) = p \end{aligned}$$

so s must be

$$s(d_1, d_2) = t(d_1),$$

and $(t+0)(d) = s(d,d) = t(d)$

The Tangent Bundle

- It is simply the collection of all tangent spaces, M^D. We write TM.
- We have a projection $\pi : TM \to M$ given by

$$t \mapsto t(0)$$

Vector Fields

- They are sections of TM; maps $X : M \to TM$ such that $(\pi \circ X)(p) = p$.
- The set of vector fields is denoted $\mathfrak{X}(M)$.

Vector Fields

- They are sections of TM; maps $X : M \to TM$ such that $(\pi \circ X)(p) = p$.
- The set of vector fields is denoted $\mathfrak{X}(M)$.
    - Each T_pM is a vector space, so by fiberwise operations so is $\mathfrak{X}(M)$.

Vector Fields

- They are sections of TM; maps $X : M \to TM$ such that $(\pi \circ X)(p) = p$.
- The set of vector fields is denoted $\mathfrak{X}(M)$.
  - Each T_pM is a vector space, so by fiberwise operations so is $\mathfrak{X}(M)$.
  - $\mathfrak{X}(M)$ is also an M^R-module, by

$$(fX)(p) = f(p)X(p)$$

Flows

- Exponentials A^X have the property that

$$B \to A^X$$

is equivalent to

$$X \times B \to A$$

$$\left(\tilde{f}(x, b) = f(b)(x)\right)$$

Flows

- Vector fields are maps $X : M \to M^D$, therefore the same as maps
$$X : D \times M \to M.$$
We write the same X because it's comfortable.

Proposition

Flows are additive.

Proof.

Let $f, g : D(2)$ be defined by

$$f(d_1, d_2) = X(d_1 + d_2, p)$$
$$g(d_1, d_2) = X(X(d_1, p), d_2)$$

we have

$$f(d, 0) = g(d, 0)$$
$$g(0, d) = f(0, d)$$

since M is microlinear, $f = g$. $\square$

Summary

- In SDG we define geometric objects directly.
- This requires working axiomatically.
- The right context for these needs is that of topos theory.
- It's possible to recover the "same" constructs from classical geometry.

Summary

- In SDG we define geometric objects directly.
- This requires working axiomatically.
- The right context for these needs is that of topos theory.
- It's possible to recover the "same" constructs from classical geometry.

- Of course, we need to construct a category which satisfies the axioms.
- These are models of SDG.

For Further Reading

- Marta C. Bunge, Felipe Gago, and Ana María San Luis.
  Synthetic differential topology.
  Submitted manuscript, not yet published, 2017.
- Anders Kock.
  Synthetic Differential Geometry.
  Cambridge University Press, 2nd edition, 2006.
- Anders Kock.
  Synthetic Geometry of Manifolds.
  Cambridge University Press, 2010.
- René Lavendhomme.
  Basic Concepts of Synthetic Differential Geometry.
  Springer Science+Business Media, B.V., 1996.