A Brief Introduction to Synthetic Differential Geometry

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Outline

- 1 Introductory theory
 - Single Variable Calculus
 - A Primer on Toposes
- 2 Microlinear Objects
 - The Tangent Bundle
 - Vector Fields and Flows

The Derivative

"The rate of change, for very small changes"

$$f(x+h) = f(x) + f'(x)h + \mathcal{O}(h^2)$$

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Axiom

R is an algebra over \mathbb{Q} .



Recreating the Derivative Axiomatically The Infinitesimals

Let D be the set

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KL Axiom

For all functions

$$f: D \rightarrow R$$
,

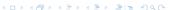
there exist unique $a, b \in R$ such that

$$f(d) = a + bd$$

for all $d \in D$.

■ The coefficient a is easily seen to be f(0).





- The derivative can now be defined.
- Let $f: R \to R$ be a function. Fix x, define $g: D \to R$ by

$$g(d)=f(x+d).$$



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$$g(d) = f(x+d) = a + bd$$

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Definition

The derivative of f, written f'(x), is the unique coefficient b.



Thus we have

$$f(x+d) = f(x) + f'(x)d \quad \forall d \in D$$

An Example

Example

Consider $f(x) = x^n$. Then

$$f(x+d) = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}d + K_2d^2 + \dots + K_nd^n$$
$$= x^n + nx^{n-1}d$$

Therefore $f'(x) = nx^{n-1}$.





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- Toposes are the correct setting for SDG.
- The objects in a topos behave like sets.

Microlinear Objects

Define "2-infinitesimals":

$$D(2) = \{(d_1, d_2) \in D \times D \mid d_1 d_2 = 0\}$$





Microlinear Objects

■ In particular, if *M* is microlinear then two maps

$$f, g: D \rightarrow M$$

define a unique map

$$h: D(2) \rightarrow M$$

such that

$$h(d,0)=f(d)$$

$$h(0,d)=g(d)$$





Microlinear Objects

- To be thought of as "manifolds".
- We can define tangent vectors (and they will form a vector space).



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Definition

Let *M* be microlinear. A tangent vector to *M* at *p* is a map

$$t:D\to M$$

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Definition

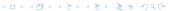
Let *M* be microlinear. A tangent vector to *M* at *p* is a map

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■ With that, the tangent space at p, T_pM , is the set of all tangent vectors at p.





- The tangent space T_pM is an R-vector space.
 - \bullet 0 = the constant map p.
 - $\lambda t(d) := t(\lambda d)$
- For addition we need to use microlinearity.

Addition of Tangent Vectors

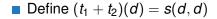
■ Microlinearity guarantees that for t_1 , t_2 tangent vectors, there exists a unique

$$s: D(2) \rightarrow M$$

such that

$$s(d,0)=t_1(d)$$

$$s(0,d)=t_2(d)$$







Addition of Tangent Vectors

The Neutral Element

- Recall we designated $0 \equiv p$ (ct.)
- Let t be a t.v. The map $s : D(2) \rightarrow M$ must satisfy

$$s(d,0) = t(d,0)$$

 $s(0,d) = 0(d) = p$

so s must be

$$s(d_1,d_2)=t(d_1),$$

and
$$(t+0)(d) = s(d,d) = t(d)$$

The Tangent Bundle

- It is simply the collection of all tangent spaces, M^D . We write TM.
- We have a projection $\pi: TM \to M$ given by

$$t \mapsto t(0)$$



Vector Fields

- They are sections of TM; maps $X : M \to TM$ such that $(\pi \circ X)(p) = p$.
- The set of vector fields is denoted $\mathfrak{X}(M)$.





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- The set of vector fields is denoted $\mathfrak{X}(M)$.
 - Each T_pM is a vector space, so by fiberwise operations so is $\mathfrak{X}(M)$.
 - \blacksquare $\mathfrak{X}(M)$ is also an M^R -module, by

$$(fX)(p) = f(p)X(p)$$





Flows

 \blacksquare Exponentials A^X have the property that

$$B \rightarrow A^X$$

is equivalent to

$$X \times B \rightarrow A$$

$$\left(\tilde{f}(x,b)=f(b)(x)\right)$$





Flows

■ Vector fields are maps $X : M \rightarrow M^D$, therefore the same as maps

$$X: D \times M \rightarrow M$$
.

We write the same *X* because it's comfortable.



Proposition

Flows are additive.

Proof.

Let f, g : D(2) be defined by

$$f(d_1, d_2) = X(d_1 + d_2, p)$$

$$g(d_1, d_2) = X(X(d_1, p), d_2)$$

we have

$$f(d,0) = g(d,0)$$

 $g(0,d) = f(0,d)$

since M is microlinear, f = g.







Summary

- In SDG we define geometric objects directly.
- This requires working axiomatically.
- The right context for these needs is that of topos theory.
- It's possible to recover the "same" constructs from classical geometry.

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- The right context for these needs is that of topos theory.
- It's possible to recover the "same" constructs from classical geometry.
- Of course, we need to construct a category which satisfies the axioms.
- These are models of SDG.



For Further Reading

- Marta C. Bunge, Felipe Gago, and Ana María San Luis. Synthetic differential topology. Submitted manuscript, not yet published, 2017.
- Anders Kock. Synthetic Differential Geometry. Cambridge University Press, 2nd edition, 2006.
- Anders Kock. Synthetic Geometry of Manifolds. Cambridge University Press, 2010.
- René Lavendhomme. Basic Concepts of Synthetic Differential Geometry. Springer Science+Business Media, B.V., 1996.



