A Brief Introduction to Synthetic Differential Geometry

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Motivation

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SDG attempts to work with the classical constructs of DG from an axiomatic standpoint. All of the backdrop of coordinate frames is done away with, and we start directly with geometric considerations.

Outline

The Derivative

- “The rate of change, for very small changes”

$$f(x+h) = f(x) + f'(x)h + \mathcal{O}(h^2)$$

Recreating the Derivative Axiomatically

- We begin with an object R. Naturally, this is the “synthetic version” of $\mathbb{R}$.

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Axiom

R is an algebra over $\mathbb{Q}$.

- Let D be the set

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KL Axiom

For all functions

$$f : D \to R,$$

there exist unique $a, b \in R$ such that

$$f(d) = a + bd$$

for all $d \in D$.

- *The coefficient a is easily seen to be $f(0)$.*

Recreating the Derivative Axiomatically

- The derivative can now be defined.
- Let $f : R \to R$ be a function. Fix x, define $g : D \to R$ by

$$g(d) = f(x + d).$$

Recreating the Derivative Axiomatically

- By the KL axiom, there exist unique a, b with

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Definition

The derivative of f, written $f'(x)$, is the unique coefficient b.

Recreating the Derivative Axiomatically

- Thus we have

$$f(x+d) = f(x) + f'(x)d \quad \forall d \in D$$

An Example

Example

Consider $f(x) = x^n$. Then

$$f(x+d) = \binom{n}{0} x^n + \binom{n}{1} x^{n-1} d + K_2 d^2 + \cdots + K_n d^n$$

$$= x^n + nx^{n-1} d$$

Therefore $f'(x) = nx^{n-1}$.

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- We have to weaken logic to only constructive logic (no LEM).
- Toposes are the correct setting for SDG.
- The objects in a topos behave like sets.

How is this possible?

[...] all notions, constructions, and proofs involved are presented as if the base category were the category of sets; in particular all constructions on the objects involved are described in terms of "elements" of them. However, it is necessary and possible to be able to understand this naive writing as referring to Cartesian closed categories. It is necessary because the basic axioms of synthetic differential geometry have no models in the category of sets (cf. I S. 1); and it is possible [...]

Weil Algebras

- A Weil algebra is something of the form

$$R[X_1, \dots, X_n]/(P_1, \cdots, P_s)$$

(with some more conditions)

Definition

Let $W = R[X_1, \ldots, X_n]/(P_1, \cdots, P_s)$ be a Weil algebra. The spectrum of W in R is

$$\mathrm{Spec}_{\mathrm{R}}\, W = \{(a_1, \ldots, a_n) \in R^n \mid P_j(a_1, \ldots, a_n) = 0,\ j = 1, \ldots, s\}$$

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Example

Consider $W = R[X]/(X^2)$. In that case

$$\mathrm{Spec}_R\, W = \{a \in R \mid a^2 = 0\} = D.$$

The General KL Axiom

- $W = R[X_1, \ldots, X_n]/(P_1, \cdots, P_s)$. Define

  $$\alpha : W \to R^{\mathrm{Spec}_{\mathrm{R}}\, W}$$

  by sending $g(X_1, \ldots, X_n)$ to the map

  $$(a_1, \ldots, a_n) \mapsto g(a_1, \ldots, a_n)$$

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KL Axiom

The map α is an R-algebra isomorphism.

The General KL Axiom

Examples

- In the case $W = R[X]/(X^2)$, this is the first axiom we gave.
- Another case is $W = R[X]/(X^{k+1})$, which gives us the Taylor expansion of order k.

Microlinear Objects

- Define “2-infinitesimals”:

$$D(2) = \{(d_1, d_2) \in D \times D \mid d_1 d_2 = 0\}$$

Microlinear Objects

- In particular, if M is microlinear then two maps

$$f, g : D \to M$$

define a unique map

$$h : D(2) \to M$$

such that

$$h(d, 0) = f(d)$$
$$h(0, d) = g(d)$$

Microlinear Objects

- To be thought of as “manifolds”.
- We can define tangent vectors (and they will form a “vector space”).

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- Classically, they are equivalence classes of paths.

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Let M be microlinear. A tangent vector to M at p is a map

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Let M be microlinear. A tangent vector to M at p is a map

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- With that, the tangent space at p, T_pM, is the set of all tangent vectors at p.

Tangent Vectors

- The tangent space T_pM is an R-module.
  - 0 = the constant map p.
  - $\lambda t(d) := t(\lambda d)$
- For addition we need to use microlinearity.

Addition of Tangent Vectors

- Microlinearity guarantees that for t_1, t_2 tangent vectors, there exists a unique

$$s : D(2) \to M$$

such that

$$s(d,0) = t_1(d)$$
$$s(0,d) = t_2(d)$$

- Define $(t_1 + t_2)(d) = s(d,d)$

Addition of Tangent Vectors
The Neutral Element

- Recall we designated $0 \equiv p$ (ct.)
- Let t be a t.v. The map $s : D(2) \to M$ must satisfy

$$s(d, 0) = t(d, 0)$$
$$s(0, d) = 0(d) = p$$

so s must be

$$s(d_1, d_2) = t(d_1),$$

and $(t + 0)(d) = s(d, d) = t(d)$

The Tangent Bundle

- It is simply the collection of all tangent spaces, M^D. We write TM.
- We have a projection $\pi : TM \to M$ given by

$$t \mapsto t(0)$$

Vector Fields

- They are sections of TM; maps $X : M \to TM$ such that $(\pi \circ X)(p) = p$.
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- The set of vector fields is denoted $\mathfrak{X}(M)$.
  - Each T_pM is an R-module, so by fiberwise operations so is $\mathfrak{X}(M)$.
  - $\mathfrak{X}(M)$ is also an M^R-module, by

$$(fX)(p) = f(p)X(p)$$

Flows

- Exponentials A^X have the property that

$$B \to A^X$$

is equivalent to

$$X \times B \to A$$

$$\left(\tilde{f}(x, b) = f(b)(x)\right)$$

Flows

- Vector fields are maps $X : M \to M^D$, therefore the same as maps
$$X : D \times M \to M.$$
We write the same X because it's comfortable.

Proposition

Flows are additive.

Proof.

Let $f, g : D(2)$ be defined by

$$f(d_1, d_2) = X(d_1 + d_2, p)$$
$$g(d_1, d_2) = X(X(d_1, p), d_2)$$

we have

$$f(d, 0) = g(d, 0)$$
$$g(0, d) = f(0, d)$$

since M is microlinear, $f = g$. $\square$

Summary

- In SDG we define geometric objects directly.
- This requires working axiomatically.
- The right context for these needs is that of topos theory.
- It's possible to recover the "same" constructs from classical geometry.

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- Of course, we need to construct a category which satisfies the axioms.
- These are models of SDG.

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