Hello everyone and thank you all for being here.

I know all of you, but just in case, my name is Gaspar and I'll be talking about the main points of my bachelor's thesis which I carried out in the past year.

This was done under the supervision of my former differential geometry professor, Xavier Gràcia.

Right now I'll briefly describe the idea behind SDG, but it's best understood by actually seeing the theory developed, so I won't spend much time on general explanations.

Now, I will say that a large part of the motivation behind SDG is somewhat inaccessible to those not involved in *algebraic* geometry. Since at least the beginning of the last century, algebraic geometry has made its home within category theory. Many of the constructs in SDG reflect those developments (toposes originated in alg. geo.). What I'm trying to say is that what I'm about to describe as the goal of SDG is far from the entire picture. One would need to study alg. geo. and cat. th. more intensively to see that picture.

Nevertheless, we can say that SDG attempts to work with the classical constructs of DG from an axiomatic standpoint. All of the backdrop of coordinate frames is dropped, and we start directly with geometric considerations.

Without further ado, let us go through the structure of the talk.

We begin with some introductory theory. As anyone who will eventually study differential geometry studies calculus, so we begin with single variable calculus.

Immediately we will run into problems interpreting the axioms naively. There we describe why topos theory is necessary.

From there we make a big jump to microlinear objects, which will be our "manifolds". As with classical manifolds, we'll construct the tangent bundle and, consequently, tangent vectors.

Now I know that you all know what a derivative is, but I'll include a description just to give some context.

In particular, the interpretation that is useful to consider now is the one we see here. We have a linear (affine) expression, followed by terms that decrease asymptotically as "h" squared.

We'll now recreate that axiomatically. We begin with R, our smooth line. In case there's any doubt, yes, R is intended to be the synthetic version of R.

So far we haven't said what R is. A precise definition will be given later. For now let us require our first axiom, **R is a an algebra over Q**.

The next part is where things start to "synthetic".

Recall the idea of "negligible quadratic terms", or terms in the 1st degree Taylor expansion which are "very small" for small increments. Instead we will posit the existence of elements of R which actually square to zero.

In a ring, we can always define this set. In a field or more generally a domain, it is always equal to the set with one element, that element being zero.

Our first axiom was rather simple, but this one will be more defining. It's referred to as the Kock-Lawvere axiom, after Bill Lawvere and Anders Kock both of whom are principle figures in the development of SDG. It says: *read slide*

That is, every map from D to R is linear, modulo a translation (we could say that every... is affine). This represents the notion that elements of D are so small that their increments can only produce linear changes.

We can now define derivatives. *read slide*. Notice, there's no restriction on f, we will comment on this later.

The KL axiom will imply that there exist unique a and b with...

read slide

This naturally leads to defining...

...giving us the first degree Taylor expansion, in synthetic terms. F of x plus d is literally equal to...

You may (or even should) be thinking that, well, that's very nice and all, but it's not very remarkable seeing as we just made it up. Furthermore, assuming we could derive results (not new results, but perhaps derive them more elegantly) in this "formal calculus", then there is still no link between them and the rest of mathematics, in particular standard calculus.

We will address this problem later on. Very briefly, essentially what one needs to do is to produce a *model* of the axioms within "normal" mathematics. What does that mean? Well, for practical purposes we can say it means within ZF(C) set theory. This is, after all, how we prove that any object-with-properties "exists" in mathematics. When we say "let a group be..." we shortly after prove that the axioms are consistent by producing an example of a group. In the case of SDG it's simply more complicated.

Back to derivatives, let's look at an easy example.

read slide

The terms with powers of d greater than or equal to two all vanish, and we're left with the expression...

By uniqueness of b (blackboard?), f'(x) = nx^(n-1).

So, again, how are we able to do this? I mentioned earlier that we have to give a model for the axioms, but the axioms themselves have also been stated in a somewhat nebulous way. We'll now go into a *bit* more detail, but only slightly so.

First of all, we have to leave the category of sets. This is simply a fact - SDG has no models in the category of sets (this has been proven). Of course, it's not typical to mention explicitly, at least in classical geometry, that one is working in the category of sets. But it is so. When we write things such as "x belonging to A", these phrases can be interpreted in terms of morphisms (which are just functions) in the category of sets.

We have to weaken logic to only constructive logic. We haven't done it but it's a simple exercise to "prove" that R = 0, if the KL axiom holds. The catch is that the crucial step in the proof assumes the truth of the law of the excluded middle, or that for any prop. p, either p is true or not p is true.

Toposes are where we want to be, or more generally "cartesian closed categories". Luckily, one can work in a topos very much in the same way as one works with sets. This is what we are doing right now. Every phrase, formula, etc. has a meaning in terms of the objects and morphisms of a topos (e.g. x\in A).

In other words objects of a topos behave like sets. Indeed, one prime example of a topos is the category of sets. I don't want to get into too much detail, but I'll read a helpful quote from Anders Kock:

read quote slide

With that out of the way, let's move on to some more interesting geometry. The goal of this last part of the talk is to define microlinear objects, and see how they are like abstracted versions of manifolds.

Again, here the fully precise definition of microlinearity requires many categorical preliminaries and definitions. Instead we look at one consequence that will be of use here. It starts with defining "2-infinitesimals". We define them because, a priori, there is no reason to believe that the product of two infinitesimals is always zero. This is a problem, because it means that the sum of two infinitesimals is not an infinitesimal (which aligns with intuition, no matter how small something is surely by adding it enough times you will get somewhere).

(blackboard?)

The property of microlinear objects that we are interested in is the following.

read slide

In fact, in Kock's book a slightly generalized version of this (n vars.) was given as microlinearity, and it was called "infinitesimal linearity". Every map from a tiny enough set (hence "micro") is linear, in the sense that it is defined by it's values on the axes or on "basis elements".

So, back to microlinear objects. They are the new "manifolds", and one way in which this is reflected is that we can talk about tangent vectors, and they do form a vector space.

Synthetic tangent vectors are… well, I find them quite elegant, because *classically* they are equivalence classes of paths (one version of them) through a point which have the same velocity. In this case, our "interval" is so small, that each path defines a single tangent vector.

read slide

They are literally infinitesimal paths through p. That which we always think of is true in this language. This has more consequences as we'll refer to shortly.

For now, the tangent space at p is defined as the set of all tangent vectors to p.

Each T__pM is an R-module, given by the following.

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To define the sum of two tangent vectors, we need to invoke microlinearity.

read slide

Thus, the sum of t__1 and t__2 is simply the action of following each of them in equal part. A linear combination such as (blackboard) is following t__1, and t__2 twice as fast.

These operations define a module structure, but obviously we won't go through all of the details. One proof we will carry out is proving that the zero we mentioned is in fact the neutral element, because it highlights yet again the role of microlinearity

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read slide

With that we may move on to the tangent bundle, which as one expects if they're coming from geometry, is just the collection of all tangent spaces. In other words, it is the set of all functions from D to M.

Since it is the tangent *bundle*, it has a projection onto the original space given by...

Moving on, we define vector fields in the most natural way possible which is as sections of the previously defined tangent bundle. *read slide*

Two things: *read slide*

Our final examination of the theory of SDG will be that of flows (induced by vector fields). Recall that (classic flow def. on blackboard). Keep that there for now.

A quick reminder on function sets:

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The tangent bundle is just a function set, and by the property we just mentioned, a vector space is equivalent to a map...

Writing the same symbol is technically an abuse of notation, but it's extremely common and very convenient.