Laboratory Report: Simulation of Mechanical Resonance for a One-Dimensional Harmonic Oscillator with RK4 Method

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1 Introduction

This report presents the numerical simulation of an one-dimensional harmonic oscillator with use of combinations of various forces acting on a particle of mass m:

- **restoring force (F_u)**, which arises from a parabolic potential $U(x)$ (Formula 1 and is given by Formula 2, where k is the spring constant;

$$U(x) = \frac{1}{2}kx^2 \tag{1}$$

$$F_u(x) = -\frac{dU}{dx} = -kx \tag{2}$$

- **friction force (F_α)**, which opposes the motion and is proportional to the particle's velocity following the Formula 3 where α is the friction coefficient;

$$F_\alpha(v,t) = -\alpha v = -\alpha\frac{dx}{dt} \tag{3}$$

- **external periodic force (F_{ext})**, which is an external driving force that varies sinusoidally with time following the Formula 4, where F_0 is the amplitude and Ω_{ext} is the angular frequency of the external force.

$$F_{ext}(t) = F_0\sin(\Omega_{ext}t) \tag{4}$$

The system's dynamics are governed by Newton's second law, leading to a second-order ordinary differential equation (ODE). The objective is to analyze the system's behavior under different damping conditions and in the presence of an external force, particularly focusing on mechanical resonance phenomena. The total force acting on the particle is the sum of the forces presented previously (Formula 5). According to Newton's second law, the equation of motion (EOM) for the system is easily presented by Formula 6.

$$F_{tot}(x,t) = F_u + F_\alpha + F_{ext} \tag{5}$$

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x - \frac{\alpha}{m}\frac{dx}{dt} + \frac{F_0}{m}\sin(\Omega_{ext}t) \tag{6}$$

For the specific case of vanishing periodic force ($F_0 = 0$), the system exhibits **damped oscillations**. Key characteristic parameters are the undamped natural frequency $\omega_0 = \sqrt{k/m}$ and the characteristic friction time scale $\tau = 2m/\alpha$. Depending on the relationship between ω_0 and τ, it is possible to observe three different regimes of damped motion:

1. **underdamped ($\omega_0\tau > 1$)**, which creates oscillatory motion with decreasing amplitude and its analytical solution for position $x(t)$ is presented in Formula 7 depending on initial position and velocity x_0 and v_0, damped angular frequency Ω_d (Formula 8) and phase ϕ (Formula 9);

$$x(t) = \sqrt{x_0^2 + \left(\frac{x_0 + v_0\tau}{\Omega_d\tau}\right)^2}\, e^{-t/\tau}\cos(\Omega_d t + \phi) \tag{7}$$

$$\Omega_d = \sqrt{\omega_0^2 - \frac{1}{\tau^2}} \tag{8}$$

$$\phi = \arctan\left(\frac{x_0 + v_0\tau}{x_0\Omega_d\tau}\right) \tag{9}$$

2. **critically damped** ($\omega_0\tau = 1$), in which system returns to equilibrium as quickly as possible without oscillating.

3. **overdamped** ($\omega_0\tau < 1$), where the system returns to equilibrium slowly without oscillating.

In this project we focus primarily on the underdamped regime. It allows for the observation of oscillations with decaying amplitudes, which is crucial for understanding energy dissipation and resonance. For systems without damping, the total energy, sum of kinetic E_{kin} and potential E_{pot} energies, should remain constant. With application of an external periodic force, the system is driven into oscillations. The amplitude of these driven oscillations depends on the frequency of the external force Ω_{ext} relative to the natural frequency ω_0. Therefore, the analytical solution for the maximum amplitude x_{max} as a function of Ω_{ext} can be presented as in Formula 10. Mechanical resonance occurs when Ω_{ext} approaches ω_0, leading to a significant increase in the amplitude of the oscillation, limited only by the damping of the system.

$$x_{max}(\Omega_{ext}) = \frac{F_0}{m\sqrt{(\omega_0^2 - \Omega_{ext}^2)^2 + \left(\frac{2\Omega_{ext}}{\tau}\right)^2}} \tag{10}$$

2 Numerical Algorithm

To solve the second-order ODE (Formula 6), the Runge-Kutta 4th order (RK4) method can be utilized. It is suitable for solving systems of first-order ODEs, therefore, the second-order EOM has to be transformed into a system of two coupled first-order ODEs by introducing a new variable - velocity $v = \frac{dx}{dt}$. The transformation is presented more specifically in Formula 11 and Formula 12, which results in the further representation of the system in vector form (Formula 13), where $\mathbf{u} = \begin{bmatrix} x \\ v \end{bmatrix}$ is the state vector, and function $\mathbf{f}(t, \mathbf{u})$ if defined by the Formula 14.

$$\frac{dx}{dt} = v \tag{11}$$

$$\frac{dv}{dt} = \frac{F_{tot}}{m} = -\frac{k}{m}x - \frac{\alpha}{m}v + \frac{F_0}{m}\sin(\Omega_{ext}t) \tag{12}$$

$$\frac{d\mathbf{u}}{dt} = \mathbf{f}(t, \mathbf{u}) \tag{13}$$

$$\mathbf{f}(t, \mathbf{u}) = \begin{bmatrix} v \\ -\frac{k}{m}x - \frac{\alpha}{m}v + \frac{F_0}{m}\sin(\Omega_{ext}t) \end{bmatrix} \tag{14}$$

The RK4 algorithm operates on each iteration on the current state $\mathbf{u}(t)$ and time t. For each time step Δt, it calculates the change steps k_1, k_2, k_3 and k_4 presented by the Formulas 15 - 18. Then, the solution is updated for further iteration using Formula 19, with simultaneous update of time (Formula 20). The implementation uses fixed parameters $k = m = \omega_0 = 1$, and initial conditions $x_0 = 1$, $v_0 = 0$. Other parameters like α, F_0, and Ω_{ext} are varied according to the specific tasks.

$$\mathbf{k_1} = \mathbf{f}(t, \mathbf{u}(t)) \tag{15}$$

$$\mathbf{k_2} = \mathbf{f}\left(t + \frac{\Delta t}{2}, \mathbf{u}(t) + \frac{\Delta t}{2}\mathbf{k_1}\right) \tag{16}$$

$$\mathbf{k_3} = \mathbf{f}\left(t + \frac{\Delta t}{2}, \mathbf{u}(t) + \frac{\Delta t}{2}\mathbf{k_2}\right) \tag{17}$$

$$\mathbf{k_4} = \mathbf{f}(t + \Delta t, \mathbf{u}(t) + \Delta t \mathbf{k_3}) \tag{18}$$

$$\mathbf{u}(t + \Delta t) = \mathbf{u}(t) + \frac{\Delta t}{6}(\mathbf{k_1} + 2\mathbf{k_2} + 2\mathbf{k_3} + \mathbf{k_4}) \tag{19}$$

$$t \leftarrow t + \Delta t \tag{20}$$

3 Results of the Practical Part

Throughout the simulations, fixed values of $k = m = \omega_0 = 1$, initial position $x_0 = 1$, and initial velocity $v_0 = 0$ were used, unless otherwise specified.

3.1 Testing Correctness of the Results

3.1.1 Undamped and Unforced Oscillator

For this scenario, parameters were set to $\alpha = 0$, $F_{ext} = 0$, $\Omega_{ext} = 0$, $t_{max} = 50$, and $N = 10^4$.

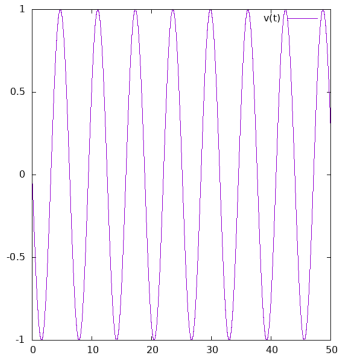

Figure 1: Dependency of the velocity v on time t for undamped oscillator

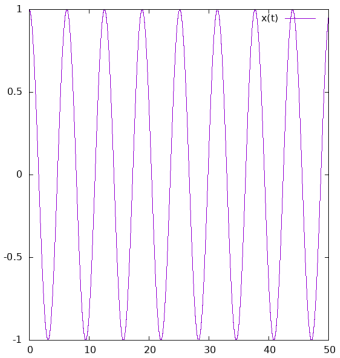

Figure 2: Dependency of the placement x on time t for undamped oscillator

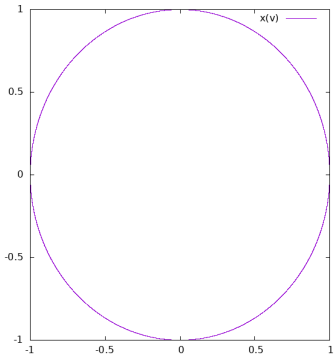

Figure 3: Dependency of the placement x on velocity v for undamped oscillator

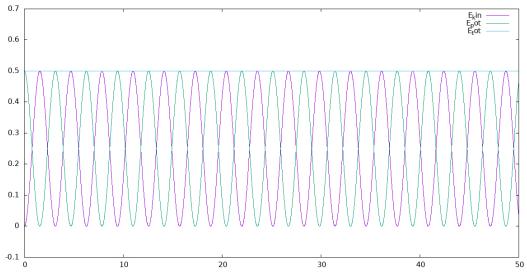

Figure 4: Energy variation for undamped oscillator for different values of friction factor α

The results follow the predictions and confirm that for $\alpha = 0$, the total energy remains constant, and the phase space trajectory is a closed ellipse, indicating the absence of dissipation.

3.1.2 Damped Oscillator ($\alpha = 0.1, F_{ext} = 0$)

The friction parameter was changed to $\alpha = 0.1$, while other parameters (F_{ext}, Ω_{ext}, t_{max}, N) remained the same. The simulated trajectory $x(t)$ is compared with the analytical solution (Formula 7).

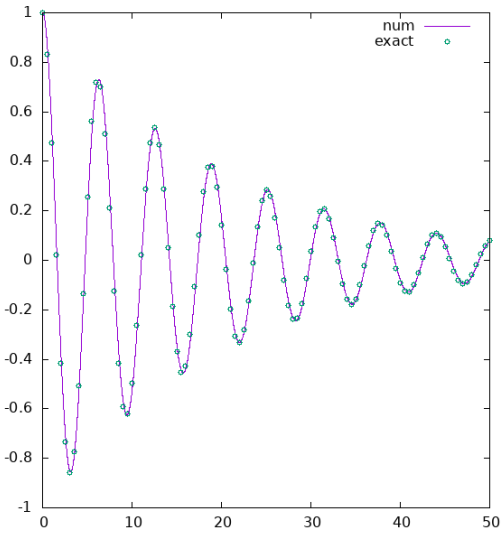

Figure 5: Dependency of the placement x on time t for damped oscillator

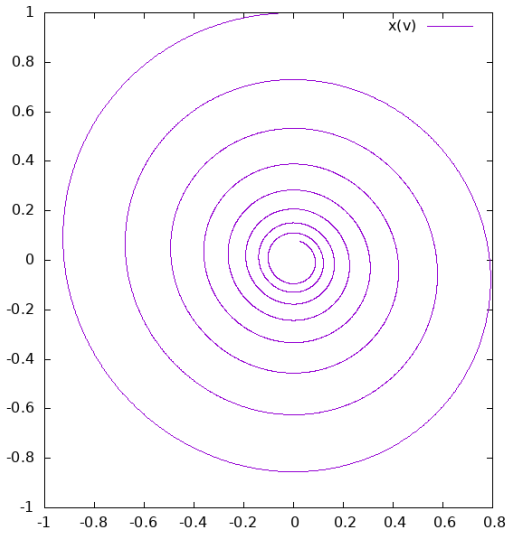

Figure 6: Dependency of the placement x on velocity v for damped oscillator

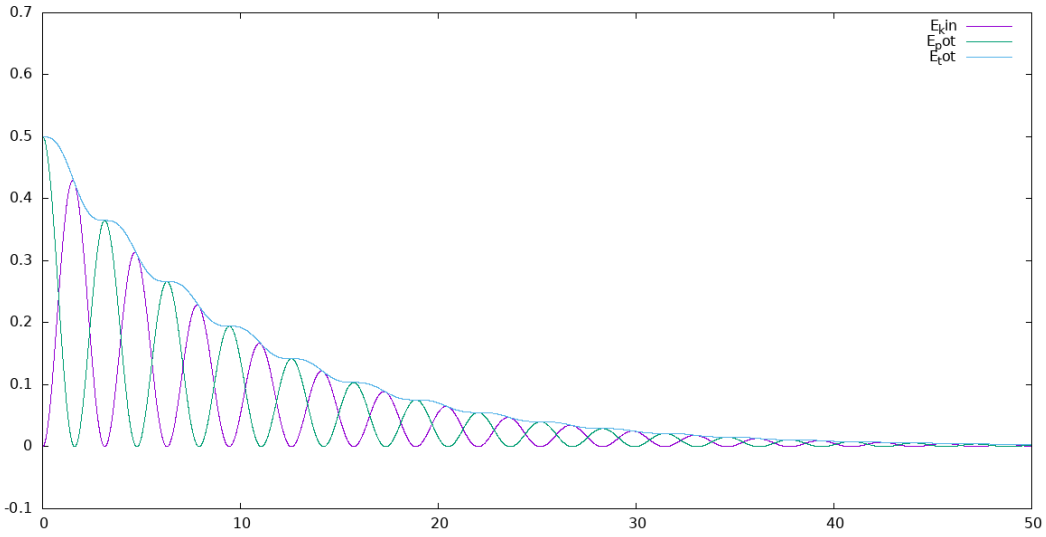

Figure 7: Energy variation for damped oscillator

The numerical solution for $x(t)$ shows well agreement with the analytical solution for $\alpha = 0.1$, ensuring the correctness of RK4 method. The phase space trajectory is visible as a decaying spiral, which clearly indicates the dissipation of the energy. Additionally, all energy components (kinetic, potential, and total) decrease over time, as expected for a damped system. The energy dissipation is continuous, with the rate depending on the instantaneous velocity and position. It is fastest when velocity is high, and slowest as the system approaches equilibrium.

3.2 Energy Dissipation

This part of simulation was performed with repeated values of F_{ext}, t_{max} and N, but for various friction parameters: $\alpha = 10^{-4}, 0.1, 0.5, 1.95$.

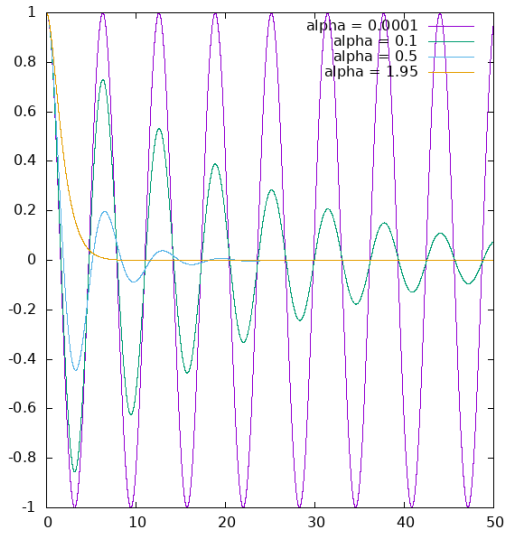

Figure 8: Trajectories $x(t)$ for different α values

As α increases, the damping effect becomes more visible. For $\alpha = 10^{-4}$, the oscillations decay very slowly, nearly copying the undamped case. As α increases to 0.1 and 0.5, the decay becomes progressively faster. For $\alpha = 1.95$, which is close to the critical damping value $\alpha_{crit} = 2\omega_0 m = 2$ (since $\omega_0 = 1, m = 1$), the oscillations are significantly suppressed and decay very quickly, approaching the critically damped regime. For $\alpha > 2$ (overdamped regime), we would expect no oscillations, and the system would return to equilibrium exponentially without crossing the equilibrium position more than once.

3.3 Driving the Damped System with External Periodic Force - Mechanical Resonance

This section investigates the system's response to an external periodic force. Parameters were set to $F_0 = 1$, $\alpha = 1.0$, $t_{max} = 10^3$, $N = 2 \cdot 10^5$.

3.3.1 Test Simulation for $\Omega_{ext} = 0.5\omega_0$

A preliminary simulation was performed with $\Omega_{ext} = 0.5\omega_0 = 0.5$.

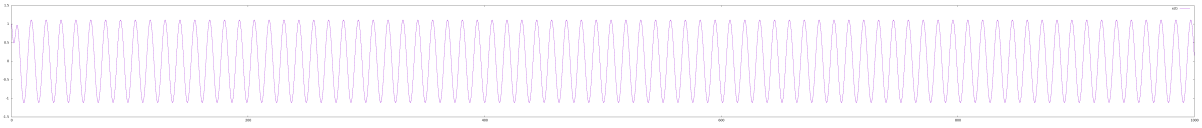

Figure 9: Trajectory $x(t)$ for a driven oscillator with $\alpha = 1.0, F_0 = 1.0, \Omega_{ext} = 0.5$

The plot shows an initial turbulent phase where the amplitude and frequency are slightly irregular, followed by a steady-state oscillation where the object synchronizes with the periodic force.

3.3.2 Dependence of Maximum Amplitude on External Frequency $x_{max}(\Omega_{ext})$

The maximum amplitude of oscillations (x_{max}) was recorded for varying external frequencies Ω_{ext} from 0.1 to 2.0 with a step of 0.01. This process was repeated for different friction parameters: $\alpha = 0.01, 0.1, 0.5, 1.0$. The amplitude was determined by finding the height of the last maximum in the steady-state phase.

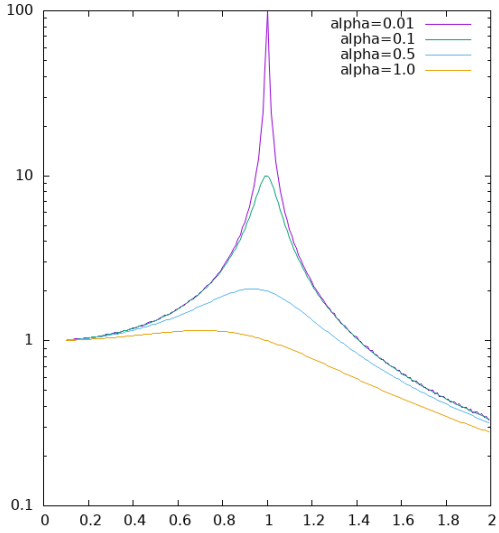

Figure 10: Maximum amplitude $x_{max}(\Omega_{ext})$ for various α values

The plot clearly illustrates the phenomenon of mechanical resonance. For each α value, there is a distinct peak in x_{max} as Ω_{ext} approaches $\omega_0 = 1$. The amplitude at resonance is highest for the smallest damping coefficient ($\alpha = 0.01$), and decreases significantly as α increases. This indicates that damping plays a crucial role in limiting the resonance amplitude, preventing the system from possible destruction.

4 Conclusions

The numerical simulations using the Runge Kutta method accurately reproduced the expected behavior of a one-dimensional harmonic oscillator under various conditions.

- **Testing correctness:** For an undamped and unforced oscillator, the total energy was conserved, and the phase space trajectory was a closed ellipse, confirming the correctness of the numerical implementation. When damping was introduced ($\alpha = 0.1$), the numerical solution for $x(t)$ closely resembled the analytical solution, and the phase space trajectory became a decaying spiral, with total energy dissipating over time.

- **Energy dissipation:** Increasing the friction parameter α consistently led to faster decay of oscillations, demonstrating the impact of damping on energy dissipation. For α approaching the critical damping value ($\alpha_{crit} = 2$) the oscillations were rapidly suppressed. This led to prediction of the behavior of an overdamped systems ($\alpha > \alpha_{crit}$), which would show non-oscillatory exponential decay.

- **Mechanical resonance:** The simulation clearly demonstrated mechanical resonance with use of an external force. The maximum oscillation amplitude x_{max} peaked when the external driving frequency Ω_{ext} was close to the natural frequency ω_0. The height of this resonance peak was inversely related to the damping coefficient α: lower damping resulted in significantly higher resonance amplitudes, nevertheless, its placement on the Ω_{ext} axis stayed the same for every value. This highlights the critical role of damping in regulating the system's response to external forces, preventing unbounded oscillations at resonance.

In conclusion, the Runge Kutta method proved to be an accurate numerical technique for simulating the dynamics of the damped, driven harmonic oscillator. It effectively captured fundamental physical phenomena such as energy dissipation and mechanical resonance.