Simulation of heat diffusion with Crank-Nicolson method

Gabriela Pyda

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1 Introduction

Heat diffusion is a fundamental physical phenomenon describing how thermal energy spreads within a material over time. In macroscopic systems, this process is governed by the diffusion equation, which relates the temporal evolution of temperature to its spatial distribution. In this project, we consider a two-dimensional room exchanging heat with its surroundings through walls and a window, and equipped with a heater supplying thermal energy at a controlled rate.

The temperature field $T(\mathbf{r}, t)$ evolves according to the diffusion equation

$$\frac{\partial T}{\partial t} = D\nabla^2 T + S, \tag{1}$$

where the diffusion constant D characterizes the thermal properties of the medium, and the term $S(\mathbf{r}, t)$ represents internal heat sources or sinks. In the setup considered, S accounts for an electric heater of finite area.

Boundary conditions are essential to correctly capture the heat exchange between the room and the external environment. On each wall and the window, the heat flux leaving the material is proportional to the difference between the local temperature and the external bath temperature. This leads to Robin-type boundary conditions, which reflect physical heat leakage.

The aim of this work is to implement a numerical solver for Eq. (1) using the Crank–Nicolson (CN) scheme, simulate heating dynamics under varying physical parameters, and analyze temperature evolution, spatial profiles, and energy flow. The final goal is to interpret how different degrees of insulation, heater thresholds, and window permeability affect the resulting temperature distributions and thermal balance.

2 Numerical Methods and Algorithms

The numerical simulation is performed on a square domain of side length L, discretized into $N \times N$ spatial nodes separated by a uniform spacing (Eq. (2)), whereas the temporal domain is discretized into steps of size Δt.

$$\Delta = \frac{L}{N-1}. \tag{2}$$

2.1 Finite Difference Approximation

The Laplacian $\nabla^2 T$ in Eq. (1) is approximated using the standard five-point stencil (Eq. (3)), whereas the heater source term is defined as $S_{i,j}$ in Eq. (4), which ensures spatially uniform heating within the heater boundaries.

$$\nabla^2 T \approx \frac{T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j}}{\Delta^2}. \tag{3}$$

$$S_{i,j} = \begin{cases} S_{\max}, & (x_i, y_j) \in [x_a, x_b] \times [y_a, y_b], \\ 0, & \text{otherwise}, \end{cases} \tag{4}$$

2.2 Boundary Conditions

Heat exchange through walls and the window is modeled using Robin boundary conditions, enforcing flux continuity, presented in Eq. (5), where h (or h_w for the window) defines the heat leakage, and T_∞ is the external temperature.

$$-D\,(\mathbf{n} \cdot \nabla T) = h(T - T_\infty), \tag{5}$$

These relations rearrange into explicit expressions for boundary temperatures. For example, for the west wall in Eq. (6). Similar formulas are applied on the remaining walls and on the window region.

$$T_{0,j} = \frac{h\Delta}{D + h\Delta} T_\infty + \frac{D}{D + h\Delta} T_{1,j}. \tag{6}$$

The corner points do not directly participate in boundary flux conditions and are set to averages of their neighbors, as in an example in Eq. (7).

$$T_{0,0} = \frac{1}{2}(T_{0,1} + T_{1,0}). \tag{7}$$

2.3 Crank–Nicolson Method

The Crank–Nicolson method is a second-order implicit time-stepping scheme that evaluates the right-hand side of Eq. (1) at times t_n and t_{n+1} with equal weights, forming the Eq. (8), and then rearranging into the form of Eq. (9), where $R_{i,j}$ is fully known, yields the implicit equation to be solved at each step.

$$\frac{T_{i,j}^{n+1} - T_{i,j}^{n}}{\Delta t} = \frac{D}{2}\left(\nabla^2 T_{i,j}^{n} + \nabla^2 T_{i,j}^{n+1}\right) + \frac{1}{2}\left(S_{i,j}^{n} + S_{i,j}^{n+1}\right) \tag{8}$$

$$L_{i,j} = R_{i,j} \tag{9}$$

2.4 Gauss–Seidel Relaxation

Since Eq. (8) leads to a large linear system at each time step, we employ Gauss–Seidel (GS) relaxation. The update formula is presented in Eq. (10). The iteration continues until the residual norm (Eq. (11)) falls below the tolerance.

$$T_{i,j}^{n+1} \leftarrow \frac{1}{1+2D\Delta t/\Delta^2}\left[\frac{D\Delta t}{2\Delta^2}(T_{i+1,j}^{n+1}+T_{i-1,j}^{n+1}+T_{i,j+1}^{n+1}+T_{i,j-1}^{n+1})+\frac{1}{2}S_{i,j}^{n+1}+R_{i,j}\right] \tag{10}$$

$$C = \sqrt{\sum_{i,j}(L_{i,j}-R_{i,j})^2\Delta^2} \tag{11}$$

2.5 Heater Control

To mimic an automated heating system, a thermostat located at (i_c, j_c) switches the heater on or off according to the below rules:

$$w = 1, \qquad \text{if } T_{i_c,j_c}^n < T_{\text{low}}, \tag{12}$$

$$w = 0, \qquad \text{if } T_{i_c,j_c}^n > T_{\text{high}}. \tag{13}$$

The source term is then replaced by $wS_{i,j}$.

2.6 Energy Balance

The energy supplied by the heater during a single time step is approximated as E_{supplied}^n in Eq. (14) and the heat lost through the window E_{window}^n calculated in Eq. (15).

$$E_{\text{supplied}}^n = \frac{1}{2}(w+w_{\text{old}})\,\Delta^2\Delta t\sum_{i,j}S_{i,j}^n \tag{14}$$

$$E_{\text{window}}^n = \Delta\,\Delta t\sum_{j=j_1}^{j_2}h_w(T_{N-1,j}^n - T_\infty) \tag{15}$$

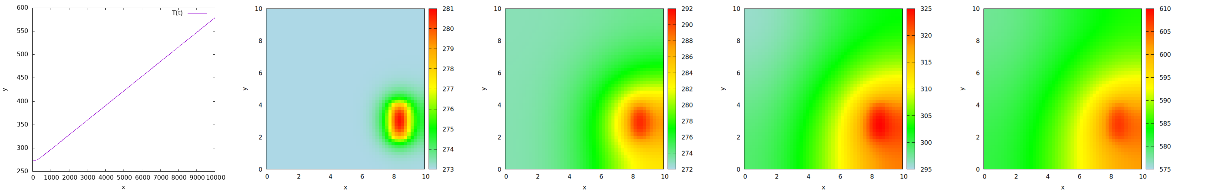
3 Results

The following parameters are used in all simulations:

- domain size: $L = 10$, grid size: $N = 51$,
- time step: $\Delta t = 10$, total time: $T_{\max} = 10^4$,
- diffusion constant: $D = 0.1$,
- heater range: $[8.0, 8.8] \times [2, 4]$, with $S_{\max} = 2$,
- thermostat location: $(x_c, y_c) = (2, 8)$, threshold $T_{\text{low}} = 293$,
- external temperature: $T_\infty = 273$,
- GS relaxation: $k_{\max} = 30$, tolerance 10^{-8}.

3.1 Perfect insulation $h = h_w = 0$ / $T_{max} = 10^4$

The first step to perform is to simulate thermally isolated room. The heater remains on due to $T_{\text{high}} = 10^4$. The results are present in Figure 1.



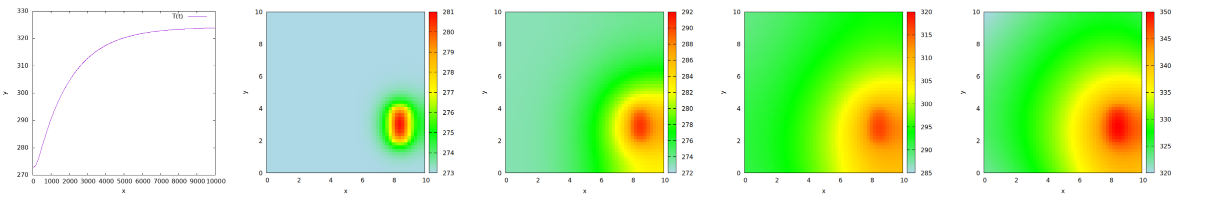
(a) Sensor temp. (b) t = 10s (c) t = 100s (d) t = 1000s (e) t = 10000s

Figure 1: Temperature distribution over the room and at the sensor for $h = h_w = 0$

The temperature increases linearly, and heat maps at selected times reveal uniform spreading of heat until the room reaches high temperatures, then the hottest parts of air are placed in the nearby of the heater.

3.2 Moderate leakage $h = h_w = 0.002$ / $T_{max} = 10^4$

The next experiment covers situation, where the window and walls allow small heat leakage. The visualization is present in Figure 2.



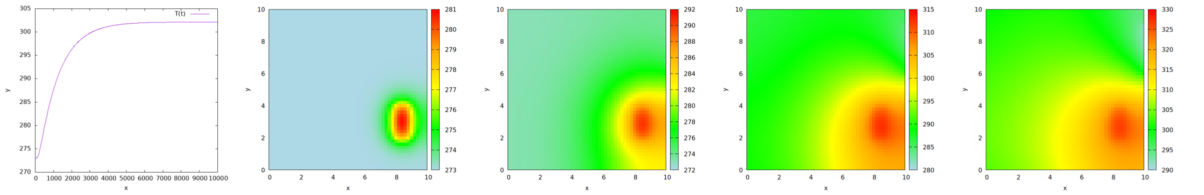
(a) Sensor temp. (b) t = 10s (c) t = 100s (d) t = 1000s (e) t = 10000s

Figure 2: Temperature distribution over the room and at the sensor for $h = h_w = 0.002$

With finite heat exchange, temperature rises visibly slower than in the previous experiment, and therefore rises to smaller maximal temperature. At the start, the change of sensor readings is rapid, but with time and increase of temperature difference between the room and the outside, the work of heater becomes harder. Spatial gradients near boundaries become visible and more accurately depict the curvature of heater's energy.

3.3 Dominant window leakage $h = 0.002, h_w = 0.01$ / $T_{max} = 10^4$

The next scenario covers window dominating heat loss over the walls and results in the Figure 3.



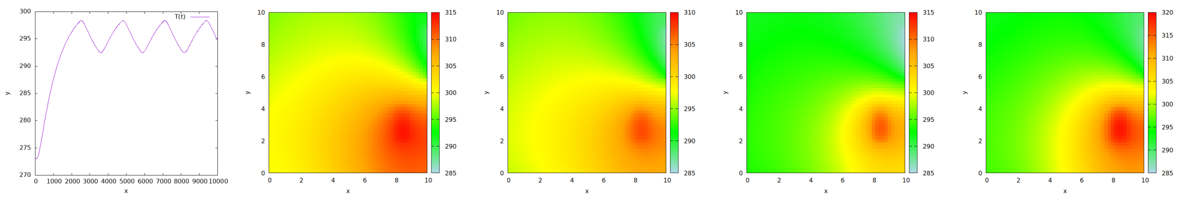
(a) Sensor temp. (b) t = 10s (c) t = 100s (d) t = 1000s (e) t = 10000s

Figure 3: Temperature distribution over the room and at the sensor for $h = 0.002, h_w = 0.01$

It is clearly visible that, while temperature near the heater significantly increases, the nearby of the window stays crucially cooler, which proves that it considerably changed the outcome of the experiment. Additionally, the temperature at the sensor started to converge at some time, which provides the information about the maximal temperature possible to obtain from the simulated heater considering some energy leaks from room.

3.4 Thermostat operation $h = 0.002, h_w = 0.03$ / $T_{max} = 298$

The heater now switches off when the control point exceeds 298 K. This is clearly visualized in Figure 4.



(a) Sensor temp. (b) t = 2500s (c) t = 3000s (d) t = 3500s (e) t = 4000s

Figure 4: Temperature distribution over the room and at the sensor for $h = 0.002, h_w = 0.03$

The sensor's temperature clearly shows the oscillations, which are the result of switching the heater on and off depending on the nearby conditions. The heat distributions are chosen in one of the oscillation phases to show the behavior of temperature mixing. When the heater stops work, higher temperature is placed mostly in the nearby of the heater and near the floor, but with time, it spreads across the rest of the room, equalizing the temperature. Then, when the heater is back on, the temperature in its surroundings rises again, repeating the process.

3.5 Supplied and lost heat $h = 0, h_w = 1.0$ / $T_{max} = 298$

The next point considers a situation with a broken window, which causes great heat loss. The evolution of heat supplied bu the heater and heat lost through the window is presented in Figure 5.

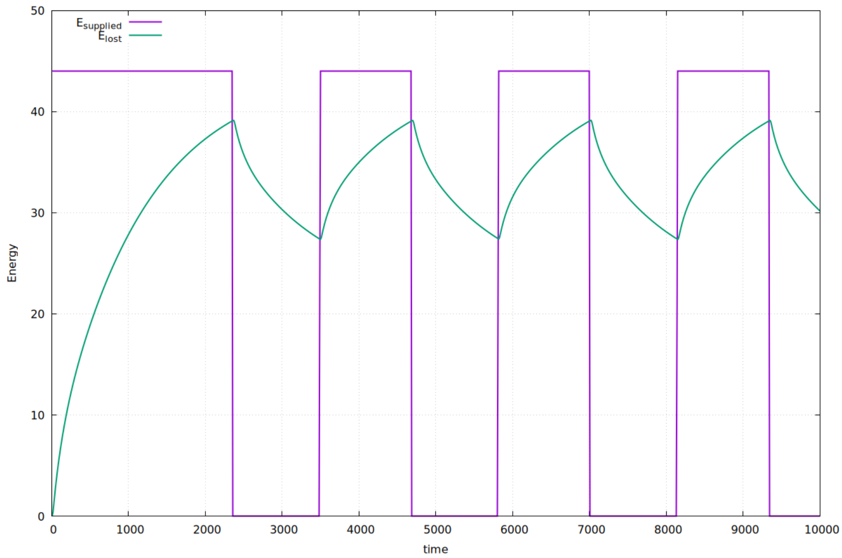


Figure 5: Energy supply and loss for $h = 0, h_w = 1.0$

The supplied heat and lost heat are compared over time, the heat supplied by the heater is visibly switched off in case of reaching the temperature threshold. Nevertheless, the experiment shows strong imbalance and large energy requirements considering large energy loss.

3.6 Energy conservation with $h = 0, h_w = 1.0$ / $T_{max} = 10^4$

The last task was to simulate the energy change for $T_{max} = 10^4$, which is presented in Figure 6.

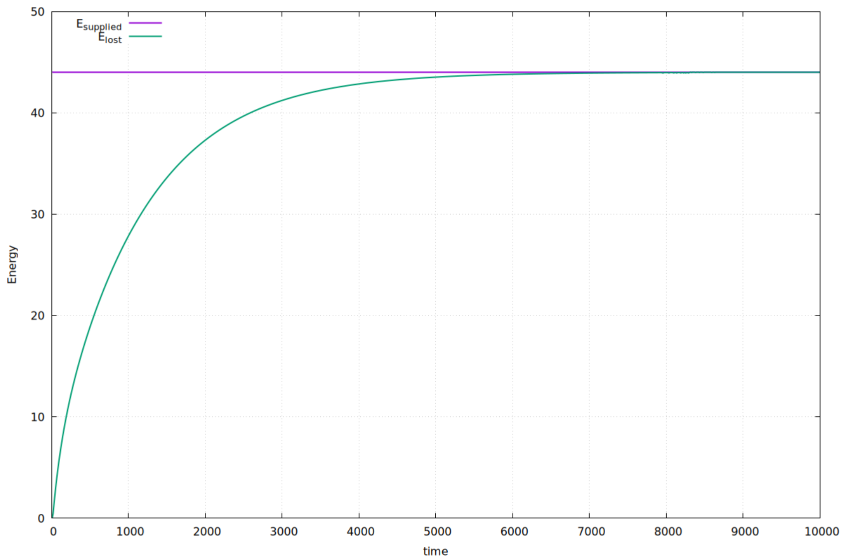


Figure 6: Energy supply and loss for $h = 0, h_w = 1.0$ and

It is clearly visible that heat flow from the window eventually balances the heater input. The two energy curves converge, confirming global conservation of energy.

4 Conclusions

The Crank-Nicolson method, combined with Gauss-Seidel relaxation and physically motivated boundary conditions, provides a stable and accurate framework for simulating heat diffusion in enclosed environments. The numerical experiments demonstrate the influence of insulation quality, window permeability, and heater control on temperature evolution.

Perfect insulation leads to continuous heating and uniform temperature rise, while introducing leakage generates spatial gradients and stabilizes the temperature at lower values. A dominant heat-loss pathway, such as a poorly insulated window, significantly alters the steady-state distribution. Implementing a thermostat allows maintaining a comfortable temperature range with cyclic heater activation.

Energy balance calculations further illustrate thermal dynamics: with strong leakage, heater input must greatly exceed losses, while under conservation conditions the heat supplied and removed eventually converge. These observations highlight the importance of insulation and control systems in efficient thermal management.