Consider the vector field $\mathbf{F} = 2x\mathbf{i} + 2y\mathbf{j} - 3\mathbf{k}$.

(a) Show that $\mathbf{F}$ is a gradient field.

(b) Describe the equipotential surfaces of $\mathbf{F}$ in words and with sketches.

Calculate the divergence of the vector field

$$\mathbf{F} = z\cos\left(e^{y^2}\right)\,\mathbf{i} + x\sqrt{z^2+1}\,\mathbf{j} + e^{2y}\sin 3x\,\mathbf{k}$$

Find the curl of the vector field

$$\mathbf{F} = x^2\,\mathbf{i} - xe^y\,\mathbf{j} + 2xyz\,\mathbf{k}$$

(a) Consider the vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and its curl. Sketch the vector field and use your picture to explain geometrically why the curl is as we calculated in class.

(b) Use geometry to determine $\nabla \times \mathbf{F}$, where

$$\mathbf{F} = \frac{(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})}{\sqrt{x^2 + y^2 + z^2}}$$

(c) For $\mathbf{F}$ as in part (b), verify your intuition by explicitly computing $\nabla \times \mathbf{F}$.

Can you tell in what portions of $\mathbb{R}^2$, the vector fields shown in Figures 3.43-3.46 have positive divergence? Negative divergence?

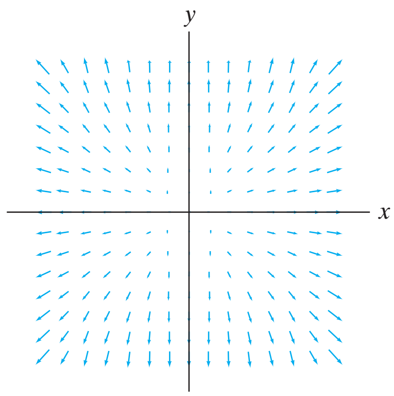

Figure 3.43 Vector field for Exercise 13(a).

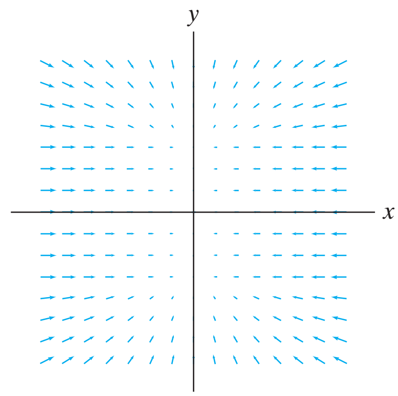

Figure 3.44 Vector field for Exercise 13(b).

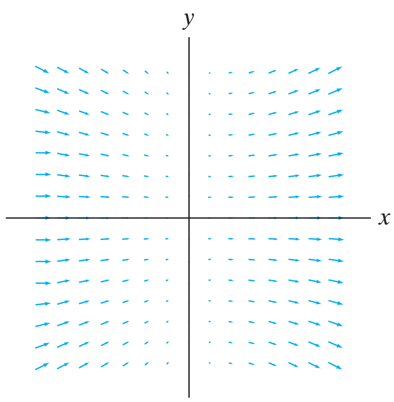

Figure 3.45 Vector field for Exercise 13(c).

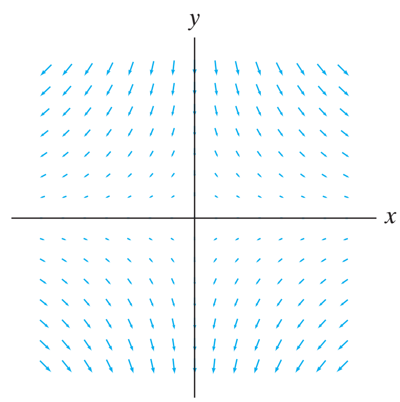

Figure 3.46 Vector field for Exercise 13(d).

THEOREM 4.4:
Let $\mathbf{F} : X \subseteq \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field of class C^2. Then $\operatorname{div}(\operatorname{curl} F) = 0$. That is, $\operatorname{curl} F$ is an incompressible vector field.

Prove Theorem 4.4.

Establish the given identity. (You may assume that any functions and vector fields are appropriately differentiable.)

$$\nabla \cdot (f\mathbf{F}) = f\nabla \cdot \mathbf{F} + \mathbf{F} \cdot \nabla f$$