Find the gradient $\nabla f(\mathbf{a})$ for

$$f(x,y,z) = \cos z \ln(x+y^2)$$
$$\mathbf{a} = \left(e, 0, \frac{\pi}{4}\right)$$

Find the matrix $D\mathbf{f}(\mathbf{a})$ of partial derivatives, for

$$\mathbf{f}(s,t) = (s^2, st, t^2)$$
$$\mathbf{a} = (-1, 1)$$

Find equations for the planes tangent to $z = x^2 - 6x + y^3$ that are parallel to the plane $4x - 12y + z = 7$.

Suppose that you have the following information concerning a differentiable function f:

$$f(2,3) = 12, \qquad f(1.98,3) = 12.1, \qquad f(2,3.01) = 12.2.$$

(a) Give an approximate equation for the plane tangent to the graph of f at (2,3,12).

(b) Use the result of part (a) to estimate $f(1.98, 2.98)$.

Verify the product and quotient rules (Proposition 4.2) for the pair of functions given below.

$$f(x,y) = x^2y + y^3$$
$$g(x,y) = \frac{x}{y}$$

PROPOSITION 4.2:
Let $f, g : X \subseteq \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\mathbf{a} \in X$. Then

1. The product function fg is also differentiable at $\mathbf{a}$, and
$$D(fg)(\mathbf{a}) = g(\mathbf{a})Df(\mathbf{a}) + f(\mathbf{a})Dg(\mathbf{a}).$$
2. If $g(a) \neq 0$, then the quotient function $\frac{f}{g}$ is differentiable at $\mathbf{a}$, and
$$D\left(\frac{f}{g}\right)(\mathbf{a}) = \frac{g(\mathbf{a})Df(\mathbf{a}) - f(\mathbf{a})Dg(\mathbf{a})}{g(\mathbf{a})^2}$$

For the function given below determine all second-order partial derivatives (including mixed partials).

$$f(x,y,z) = x^2e^y + e^{2z}$$

Let $f(x,y) = ye^{3x}$. Give general formulas for $\frac{\partial^n f}{\partial x^n}$ and $\frac{\partial^n f}{\partial y^n}$ where $n \geq 2$.

The three-dimensional heat equation is the partial differential equation

$$k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right) = \frac{\partial T}{\partial t},$$

where k is a positive constant. It models the temperature $T(x,y,z,t)$ at the point (x,y,z) and time t of a body in space.

(a) We examine a simplified version of the heat equation. Consider a straight wire "coordinatized" by x. Then the temperature $T(x,t)$ at time t and position x along the wire is modeled by the one-dimensional heat equation

$$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

Show that the function $T(x,t) = e^{-kt}\cos x$ satisfies this equation. Note that if t is held constant at value t_0, then $T(x,t_0)$ shows how the temperature varies along the wire at time t_0. Graph the curves $z = T(x,t_0)$ for $t_0 = \{0,1,10\}$, and use them to understand the graph of the surface $z = T(x,t)$ for $t \geq 0$. Explain what happens to the temperature of the wire after a long period of time.