

*Instructor's Note:* I recommend that you also look at the Chapter Review on pages 527-528 of Poole and skim the problems to see if there are any concepts or problems that seem challenging to you. Try some of these problems for more practice.

Let  $V$  be a vector space with subspaces  $U$  and  $W$ . Give an example with  $V = \mathbb{R}^2$  to show that  $U \cup W$  need not be a subspace of  $V$ .

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Let  $\mathcal{B}$  and  $\mathcal{C}$  be bases for  $\mathcal{P}_2$ . If  $\mathcal{B} = \{x, 1+x, 1-x+x^2\}$  and the change-of-basis matrix from  $\mathcal{B}$  to  $\mathcal{C}$  is

$$P_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix},$$

find  $\mathcal{C}$ .

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Determine whether  $T : M_{nn} \rightarrow M_{nn}$ , defined by  $T(A) = AB - BA$ , where  $B$  is a fixed  $n \times n$  matrix, is a linear transformation.

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THEOREM 6.14:

Let  $T : U \rightarrow V$  be a linear transformation. Then,

- a.  $T(\mathbf{0}) = \mathbf{0}$ .
- b.  $T(-\mathbf{v}) = -T(\mathbf{v})$  for all  $\mathbf{v}$  in  $V$ .
- c.  $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$  for all  $\mathbf{u}$  and  $\mathbf{v}$  in  $V$ .

Prove Theorem 6.14(b).

$T : M_{22} \rightarrow M_{22}$  defined by  $T(A) = AB - BA$ , where  $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Find either the nullity or the rank of  $T$  and then use the Rank Theorem to find the other.

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THEOREM 6.26:

Let  $V$  and  $W$  be two finite-dimensional vector spaces with bases  $\mathcal{B}$  and  $\mathcal{C}$  respectively, where  $\mathcal{B} = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ . If  $T : V \rightarrow W$  is a linear transformation, then the  $m \times n$  matrix  $A$  defined by

$$A = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

satisfies

$$A[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}}$$

for every vector  $\mathbf{v}$  in  $V$ .

Find the matrix  $[T]_{\mathcal{C} \leftarrow \mathcal{B}}$  of the linear transformation  $T : V \rightarrow W$  with respect to the bases  $\mathcal{B}$  and  $\mathcal{C}$  of  $V$  and  $W$ , respectively. Verify Theorem 6.26 for the vector  $\mathbf{v}$  by computing  $T(\mathbf{v})$  directly and using the theorem.

$T : \mathcal{P}_1 \rightarrow \mathcal{P}_1$  defined by

$$T(a + bx) = b - ax,$$

$$\mathcal{B} = \{1 + x, 1 - x\},$$

$$\mathcal{C} = \{1, x\},$$

$$\mathbf{v} = p(x) = 4 + 2x$$

Use the method of Example 4.29 to compute

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}^k$$

(assume that  $k$  is a positive integer)

[*Hint:* Example 4.29 uses the equality  $M^n = PD^nP^{-1}$ .]

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Let  $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$  be a set of vectors in an  $n$ -dimensional vector space  $V$  and let  $\mathcal{B}$  be a basis for  $V$ . Let  $S = \{[\mathbf{u}_1]_{\mathcal{B}}, \dots, [\mathbf{u}_m]_{\mathcal{B}}\}$  be on the set of coordinate vectors of  $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$  with respect to  $\mathcal{B}$ . Prove that  $\text{span}(\mathbf{u}_1, \dots, \mathbf{u}_m) = V$  if and only if  $\text{span}(S) = \mathbb{R}^n$ .

(Remember that to prove an if-and-only-if theorem, you need to prove both directions.)

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