Determine whether T is a linear transformation:

$T : M_{22} \to M_{22}$ defined by

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & 0 \\ 0 & c+d \end{bmatrix}$$

Determine whether T is a linear transformation:

$T : M_{22} \to M_{22}$ defined by

$$T \begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} 1 & w-z \\ x-y & 1 \end{bmatrix}$$

Determine whether T is a linear transformation:

$T : \mathscr{P}_2 \to \mathscr{P}_2$ defined by

$$T(a + bx + cx^2) = (a + 1) + (b + 1)x + (c + 1)x^2$$

Determine whether T is a linear transformation:

$T : \mathscr{P}_2 \to \mathscr{P}_2$ defined by

$$T(a + bx + cx^2) = a + b(x+1) + b(x+1)^2$$

Let $T : \mathbb{R}^2 \to \mathscr{P}_2$ be a linear transformation for which

$$T\begin{bmatrix}1\\1\end{bmatrix} = 1 - 2x \qquad \text{and} \qquad T\begin{bmatrix}3\\-1\end{bmatrix} = x + 2x^2$$

Find $T\begin{bmatrix}-7\\9\end{bmatrix}$ and $T\begin{bmatrix}a\\b\end{bmatrix}$.

Define linear transformations $S : \mathscr{P}_n \to \mathscr{P}_n$ and $T : \mathscr{P}_n \to \mathscr{P}_n$ by

$$S(p(x)) = p(x+1) \qquad \text{and} \qquad T(p(x)) = p'(x)$$

Find $(S \circ T)(p(x))$ and $(T \circ S)(p(x))$. [*Hint*: Remember the Chain Rule.]

Let $T : \mathscr{P}_2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(a + bx + cx^2) = \begin{bmatrix} a - b \\ b + c \end{bmatrix}$$

(a) Which, if any, of the following polynomials are in $\ker(T)$?

(i) $1 + x$

(ii) $x - x^2$

(iii) $1 + x - x^2$

(b) Which, if any, of the following vectors are in $\operatorname{range}(T)$?

(i) $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

(ii) $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(iii) $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

(c) Describe $\ker(T)$ and $\text{range}(T)$

Let $T : \mathscr{P}_2 \to \mathbb{R}^2$ be the linear transformation defined by

$$T(a+bx+cx^2) = \begin{bmatrix} a-b \\ b+c \end{bmatrix}$$

(Note that this is the same T as in the previous problem.) Find bases for the kernel and range of T. State the nullity and rank of T and verify the Rank Theorem.

Find either the nullity or the rank of T and then use the Rank Theorem to find the other.

$T : M_{33} \to M_{33}$ defined by $T(A) = A - A^T$

Determine whether V and W are isomorphic. If they are, give an explicit isomorphism $T : V \to W$.

$V = S_3$ (symmetric 3×3 matrices), $W = U_3$ (upper-triangular 3×3 matrices).