

Consider the inhomogeneous linear system of differential equations

$$\mathbf{x}' = \begin{bmatrix} 5 & -4 \\ 6 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}.$$

Note that the system matrix has eigenvalues  $\lambda_1 = 1$  and  $\lambda_2 = -1$  with corresponding eigenvectors  $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$  and  $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ .

- (a) Assume that the solution to this system  $\mathbf{x}$  has the form  $\begin{bmatrix} a_1 e^{-t} + b_1 e^t + c_1 t e^t \\ a_2 e^{-t} + b_2 e^t + c_2 t e^t \end{bmatrix}$  for constants  $a, b, c, d, e$  and  $f$ . Use the method of undetermined coefficients to find the general solution to this system.
- (b) Why was the assumption made in part (a) reasonable given what you know about the matrix  $A$ ?
- (c) Recalculate the solution using the integrating factor formula. Reconcile your answer from this calculation with your previous calculation to show they are equivalent.
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Solve the initial value problem

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \cos(\omega t) \\ -\sin(\omega t) \end{bmatrix}$$

where  $\omega$  is a nonzero constant and  $\mathbf{x}(0) = \begin{bmatrix} a \\ b \end{bmatrix}$ . Make sure your solution is valid for all values of  $\omega \neq 0$ . In particular, be careful not to divide by 0 when  $\omega = -2$ . Work out the solution in the case when  $\omega \neq -2$ , then work out the solution in the case when  $\omega = -2$ . You may use either the undetermined coefficients or integrating factor method. You will get extra credit if you perform both calculations and show that they are equivalent.

**Hint:** If you're using the undetermined coefficients method, think about what we did in Math 45 when the forcing function took on the same form as the homogeneous solution.

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As we will discuss in class on Friday, closed-form solutions to non-constant-coefficient linear systems of differential equations are generally unavailable. However, if you are lucky enough to solve a homogeneous (unforced) non-constant-coefficient linear system, solving the inhomogeneous (forced) version is relatively easy. In this exercise, we'll walk you through how this works.

Consider the following initial value problem:

$$\mathbf{x}'(t) = A(t)\mathbf{x} + \mathbf{F}(t) = \begin{bmatrix} 2 & -2e^{-t} \\ e^t & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 4 \\ 6 \end{bmatrix} \quad \text{with} \quad \mathbf{x}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

**Hint:** Throughout this problem, you should use the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

to calculate the inverse of a  $2 \times 2$  matrix. (Unfortunately, there are no simple formulas for larger matrices.) **Note:** To learn more about fundamental matrices and the derivation of the variation of parameters formula, read Section 7.9 of Boyce and diPrima.

- (a) First, verify that

$$\mathbf{x}_h(t) = c_1 \begin{bmatrix} 2e^t \\ e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ e^t \end{bmatrix}$$

is the general solution to the homogeneous version of the ODE (that is, the version of the ODE without the forcing term  $\mathbf{F}$ ) for any  $c_1$  and  $c_2$ .

- (b) Arrange the two linearly independent solutions from part (a) as columns in a matrix  $\Psi(t)$ , which is called a *fundamental matrix*. Verify that  $\Psi'(t) = A(t)\Psi(t)$ .
- (c) The general solution to homogeneous equation is  $\mathbf{x}_h(t) = \Psi(t)\mathbf{c}$  for some constant vector  $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ . What must  $\mathbf{c}$  be to satisfy the initial condition  $\mathbf{x}_h(t_0) = \mathbf{x}_0$ ?
- (d) The *transition matrix* for this system is defined to be  $\Phi(t, s) = \Psi(t)\Psi(s)^{-1}$ . Calculate it. **Note:** The reason why we call this a “transition matrix” is that the effect of multiplication by the matrix  $\Phi(t, t_0)$  is to transition the homogeneous solution forward in time from  $t_0$  to  $t$ .
- (e) Armed with the transition matrix corresponding to the homogeneous ODE, we can now calculate the solution to the inhomogeneous ODE using the variation of

parameters formula

$$\mathbf{x}(t) = \Phi(t, 0)\mathbf{x}(0) + \int_0^t \Phi(t, s)\mathbf{F}(s) \, ds.$$

Use this formula to calculate the solution to the IVP.

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