Intructor's Note: I recommend that you also look at the Chapter Review on pages 527-528 of Poole and skim the problems to see if there are any concepts or problems that seem challenging to you. Try some of these problems for more practice.

Let V be a vector space with subspaces U and W. Give an example with $V = \mathbb{R}^2$ to show that $U \cup W$ need not be a subspace of V.

Let $\mathcal{B}$ and $\mathcal{C}$ be bases for $\mathscr{P}_2$. If $\mathcal{B} = \{x, 1+x, 1-x+x^2\}$ and the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$ is

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix},$$

find $\mathcal{C}$.

Determine whether $T : M_{nn} \to M_{nn}$, defined by $T(A) = AB - BA$, where B is a fixed $n \times n$ matrix, is a linear transformation.

THEOREM 6.14:
Let $T : U \to V$ be a linear transformation. Then,

a. $T(\mathbf{0}) = \mathbf{0}$.
b. $T(-\mathbf{v}) = -T(\mathbf{v})$ for all $\mathbf{v}$ in V.
c. $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in V.

Prove Theorem 6.14(b).

$T : M_{22} \to M_{22}$ defined by $T(A) = AB - BA$, where $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Find either the nullity or the rank of T and then use the Rank Theorem to find the other.

THEOREM 6.26:
Let V and W be two finite-dimensional vector spaces with bases $\mathcal{B}$ and $\mathcal{C}$ respectively, where $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. If $T : V \to W$ is a linear transformation, then the $m \times n$ matrix A defined by

$$A = \Big[[T(\mathbf{v}_1)]_{\mathcal{C}} \quad [T(\mathbf{v}_2)]_{\mathcal{C}} \quad \cdots \quad [T(\mathbf{v}_n)]_{\mathcal{C}} \Big]$$

satisfies

$$A[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}}$$

for every vector $\mathbf{v}$ in V.

Find the matrix $[T]_{\mathcal{C} \leftarrow \mathcal{B}}$ of the linear transformation $T : V \to W$ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$ of V and W, respectively. Verify Theorem 6.26 for the vector $\mathbf{v}$ by computing $T(\mathbf{v})$ directly and using the theorem.

$T : \mathscr{P}_1 \to \mathscr{P}_1$ defined by

$$\begin{aligned} T(a + bx) &= b - ax, \\ \mathcal{B} &= \{1 + x, 1 - x\}, \\ \mathcal{C} &= \{1, x\}, \\ \mathbf{v} &= p(x) = 4 + 2x \end{aligned}$$

Use the method of Example 4.29 to compute

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}^k$$

(assume that k is a positive integer)

[*Hint*: Example 4.29 uses the equality $M^n = PD^nP^{-1}$.]

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be a set of vectors in an n-dimensional vector space V and let $\mathcal{B}$ be a basis for V. Let $S = \{[\mathbf{u}_1]_{\mathcal{B}}, \ldots, [\mathbf{u}_m]_{\mathcal{B}}\}$ be on the set of coordinate vectors of $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ with respect to $\mathcal{B}$. Prove that $\operatorname{span}(\mathbf{u}_1, \ldots, \mathbf{u}_m) = V$ if and only if $\operatorname{span}(S) = \mathbb{R}^n$.

(Remember that to prove an if-and-only-if theorem, you need to prove both directions.)