For each system of DEs shown below, explain whether it is

- linear or nonlinear
- homogeneous (undriven) or inhomogeneous (driven)
- autonomous or nonautonomous.

Also, for any linear system of DEs, rewrite the system using vector & matrix notation.

(a) $$\begin{cases} x' &= \sin(t)x + e^{ty}y + 3t^2 \\ y' &= \cos(t)x + e^{-ty}y \end{cases}$$

(b) $$\begin{cases} x' &= 3x + 4xy \\ y' &= 2x - 3xy \end{cases}$$

(c) $\begin{cases} x' &= 3tx + 4ty \\ y' &= 6t^2x + \sin(t)y \end{cases}$

(d) $\begin{cases} x' &= 3x + 4y + \sqrt{t} \\ y' &= \quad\; - 3y - \sqrt{t} \end{cases}$

(e) $\begin{cases} x' &= 3x + 2y \\ y' &= -x - y \end{cases}$

Here is a general nth-order ODE for $y(x)$.

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y(x) = 0$$

Write it as a system of first-order ODEs, in matrix form.

Consider the second-order ODE $y''(t)+2y'(t)+2y(t)=0$.

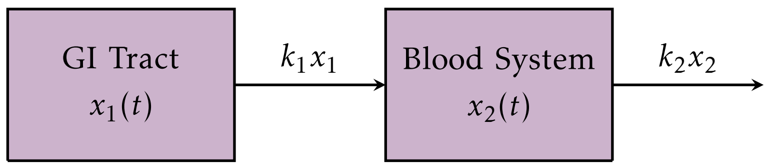
(a) First, solve this ODE using techniques that you learned in Math 45. Express the general solution in two forms: (1) complex exponentials and (2) sines and cosines (with no complex numbers).

(b) Next, convert this second-order ODE to an equivalent first-order DE system. Find the general solution to this system of equations. Use Euler's Identity to rearrange things so that you get a real-valued solution in the end. You should find that your work agrees with your answer from part (a).

Solve the initial-value problem $\mathbf{x}'(t) = A\mathbf{x}(t)$ with $A = \begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix}$ and $\mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

Express your answer as a real-valued function.

At $t = 0$ (i. e. noon) a student takes a fast-dissolving antihistamine capsule. The antihistamine is absorbed from the GI tract (stomach and intestines) into the blood system and then excreted. Let x_1 be the amount of antihistamine in the GI tract and x_2 be the amount in the blood system. Assume that the rate of absorption from the GI tract into the blood system is $k_1 x_1$ and the rate of excretion from the bloodstream (via the kidneys) is $k_2 x_2$, corresponding to the following compartment diagram. Also, assume that $k_2 < k_1$ (the rate of excretion is faster than the rate of absorbtion).



(a) Explain why the amount of antihistamine in the body satisfies the system

$$\frac{dx_1}{dt} = -k_1 x_1$$
$$\frac{dx_2}{dt} = k_1 x_1 - k_2 x_2$$

together with the initial conditions $x_1(0) = \alpha$ and $x_2(0) = 0$ where α is the initial amount of antihistamine in the GI tract just after the capsule has dissolved.

(b) This system of equations is a *cascading* system of equations in that the first equation only involves x_1 and the second equation involves both x_1 and x_2. Therefore, you can solve the first equation by itself, then plug in the solution for x_1 into the second equation and solve for x_2. Solve the system in this fashion.

(c) Next solve the system of equations again using linear algebra (eigenvalues and eigenvectors of the system matrix).

(d) When does the amount of antihistamine in the blood system reach a maximum? What is the maximum amount? Your answers will be in terms of α, k_1, and k_2 (**Hint:** Your final answer for the maximum amount can be written quite simply.)

Make up an initial-value problem involving a system of linear, first-order differential equations that has the property that its solution exists only for $a < t < b$, where a and b are numbers of your choosing. Use ODEToolkit (`http://odetoolkit.hmc.edu`) to draw the solution trajectories. Make sure you label your axes and show that the solution only exists for $a < b < t$.