

For each system of DEs shown below, explain whether it is

- linear or nonlinear
- homogeneous (undriven) or inhomogeneous (driven)
- autonomous or nonautonomous.

Also, for any linear system of DEs, rewrite the system using vector & matrix notation.

$$(a) \begin{cases} x' = \sin(t)x + e^{ty}y + 3t^2 \\ y' = \cos(t)x + e^{-ty}y \end{cases}$$

$$(b) \begin{cases} x' = 3x + 4xy \\ y' = 2x - 3xy \end{cases}$$

$$(c) \begin{cases} x' = 3tx + 4ty \\ y' = 6t^2x + \sin(t)y \end{cases}$$

$$(d) \begin{cases} x' = 3x + 4y + \sqrt{t} \\ y' = -3y - \sqrt{t} \end{cases}$$

$$(e) \begin{cases} x' = 3x + 2y \\ y' = -x - y \end{cases}$$

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Here is a general  $n$ th-order ODE for  $y(x)$ .

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y(x) = 0$$

Write it as a system of first-order ODEs, in matrix form.

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Consider the second-order ODE  $y''(t) + 2y'(t) + 2y(t) = 0$ .

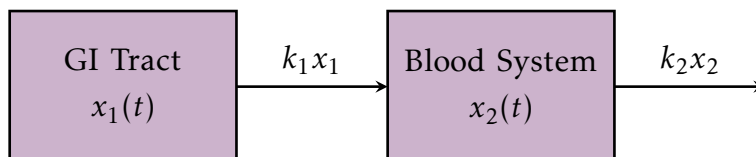
- (a) First, solve this ODE using techniques that you learned in Math 45. Express the general solution in two forms: (1) complex exponentials and (2) sines and cosines (with no complex numbers).
  - (b) Next, convert this second-order ODE to an equivalent first-order DE system. Find the general solution to this system of equations. Use Euler's Identity to rearrange things so that you get a real-valued solution in the end. You should find that your work agrees with your answer from part (a).
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Solve the initial-value problem  $\mathbf{x}'(t) = A\mathbf{x}(t)$  with  $A = \begin{bmatrix} 5 & 5 & 2 \\ -6 & -6 & -5 \\ 6 & 6 & 5 \end{bmatrix}$  and  $\mathbf{x}(0) = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ .

Express your answer as a real-valued function.

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At  $t = 0$  (i. e. noon) a student takes a fast-dissolving antihistamine capsule. The antihistamine is absorbed from the GI tract (stomach and intestines) into the blood system and then excreted. Let  $x_1$  be the amount of antihistamine in the GI tract and  $x_2$  be the amount in the blood system. Assume that the rate of absorption from the GI tract into the blood system is  $k_1x_1$  and the rate of excretion from the bloodstream (via the kidneys) is  $k_2x_2$ , corresponding to the following compartment diagram. Also, assume that  $k_2 < k_1$  (the rate of excretion is faster than the rate of absorption).



- (a) Explain why the amount of antihistamine in the body satisfies the system

$$\begin{aligned}\frac{dx_1}{dt} &= -k_1x_1 \\ \frac{dx_2}{dt} &= k_1x_1 - k_2x_2\end{aligned}$$

together with the initial conditions  $x_1(0) = \alpha$  and  $x_2(0) = 0$  where  $\alpha$  is the initial amount of antihistamine in the GI tract just after the capsule has dissolved.

- (b) This system of equations is a *cascading* system of equations in that the first equation only involves  $x_1$  and the second equation involves both  $x_1$  and  $x_2$ . Therefore, you can solve the first equation by itself, then plug in the solution for  $x_1$  into the second equation and solve for  $x_2$ . Solve the system in this fashion.
- (c) Next solve the system of equations again using linear algebra (eigenvalues and eigenvectors of the system matrix).
- (d) When does the amount of antihistamine in the blood system reach a maximum? What is the maximum amount? Your answers will be in terms of  $\alpha$ ,  $k_1$ , and  $k_2$  (**Hint:** Your final answer for the maximum amount can be written quite simply.)

Make up an initial-value problem involving a system of linear, first-order differential equations that has the property that its solution exists only for  $a < t < b$ , where  $a$  and  $b$  are numbers of your choosing. Use ODEToolkit (<http://odetoolkit.hmc.edu>) to draw the solution trajectories. Make sure you label your axes and show that the solution only exists for  $a < b < t$ .

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