Consider the planar autonomous system

$$x'(t) = 1 - y^2$$
$$y'(t) = 1 - x^2$$

(a) What are the x- and y-nullclines of the DEs? Use these nullclines to determine the regions of the plane in which the solution orbits would be moving left, right, up, and down.

(b) Locate all the equilibrium points of the system.

(c) For each equilibrium point, (i) write down the linearized differential equation about that point, (ii) calculate the eigenvalues of the linearized system, (iii) use the eigenvalues to determine the behavior (unstable/stable, node/spiral, etc.) of the linearized system near each point.

(d) Print out a phase plane portrait of the DE in the rectangle $|x| \le 3, |y| \le 3$ using either pplane[1] or ODEToolkit[2]. Explain how this phase plane portrait is consistent with all of the observations you've made on this problem.

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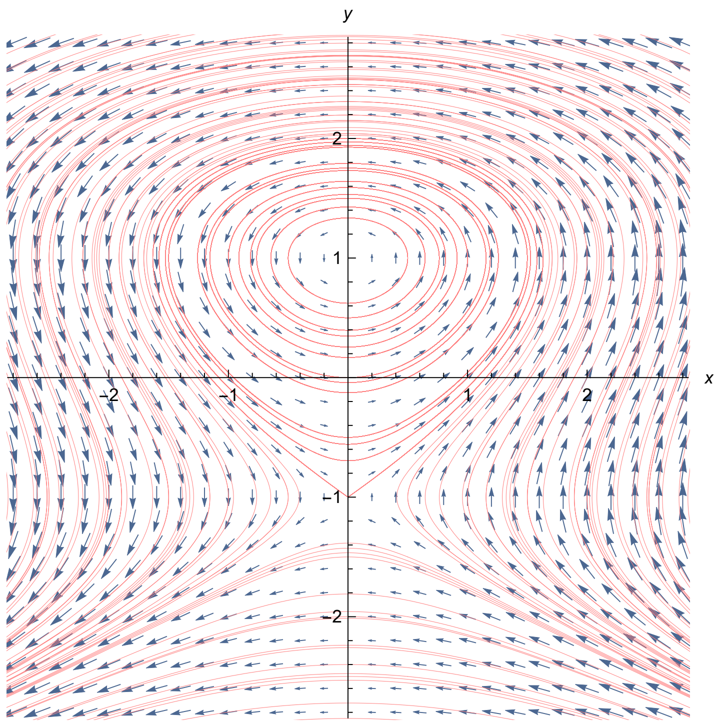
[1] `http://math.rice.edu/~dfield/dfpp.html`
[2] `http://odetoolkit.hmc.edu`

Here's a phase plane portrait for a planar, autonomous system of DEs. Find a system of DEs that matches this phase plane portrait as best as you can. Use software to print out your phase plane portrait. Explain thoroughly (using nullclines, where the solutions are moving left/right/up/down, equilibrium points) why your system of DEs matches the given phase plane portrait.



Consider the linear, constant-coefficient, homogeneous differential equation system

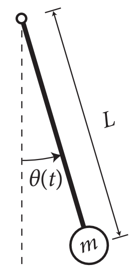
$$\mathbf{x}'(t) = \begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}.$$

Pick values of a, b, c, and d so that the equilibrium point at the origin is

- an unstable saddle point,
- an improper node (asymptotically stable or unstable, your choice),
- a center (neutrally stable), and
- a spiral (asymptotically stable or unstable, your choice).

Use software to print out a phase plane portrait for each case. On top of your portrait, plot the nullclines and show that solution curves cross them in the proper way.

A m kg mass is suspended at the end of a (massless) rod with length L meters. The other end of the rod is fixed at a pivot point. This arrangement that leads to a simple pendulum that swings in a plane. Let $\theta(t)$ be the angular position of the pendulum, measured so that the resting position of the pendulum corresponds to $\theta = 0$ and positive θ is in the direction of the arrow below.



There are two forces acting on the pendulum: gravity and friction. You may assume that the friction (due to air resistance or friction in the pivot) causes a torque that is proportional to the angular velocity, $\dot{\theta}$.

(a) First, derive a governing equation using the rotational analog of Newton's second law, $\tau = I\ddot{\theta}$. (See Chapter 18 of your Physics 24 textbook.) You can assume that the mass is concentrated at a point and the rod is massless, so the moment of inertia of the mass is $I = mL^2$. You should obtain the differential equation

$$\ddot{\theta} + c\dot{\theta} + \frac{g}{L}\sin\theta = 0.$$

What must be the units of the damping constant c?

(b) Write your second-order differential equation as a system of first-order equations by defining $\omega(t) = \dot{\theta}(t)$. Are they linear or nonlinear? Autonomous or nonautonomous?

(c) Locate the equilibrium points of this system. (There are infinitely many of them.) What physical arrangement of the pendulum corresponds to each equilibrium point?

(d) Use software to sketch phase plane portraits for two cases: zero damping and some positive damping. (You will need to pick some reasonable values for your parameters.) Draw nullclines on top of your phase plane portraits and verify that the solution curves pass through them in the proper way.

In class, we considered the competitive species population model.

$$x'(t) = x(r_1 - a_1 x - b_1 y)$$
$$y'(t) = y(r_2 - a_2 y - b_2 x)$$

All of the parameters in the DE above (r_1, a_1, b_1, r_2, a_2, b_2) are positive, and we restrict our attention to $x \geq 0$ and $y \geq 0$ only. Depending on these four constants, there were four different (nondegerate) cases to be considered:

- In case 1, the x-nullcline (blue) is completely below the y-nullcline in the first quadrant and the species represented by x always goes extinct for any positive initial population values.
- In case 2, the y-nullcline (red) is completely below the x-nullcline in the first quadrant and the species represented by y always goes extinct for any positive initial population values.
- In case 3, the two nullclines intersect at a point in the first quadrant. That point is a stable equilibrium point. The two populations always tend to a state where both populations coexist.
- In case 4, the two nullclines intersect at a point in the first quadrant. That point is an unstable saddle point. One population will always go extinct but which population goes extinct depends on the initial condition.

In this problem, you will complete the local stability analysis of this system of DEs.

(a) In all four cases, there is always one boring equilibrium point in which both populations go extinct ($x = y = 0$). Calculate the Jacobian matrix for the DEs and evaluate it at this equilibrium point. What do the eigenvalues of this matrix tell you about the stability of the linearized system near this equilibrium point?

(b) In all four cases, there are two other equilibrium points: one in which the x-population goes extinct but the other doesn't, and one in which the y-population goes extinct but the other doesn't. Calculate the locations of these two equilibrium points.

(c) Sketch the nullclines for case 1 and locate the x- and y-intercepts of both nullclines. What two inequalities must hold true for the x-nullcline to be completely below the y-nullcline in the first quadrant? Calculate the eigenvalues of $D\mathbf{f}(\mathbf{x}_{\text{eq}})$ for both equilibrium points you found in part (b). What do these eigenvalues tell you about the stability of the linearized system near these equilibrium points?

(d) Repeat part (c) for case 2.

(e) Determine the pairs of inequalities that must hold true to get cases 3 and 4. (Refer to the lecture notes to see how the nullclines intersect each other.)

(f) Consider the parameter values $r_1 = a_1 = b_1 = a_2 = 1$, $r_2 = 1/2$, and $b_2 = 1/4$. According to your work from part (e), which case does this set of parameter values fall under? Locate the equilibrium point for which $x > 0$ and $y > 0$. Calculate the eigenvalues of $D\mathbf{f}(\mathbf{x}_{\text{eq}})$ for this equilibrium point to determine the behavior of solutions near it. (It is easier to use the parameter values that are given rather than perform the calculation for a general set of parameter values.)

(g) (Extra credit) For an arbitrary set of parameters in cases 3 and 4, calculate the location of equilibrium point for which $x > 0$ and $y > 0$ and determine the stability of the linearized system about this equilibrium point.