For each linear system of DEs given below,

(i) Classify the origin as a saddle, center, spiral, deficient node, improper node or star node,

(ii) Identify it as neutrally stable, unstable or asymptotically stable (you may refer to Tables 6.5.1 and 6.5.2 in Borrelli & Coleman, which is available as a handout on our Sakai site),

(iii) Sketch the orbital portrait for each system. You may use ODEToolkit (`http://odetoolkit.hmc.edu`), Mathematica, or some other program to help you, but you should try to see if you can draw it yourself first without hints from the computer.

(a) $\begin{cases} x' &= 4y \\ y' &= -x \end{cases}$

Eigenvalues are $\lambda_1 = 2i$, $\lambda_2 = -2i$
with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ i \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -i \end{bmatrix}$.

(b) $\begin{cases} x' &= -x - 4y \\ y' &= x - y \end{cases}$

Eigenvalues are $\lambda_1 = -1 + 2i$, $\lambda_2 = -1 - 2i$
with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 2 \\ -i \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ i \end{bmatrix}$.

(c) $\begin{cases} x' &= 2x + y \\ y' &= -x + 4y \end{cases}$

Eigenvalues are $\lambda_1 = \lambda_2 = 3$
with corresponding eigenspace spanned by the eigenvector $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.
Hence the eigenspace is deficient. A generalized eigenvector is $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

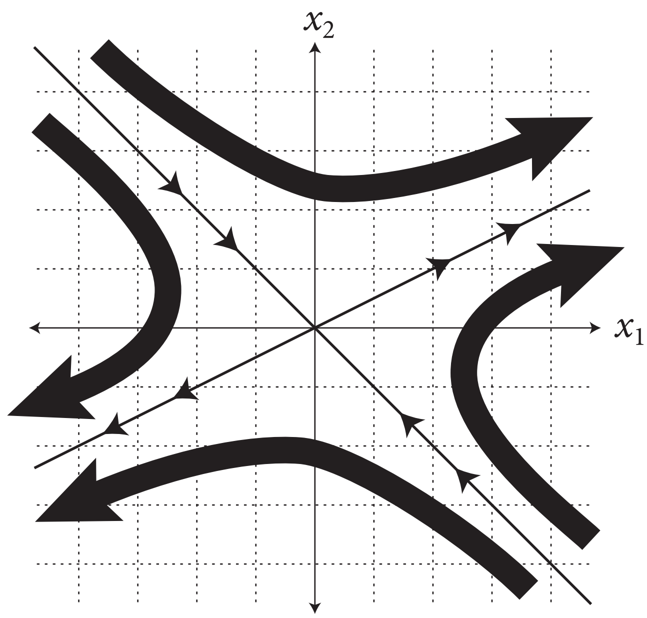
(d) $\begin{cases} x' &= x - 2y \\ y' &= x + 4y \end{cases}$

Eigenvalues are $\lambda_1 = 3$, $\lambda_2 = 2$
with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

(a) Consider a 2×2 system of first-order linear, constant-coefficient, homogeneous differential equations where the system matrix A is real and has *exactly one zero eigenvalue*. Write the general form for the set of solutions to any such system.

(b) Sketch orbital portraits for the various distinct cases where exactly one eigenvalue is 0. Explain why your gallery of portraits represents all the possible cases.

Write down a system of linear, homogeneous, first-order, constant-coefficient differential equations that could support the following graph. The straight trajectories (and their orientations) are the only ones that you must preserve. The large arrows are just meant to show the general trend of the trajectories in the rest of the phase plane. Also, use ODEToolkit or Mathematica or some other computer program to draw some solution trajectories for your differential equations to show that they match the diagram below.



Consider the system of differential equations

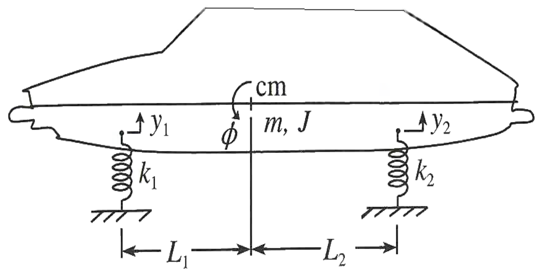
$$\mathbf{x}'(t) = \begin{bmatrix} -1 & 2 \\ -6 & -9 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 7 \\ 21 \end{bmatrix}.$$

(a) Find the equilibrium point for this system of differential equations. In other words, find values for x_1 and x_2 such that $x_1' = 0$ and $x_2' = 0$. Let those values of x_1 and x_2 be denoted as a and b.

(b) Let $u_1(t) = x_1(t) - a$ and $u_2(t) = x_2(t) - b$. What new system of differential equations results for u_1 and u_2? Solve it by hand using your favorite method.

(c) Find the solution to the original system of equations for the initial condition $x_1(0) = 3$ and $x_2(0) = -1$. What is the behavior of your solution as $t \to \infty$? How could you have predicted that long-term behavior simply based on the eigenvalues of the matrix alone?

Here is a simplified diagram of a car. The suspension at each axle (springs, tires, and shock absorbers) is modeled by a spring and damper. Only a spring is shown in the figure below, but assume that there is some damping there too.



The motion of this system is governed by these four coupled differential equations. (You don't have to derive them— just appreciate that someone has gone through the mechanics for you.)

$$\dot{y_1} = \frac{p_m}{m} - \frac{L_1 p_\psi}{J}$$
$$\dot{y_2} = \frac{p_m}{m} + \frac{L_2 p_\psi}{J}$$
$$\dot{p_m} = -k_1 y_1 - k_2 y_2 - \frac{R_1 + R_2}{m} p_m - \frac{L_2 R_2 - L_1 R_1}{J} p_\psi$$
$$\dot{p_\psi} = L_1 k_1 y_1 - L_2 k_2 y_2 - \frac{L_2 R_2 - L_1 R_1}{m} p_m - \frac{L_2^2 R_2 + L_1^2 R_1}{J} p_\psi$$

The four dependent variables are $y_1(t)$ and $y_2(t)$, the vertical displacements of the front and rear axles, respectively, and $p_m(t)$ and $p_\psi(t)$, the linear momentum and angular momentum about the mass center of the system, respectively. Here, J is the moment of inertia of the car about its center of mass, L_1 and L_2 are the horizontal distances between the front and rear axles and the center of mass, k_1 and k_2 are the spring constants, and R_1 and R_2 are the damping resistance constants.

Use the following parameter values: $J = mr^2$ with $m = 1250\,\mathrm{kg}$ and $r^2 = 1.6\,\mathrm{m}^2$, $k_1 = 30\,\frac{\mathrm{kN}}{\mathrm{m}}$, $k_2 = 32\,\frac{\mathrm{kN}}{\mathrm{m}}$, $L_1 = 1.4\,\mathrm{m}$ and $L_2 = 1.6\,\mathrm{m}$.

(a) This is a system of four first-order, linear, homogeneous, autonomous differential equations. Write it in linear algebra form. Breathe a sign of relief that in this entire problem you are not being asked to calculate $Pe^{Dt}P^{-1}$ to solve these ODEs.

(b) You already know that the solution to this system of ODEs is going to involve terms like $e^{\lambda_1 t}$, $e^{\lambda_2 t}$, $e^{\lambda_3 t}$, $e^{\lambda_4 t}$, where the λ_i are the eigenvalues to the system matrix. One common misconception is that y_1 involves only $e^{\lambda_1 t}$, that y_2 involves only $e^{\lambda_2 t}$, etc. By imagining how the calculation $Pe^{Dt}P^{-1}$ would be performed (but not actually performing it), explain why it is more likely that all four dependent variables will consist of a linear combination of all four exponential terms.

(c) Suppose that there is no damping in the system ($R_1 = R_2 = 0$). Use a technology to help you calculate the eigenvalues of the system matrix. What kind of motion (undamped, underdamped, critically damped, or overdamped oscillation) characterizes the behavior of the system? What are its vibrational frequencies? (In other words, if there is a term like $\cos(\sigma t)$ in the solution, what is the σ?) Report these frequencies with the proper units.

(d) Now suppose $R_1 = R_2 = 4000\,\mathrm{Ns^2/m}$. What kind of motion characterizes the behavior of the system?

(e) Now suppose $R_1 = R_2 = R$. Find a value of R for which the behavior of the system is no longer oscillatory. (You don't have to do this in a systematic way. Just picking numbers until you find what you need is fine.)