Consider the vector space C^n, the set of all real-valued functions $f(x)$ for which $f', f'', \ldots, f^{(n)}$ exist and are continuous, over $\mathbb{R}$. Show the differential operator

$$\mathcal{L}[y(x)] = a_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + a_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + a_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + a_0(x)y(x)$$

is a linear transformation, where $a_0(x), \ldots, a_n(x)$ are also C^n functions.

Let $S : V \to W$ and $T : U \to V$ be linear transformations.

(a) Prove that if S and T are both one-to-one, so is $S \circ T$.

(b) Prove that if S and T are both onto, so it $S \circ T$.

Let $T : V \to W$ be a linear transformation between two finite-dimensional vector spaces.

(a) Prove that if $\dim V < \dim W$, then T cannot be onto.

(b) Prove that if $\dim V > \dim W$, then T cannot be one-to-one.

Find the matrix $[T]_{\mathcal{C} \to \mathcal{B}}$ of the linear transformation $T : V \to W$ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$ of V and W, respectively. Verify Theorem 6.26 for the vector $\mathbf{v}$ by computing $T(\mathbf{v})$ directly and using the theorem.

$T : \mathscr{P}_2 \to \mathscr{P}_2$ defined by

$$\begin{aligned} T(p(x)) &= p(x+2), \\ \mathcal{B} &= \{1, x+2, (x+2)^2\}, \\ \mathcal{C} &= \{1, x, x^2\}, \\ \mathbf{v} &= p(x) = a + bx + cx^2 \end{aligned}$$

Find the matrix $[T]_{\mathcal{C}\to\mathcal{B}}$ of the linear transformation $T : V \to W$ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$ of V and W, respectively. Verify Theorem 6.26 for the vector $\mathbf{v}$ by computing $T(\mathbf{v})$ directly and using the theorem.

$T : M_{22} \to M_{22}$ defined by

$$T(A) = A - A^T,$$
$$\mathcal{B} = \mathcal{C} = \{E_{11}, E_{12}, E_{21}, E_{22}\},$$
$$\mathbf{v} = A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Determine whether the linear transformation T is invertible by considering its matrix with respect to the standard bases. If T is invertible, use Theorem 6.28 and the method of Example 6.82 to find T^{-1}.

$T : \mathscr{P}_2 \to \mathscr{P}_2$ defined by

$$T(p(x)) = p'(x)$$

Verify that S and T are inverses.

$S : \mathscr{P}_1 \to \mathscr{P}_1$ defined by

$$S(a+bx) = (-4a+b) + 2ax$$

$T : \mathscr{P}_1 \to \mathscr{P}_1$ defined by

$$T(a+bx) = \frac{b}{2} + (a+2b)x$$

In addition, calculate $[S]_{\mathcal{B}}$ and $[T]_{\mathcal{B}}$ for some basis $\mathcal{B}$ (of your choice) for the vector space in question. Then show that the matrices are the inverses of each other.

A linear transformation $T : V \to V$ is given. If possible, find a basis $\mathcal{C}$ for V such that the matrix $[T]_{\mathcal{C}}$ of T with respect to $\mathcal{C}$ is diagonal.

$T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a - b \\ a + b \end{bmatrix}$$