Box #________________

Math 65

HW 3

Due May 19, 2016

Problems: 6.1.47, 6.3.16, 6.4.{4, 21}, 6.5.12, 6.6.2, 4.4.22, EC: 6.2.33

*Intructor's Note:* I recommend that you also look at the Chapter Review on pages 527-528 of Poole and skim the problems to see if there are any concepts or problems that seem challenging to you. Try some of these problems for more practice.

**Problem 6.1.47**

Let $V$ be a vector space with subspaces $U$ and $W$. Give an example with $V = \mathbb{R}^2$ to show that $U \cup W$ need not be a subspace of $V$.

**Problem 6.3.16**

Let $\mathcal{B}$ and $\mathcal{C}$ be bases for $\mathscr{P}_2$. If $\mathcal{B} = \{x, 1+x, 1-x+x^2\}$ and the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$ is

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix},$$

find $\mathcal{C}$.

**Problem 6.4.4**

Determine whether $T : M_{nn} \to M_{nn}$, defined by $T(A) = AB - BA$, where $B$ is a fixed $n \times n$ matrix, is a linear transformation.

**Problem 6.4.21**

THEOREM 6.14:

Let $T : U \to V$ be a linear transformation. Then,

a. $T(\mathbf{0}) = \mathbf{0}$.

b. $T(-\mathbf{v}) = -T(\mathbf{v})$ for all $\mathbf{v}$ in $V$.

c. $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in $V$.

Prove Theorem 6.14(b).

**Problem 6.5.12**

$T : M_{22} \to M_{22}$ defined by $T(A) = AB - BA$, where $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Find either the nullity or the rank of $T$ and then use the Rank Theorem to find the other.

**Problem 6.6.2**

THEOREM 6.26:

Let $V$ and $W$ be two finite-dimensional vector spaces with bases $\mathcal{B}$ and $\mathcal{C}$ respectively, where $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. If $T : V \to W$ is a linear transformation, then the $m \times n$ matrix $A$ defined by

$$A = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{C}} & [T(\mathbf{v}_2)]_{\mathcal{C}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{C}} \end{bmatrix}$$

satisfies

$$A[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}}$$

for every vector $\mathbf{v}$ in $V$.

Find the matrix $[T]_{\mathcal{C}\leftarrow\mathcal{B}}$ of the linear transformation $T : V \to W$ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$ of $V$ and $W$, respectively. Verify Theorem 6.26 for the vector $\mathbf{v}$ by computing $T(\mathbf{v})$ directly and using the theorem.

$T : \mathscr{P}_1 \to \mathscr{P}_1$ defined by

$$T(a + bx) = b - ax,$$
$$\mathcal{B} = \{1 + x, 1 - x\},$$
$$\mathcal{C} = \{1, x\},$$
$$\mathbf{v} = p(x) = 4 + 2x$$

**Problem 4.4.22**

Use the method of Example 4.29 to compute

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}^k$$

(assume that $k$ is a positive integer)

[*Hint*: Example 4.29 uses the equality $M^n = PD^nP^{-1}$.]

**Extra Credit: 6.2.33**

Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be a set of vectors in an $n$-dimensional vector space $V$ and let $\mathcal{B}$ be a basis for $V$. Let $S = \{[\mathbf{u}_1]_{\mathcal{B}}, \ldots, [\mathbf{u}_m]_{\mathcal{B}}\}$ be on the set of coordinate vectors of $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ with respect to $\mathcal{B}$. Prove that $\text{span}(\mathbf{u}_1, \ldots, \mathbf{u}_m) = V$ if and only if $\text{span}(S) = \mathbb{R}^n$.

(Remember that to prove an if-and-only-if theorem, you need to prove both directions.)