Problems: 6.1.47, 6.3.16, 6.4.{4, 21}, 6.5.12, 6.6.2, 4.4.22, EC: 6.2.33

Intructor's Note: I recommend that you also look at the Chapter Review on pages 527-528 of Poole and skim the problems to see if there are any concepts or problems that seem challenging to you. Try some of these problems for more practice.

Problem 6.1.47

Let V be a vector space with subspaces U and W. Give an example with $V = \mathbb{R}^2$ to show that $U \cup W$ need not be a subspace of V.

Problem 6.3.16

Let $\mathcal{B}$ and $\mathcal{C}$ be bases for $\mathscr{P}_2$. If $\mathcal{B} = \{x, 1+x, 1-x+x^2\}$ and the change-of-basis matrix from $\mathcal{B}$ to $\mathcal{C}$ is

$$P_{\mathcal{C}\leftarrow\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ -1 & 1 & 1 \end{bmatrix},$$

find $\mathcal{C}$.

Problem 6.4.4

Determine whether $T : M_{nn} \to M_{nn}$, defined by $T(A) = AB - BA$, where B is a fixed $n \times n$ matrix, is a linear transformation.

Problem 6.4.21

THEOREM 6.14:

Let $T : V \to W$ be a linear transformation. Then,

    a. $T(\mathbf{0}) = \mathbf{0}$.

    b. $T(-\mathbf{v}) = -T(\mathbf{v})$ for all $\mathbf{v}$ in V.

    c. $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$ for all $\mathbf{u}$ and $\mathbf{v}$ in V.

Prove Theorem 6.14(b).

Problem 6.5.12

$T : M_{22} \to M_{22}$ defined by $T(A) = AB - BA$, where $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Find either the nullity or the rank of T and then use the Rank Theorem to find the other.

Problem 6.6.2

THEOREM 6.26:

Let V and W be two finite-dimensional vector spaces with bases $\mathcal{B}$ and $\mathcal{C}$ respectively, where $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$. If $T : V \to W$ is a linear transformation, then the $m \times n$ matrix A defined by

$$A = \left[[T(\mathbf{v}_1)]_{\mathcal{C}} \quad [T(\mathbf{v}_2)]_{\mathcal{C}} \quad \cdots \quad [T(\mathbf{v}_n)]_{\mathcal{C}} \right]$$

satisfies

$$A[\mathbf{v}]_{\mathcal{B}} = [T(\mathbf{v})]_{\mathcal{C}}$$

for every vector $\mathbf{v}$ in V.

Find the matrix $[T]_{\mathcal{C} \leftarrow \mathcal{B}}$ of the linear transformation $T : V \to W$ with respect to the bases $\mathcal{B}$ and $\mathcal{C}$ of V and W, respectively. Verify Theorem 6.26 for the vector $\mathbf{v}$ by computing $T(\mathbf{v})$ directly and using the theorem.

$T : \mathscr{P}_1 \to \mathscr{P}_1$ defined by

$$T(a + bx) = b - ax,$$

$$\mathcal{B} = \{1 + x, 1 - x\},$$

$$\mathcal{C} = \{1, x\},$$

$$\mathbf{v} = p(x) = 4 + 2x$$

Problem 4.4.22

Use the method of Example 4.29 to compute

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}^k$$

(assume that k is a positive integer)

[*Hint*: Example 4.29 uses the equality $M^n = PD^nP^{-1}$.]

Extra Credit: 6.2.33

Let $\{\mathbf{u}_1, ..., \mathbf{u}_m\}$ be a set of vectors in an n-dimensional vector space V and let $\mathcal{B}$ be a basis for V. Let $S = \{[\mathbf{u}_1]_\mathcal{B}, ..., [\mathbf{u}_m]_\mathcal{B}\}$ be on the set of coordinate vectors of $\{\mathbf{u}_1, ..., \mathbf{u}_m\}$ with respect to $\mathcal{B}$. Prove that $\text{span}(\mathbf{u}_1, ..., \mathbf{u}_m) = V$ if and only if $\text{span}(S) = \mathbb{R}^n$.

(Remember that to prove an if-and-only-if theorem, you need to prove both directions.)