Consider the inhomogeneous linear system of differential equations

$$\mathbf{x}' = \begin{bmatrix} 5 & -4 \\ 6 & -5 \end{bmatrix} \mathbf{x} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}.$$

Note that the system matrix has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$ with corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

(a) Assume that the solution to this system $\mathbf{x}$ has the form $\begin{bmatrix} a_1 e^{-t} + b_1 e^t + c_1 te^t \\ a_2 e^{-t} + b_2 e^t + c_2 te^t \end{bmatrix}$ for constants a, b, c, d, e and f. Use the method of undetermined coefficients to find the general solution to this system.

(b) Why was the assumption made in part (a) reasonable given what you know about the matrix A?

(c) Recalculate the solution using the integrating factor formula. Reconcile your answer from this calculation with your previous calculation to show they are equivalent.

Solve the initial value problem

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} \cos(\omega t) \\ -\sin(\omega t) \end{bmatrix}$$

where ω is a nonzero constant and $\mathbf{x}(0) = \begin{bmatrix} a \\ b \end{bmatrix}$. Make sure your solution is valid for all values of $\omega \neq 0$. In particular, be careful not to divide by 0 when $\omega = -2$. Work out the solution in the case when $\omega \neq -2$, then work out the solution in the case when $\omega = -2$. You may use either the undetermined coefficients or integrating factor method. You will get extra credit if you perform both calculations and show that they are equivalent.

Hint: If you're using the undetermined coefficients method, think about what we did in Math 45 when the forcing function took on the same form as the homogeneous solution.

As we will discuss in class on Friday, closed-form solutions to non-constant-coefficient linear systems of differential equations are generally unavailable. However, if you are lucky enough to solve a homogeneous (unforced) non-constant-coefficient linear system, solving the inhomogeneous (forced) version is relatively easy. In this exercise, we'll walk you through how this works.

Consider the following initial value problem:

$$\mathbf{x}'(t) = A(t)\mathbf{x} + \mathbf{F}(t) = \begin{bmatrix} 2 & -2e^{-t} \\ e^{t} & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 4 \\ 6 \end{bmatrix} \qquad \text{with} \qquad \mathbf{x}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Hint: Throughout this problem, you should use the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

to calculate the inverse of a 2×2 matrix. (Unfortunately, there are no simple formulas for larger matrices.) **Note:** To learn more about fundamental matrices and the derivation of the variation of parameters formula, read Section 7.9 of Boyce and diPrima.

(a) First, verify that

$$\mathbf{x}_h(t) = c_1\begin{bmatrix} 2e^{t} \\ e^{2t} \end{bmatrix} + c_2\begin{bmatrix} 1 \\ e^{t} \end{bmatrix}$$

is the general solution to the homogeneous version of the ODE (that is, the version of the ODE without the forcing term $\mathbf{F}$) for any c_1 and c_2.

(b) Arrange the two linearly independent solutions from part (a) as columns in a matrix $\Psi(t)$, which is called a *fundamental matrix*. Verify that $\Psi'(t) = A(t)\Psi(t)$.

(c) The general solution to homogeneous equation is $\mathbf{x}_h(t) = \Psi(t)\mathbf{c}$ for some constant vector $\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$. What must $\mathbf{c}$ be to satisfy the initial condition $\mathbf{x}_h(t_0) = \mathbf{x}_0$?

(d) The *transition matrix* for this system is defined to be $\Phi(t,s) = \Psi(t)\Psi(s)^{-1}$. Calculate it. **Note:** The reason why we call this a "transition matrix" is that the effect of multiplication by the matrix $\Phi(t,t_0)$ is to transition the homogeneous solution forward in time from t_0 to t.

(e) Armed with the transition matrix corresponding to the homogeneous ODE, we can now calculate the solution to the inhomogeneous ODE using the variation of parameters formula

$$\mathbf{x}(t) = \Phi(t,0)\mathbf{x}(0) + \int_0^t \Phi(t,s)\mathbf{F}(s)\, ds.$$

Use this formula to calculate the solution to the IVP.