C 1.2 Quadrotors modeling

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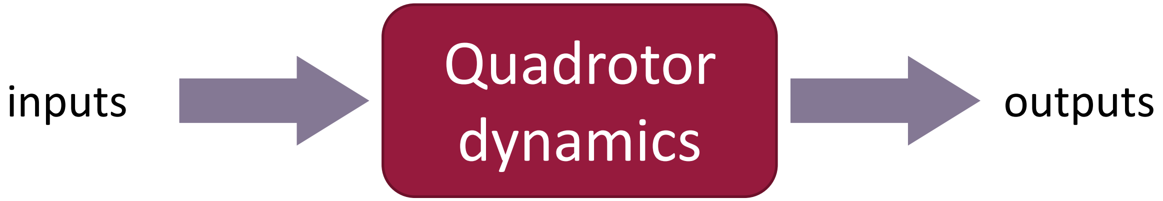
Quadrotor system

Model inputs and outputs



Physical system input/outputs vs model input outputs

Depending on problem/task: higher or lower model inputs and outputs levels

Cascade control

Input examples

- Motor voltages or duty cycles
- Thrust and torques
- 3D orientation + vertical acceleration

Output example

- 3D angular velocity + collective thrust
- 3D orientation + vertical speed
- 3D position + heading

Rigid body dynamics

Equations of motion – Fundamental Principle of Dynamics

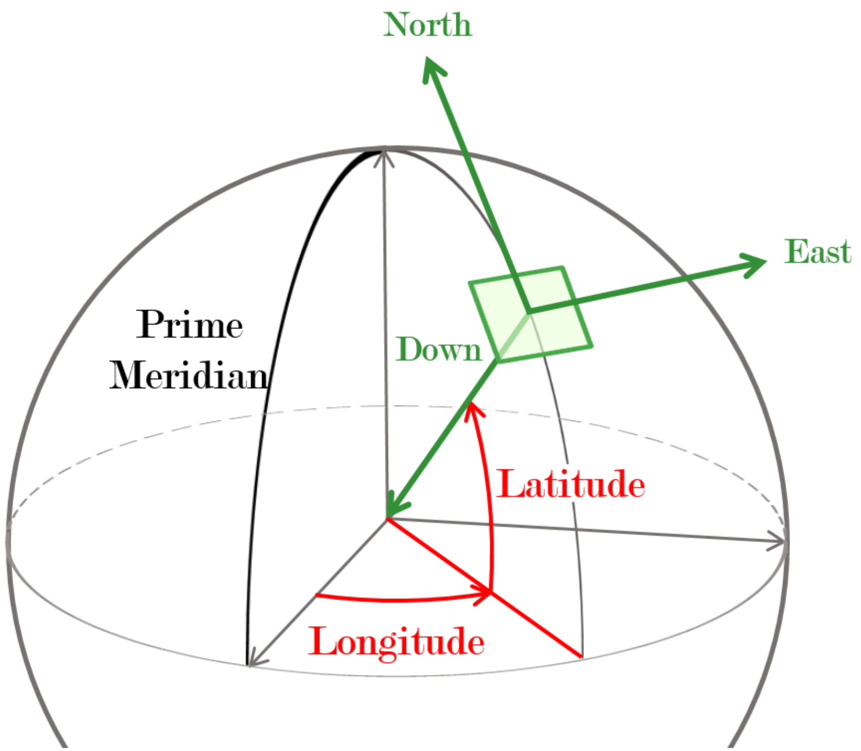
Assumption 1. Flat Earth

Assumption 2. Rigid bodies

Mechanical actions

- Gravity
- Air mass actions
- Propellers actions: thrust, drag torques

  (+reaction and gyroscopic torques)



Rigid body dynamics

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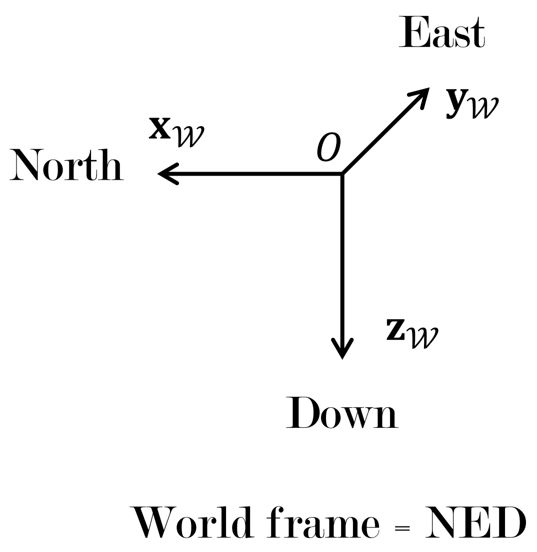
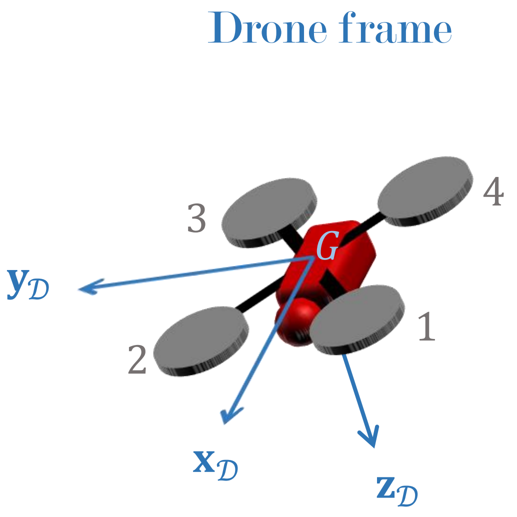




Rigid body dynamics

Equations of motion – Fundamental Principle of Dynamics

Drone parameters and variables

Drone mass	m
Drone inertia / drone center of mass in $\mathcal{D}$	$\mathbf{J}$
Position / NED frame in $\mathcal{W}$	$\boldsymbol{\zeta} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$
Attitude / NED frame	$\mathbf{R}$
Angular velocity / NED frame in $\mathcal{D}$	$\boldsymbol{\Omega} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$

FPD

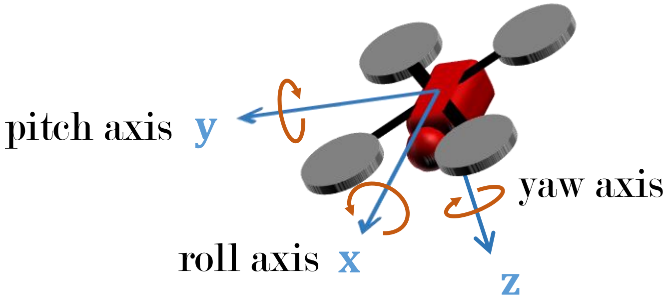
$$m\,\ddot{\boldsymbol{\zeta}} = \sum \mathbf{f}_{\text{ext}}$$

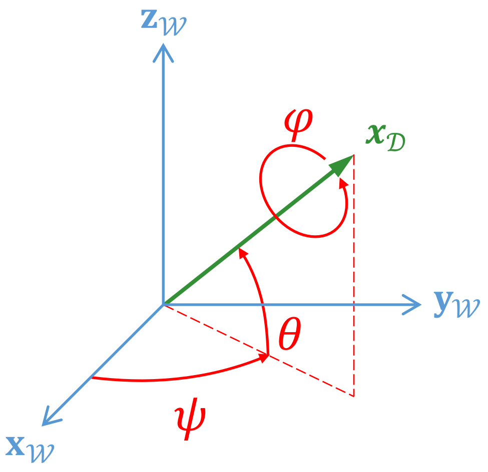
$$\mathbf{J}\,\dot{\boldsymbol{\Omega}} = \sum \boldsymbol{\tau}_{\text{ext}} - \widehat{\boldsymbol{\Omega}}\,\mathbf{J}\,\boldsymbol{\Omega}$$

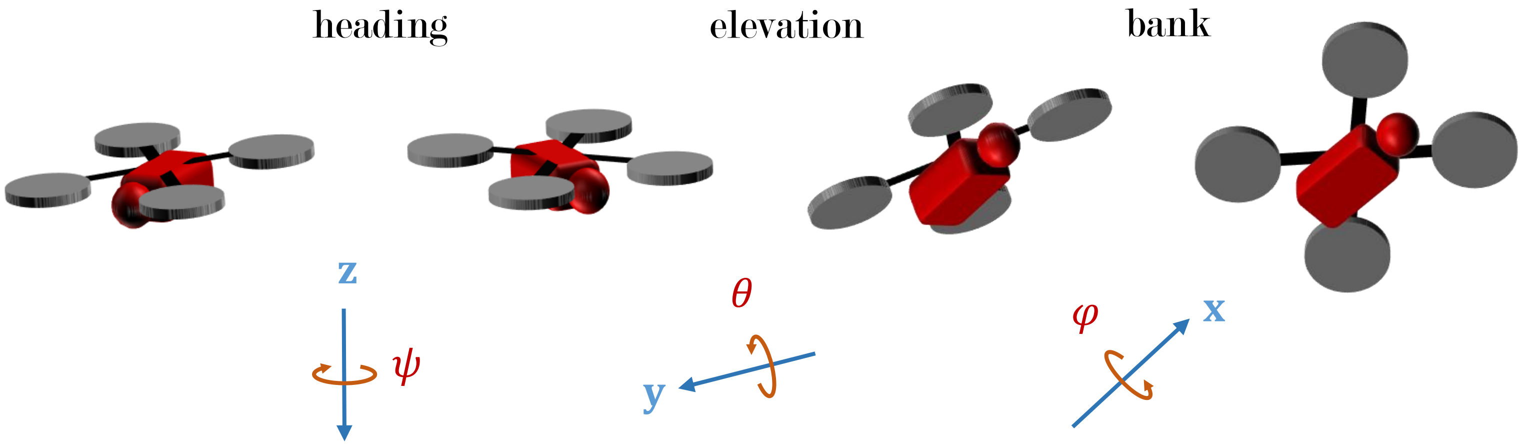
Rigid body dynamics

Attitude representation – Euler Angles ZYX

- Intuitive representation
- 3 parameters





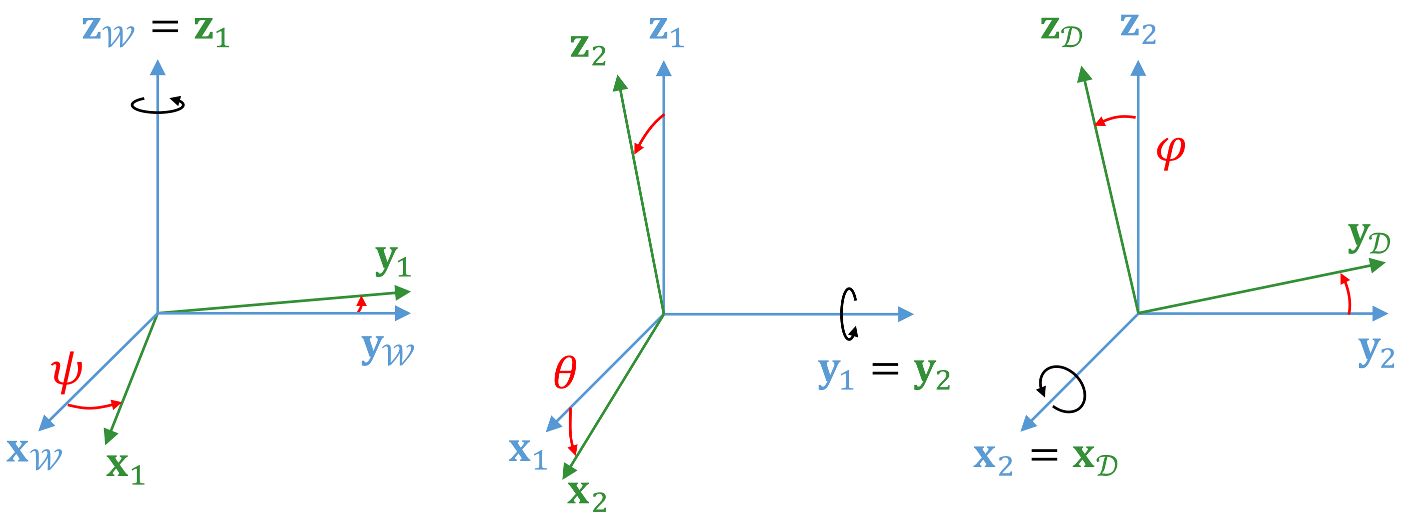


Rigid body dynamics

Attitude representation – Euler Angles ZYX

- ❌ Non unique - or - discontinuous
- ❌ Trigonometric functions

$\mathbf{R}_\psi$	$\mathbf{R}_\theta$	$\mathbf{R}_\varphi$
$\psi \in \left]-\pi, \pi\right]$	$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	$\varphi \in \left]-\pi, \pi\right]$



Rigid body dynamics

Attitude representation – Euler Angles ZYX

$$\mathbf{R}_\psi \qquad \mathbf{R}_\theta \qquad \mathbf{R}_\varphi$$

$$\psi \in \left]-\pi, \pi\right] \qquad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \qquad \varphi \in \left]-\pi, \pi\right]$$

$$\begin{pmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_\varphi & -s_\varphi \\ 0 & s_\varphi & c_\varphi \end{pmatrix}$$

- ❌ Non unique - or - discontinuous
- ❌ Trigonometric functions

$$\mathbf{R}(\varphi, \theta, \psi) = \mathbf{R}_\psi \mathbf{R}_\theta \mathbf{R}_\varphi = \begin{pmatrix} c_\theta c_\psi & s_\varphi s_\theta c_\psi - c_\varphi s_\psi & c_\varphi s_\theta c_\psi + s_\varphi s_\psi \\ c_\theta s_\psi & s_\varphi s_\theta s_\psi + c_\varphi c_\psi & c_\varphi s_\theta s_\psi - s_\varphi c_\psi \\ -s_\theta & s_\varphi c_\theta & c_\varphi c_\theta \end{pmatrix}$$

Rigid body dynamics

Attitude representation – Euler Angles ZYX

$$\boldsymbol{\Omega} = \dot{\psi}\,\mathbf{z}_{\mathcal{W}} + \dot{\theta}\,\mathbf{R}_{\psi}^{\intercal}\,\mathbf{y}_{\mathcal{H}} + \dot{\varphi}\,\mathbf{R}_{\psi}^{\intercal}\mathbf{R}_{\theta}^{\intercal}\,\mathbf{z}_{\mathcal{D}}$$

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} 1 & 0 & -s_\theta \\ 0 & c_\varphi & s_\varphi c_\theta \\ 0 & -s_\varphi & c_\varphi c_\theta \end{pmatrix} \begin{pmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix}$$

❌ Singular (gimbal lock)

$$\begin{pmatrix} \dot{\varphi} \\ \dot{\theta} \\ \dot{\psi} \end{pmatrix} = \begin{pmatrix} 1 & s_\varphi t_\theta & c_\varphi t_\theta \\ 0 & c_\varphi & -s_\varphi \\ 0 & s_\varphi / c_\theta & c_\varphi / c_\theta \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

Rigid body dynamics

Attitude representation – Quaternions

Extension of $\mathbb{C}$

$$\mathbf{q} = (q_0, q_1, q_2, q_3) \in \mathbb{H}, \qquad \mathbf{q} = q_0 + q_1\,\mathrm{i} + q_2\,\mathrm{j} + q_3\,\mathrm{k}$$

4 parameters

$$\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1$$

$$w \in \mathbb{R}, \qquad \mathbf{v} = (x, y, z) \in \mathbb{R}^3$$

$$\mathbf{q} = w + \mathbf{v} \in \mathbb{H}$$

$$\mathbf{q}_1\mathbf{q}_2 = (w_1 w_2 + \mathbf{v}_2 \cdot \mathbf{v}_2) + (w_1\,\mathbf{v}_2 + w_2\,\mathbf{v}_1 + \mathbf{v}_1 \times \mathbf{v}_2)$$

$$\mathbf{q}^* = w - \mathbf{v}$$

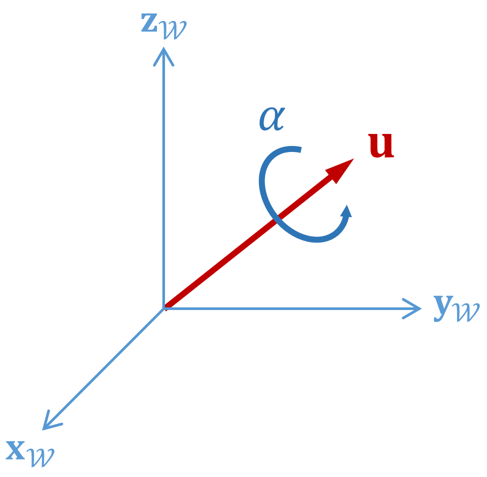
Rigid body dynamics

Attitude representation – Quaternions

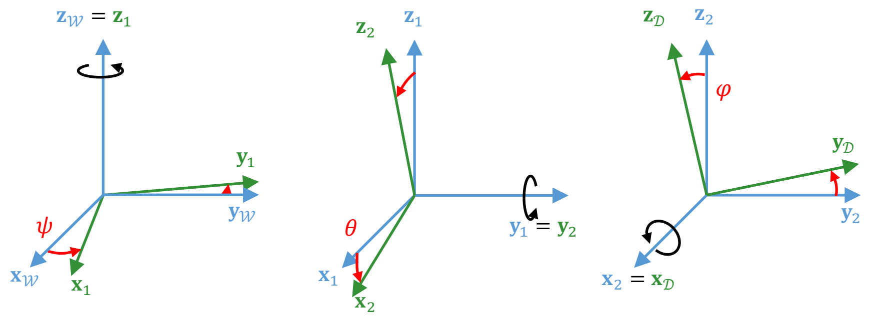
Rotations can be represented using **unit quaternions**

Rotation of angle α around unit vector $\mathbf{u} = (u_x, u_y, u_z)$

$$\|\mathbf{q}\| = 1, \qquad \mathbf{q} = c_{\alpha/2} + s_{\alpha/2}\mathbf{u}$$



Example: yaw/pitch/roll decomposition



Yaw angle ψ, axis $(0,0,1)$ ➡ $\mathbf{q}_\psi = (c_{\psi/2}, 0, 0, s_{\psi/2})$

Pitch angle θ, axis $(0,1,0)$ ➡ $\mathbf{q}_\theta = (c_{\theta/2}, 0, s_{\theta/2}, 0)$

Roll angle φ, axis $(1,0,0)$ ➡ $\mathbf{q}_\varphi = (c_{\varphi/2}, s_{\varphi/2}, 0, 0)$

$$\mathbf{q} = \mathbf{q}_\psi \mathbf{q}_\theta\, \mathbf{q}_\varphi$$

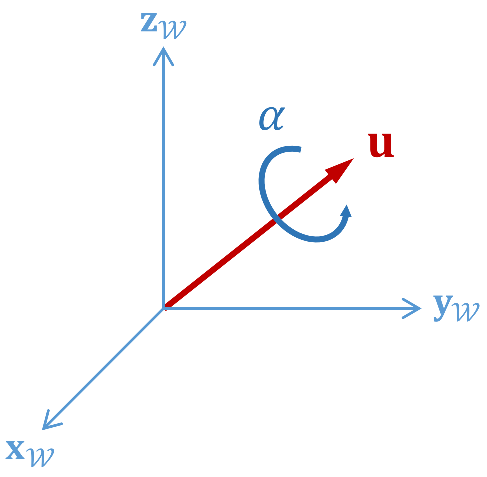
Rigid body dynamics

Attitude representation – Quaternions

Rotations can be represented using **unit quaternions**

Rotation of angle α around unit vector $\mathbf{u} = (u_x, u_y, u_z)$

$$\|\mathbf{q}\| = 1, \qquad \mathbf{q} = c_{\alpha/2} + s_{\alpha/2}\mathbf{u}$$



$$\mathbf{a} = (a_x, a_y, a_z) \in \mathbb{R}^3 \Rightarrow \mathbf{a}_\mathbb{H} = (0, a_x, a_y, a_z) \in \mathbb{H}$$

$$\mathbf{R} \in SO(3) \Rightarrow \mathbf{q} \in \mathbb{H}, \qquad \|\mathbf{q}\| = 1$$

$$\mathbf{b} = \mathbf{R}\,\mathbf{a} \Rightarrow \mathbf{b}_\mathbb{H} = \mathbf{q}^*\mathbf{a}_\mathbb{H}\mathbf{q}$$

- ❌ Non intuitive
- ❌ 1 constraint

Rigid body dynamics

Attitude representation – Quaternions

Equivalent rotation matrix

$$\mathbf{R}(\mathbf{q}) = \begin{pmatrix} 1-2(q_2^2+q_3^2) & 2(q_1q_2-q_0q_3) & 2(q_0q_2+q_1q_3) \\ 2(q_0q_3+q_1q_2) & 1-2(q_1^2+q_3^2) & 2(q_2q_3-q_0q_1) \\ 2(q_1q_3-q_0q_2) & 2(q_0q_1+q_2q_3) & 1-2(q_1^2+q_2^2) \end{pmatrix}$$

Non unique

$$\mathbf{R}(\mathbf{q}) = \mathbf{R}(-\mathbf{q})$$

Derivative

$$\dot{\mathbf{q}} = \frac{1}{2}\mathbf{q}\mathbf{\Omega}_{\mathbb{H}} \qquad \mathbf{\Omega}_{\mathbb{H}} = (0, p, q, r)$$

Regular

Rigid body dynamics

Attitude representation – Rotation matrix

Special orthogonal group $SO(3)$

$$\mathbf{R} = \begin{pmatrix} R_{1,1} & R_{1,2} & R_{1,3} \\ R_{2,1} & R_{2,2} & R_{2,3} \\ R_{3,1} & R_{3,2} & R_{3,3} \end{pmatrix}$$

$$\mathbf{R}\mathbf{R}^{\top} = \mathbf{R}^{\top}\mathbf{R} = \mathbf{I} \qquad |\mathbf{R}| = 1$$

- ✗ 9 parameters
- ✓ unique
- ✗ Non intuitive
- ✗ 6 constraints

Hat map

$$\mathbf{a} = (a_x, a_y, a_z) \in \mathbb{R}^3 \Rightarrow \hat{\mathbf{a}} = \begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix}$$

$$\mathbf{a} \times \mathbf{b} \Rightarrow \hat{\mathbf{a}}\,\mathbf{b}$$

Derivative

$$\dot{\mathbf{R}} = \mathbf{R}\,\widehat{\boldsymbol{\Omega}}$$

- ✓ Regular

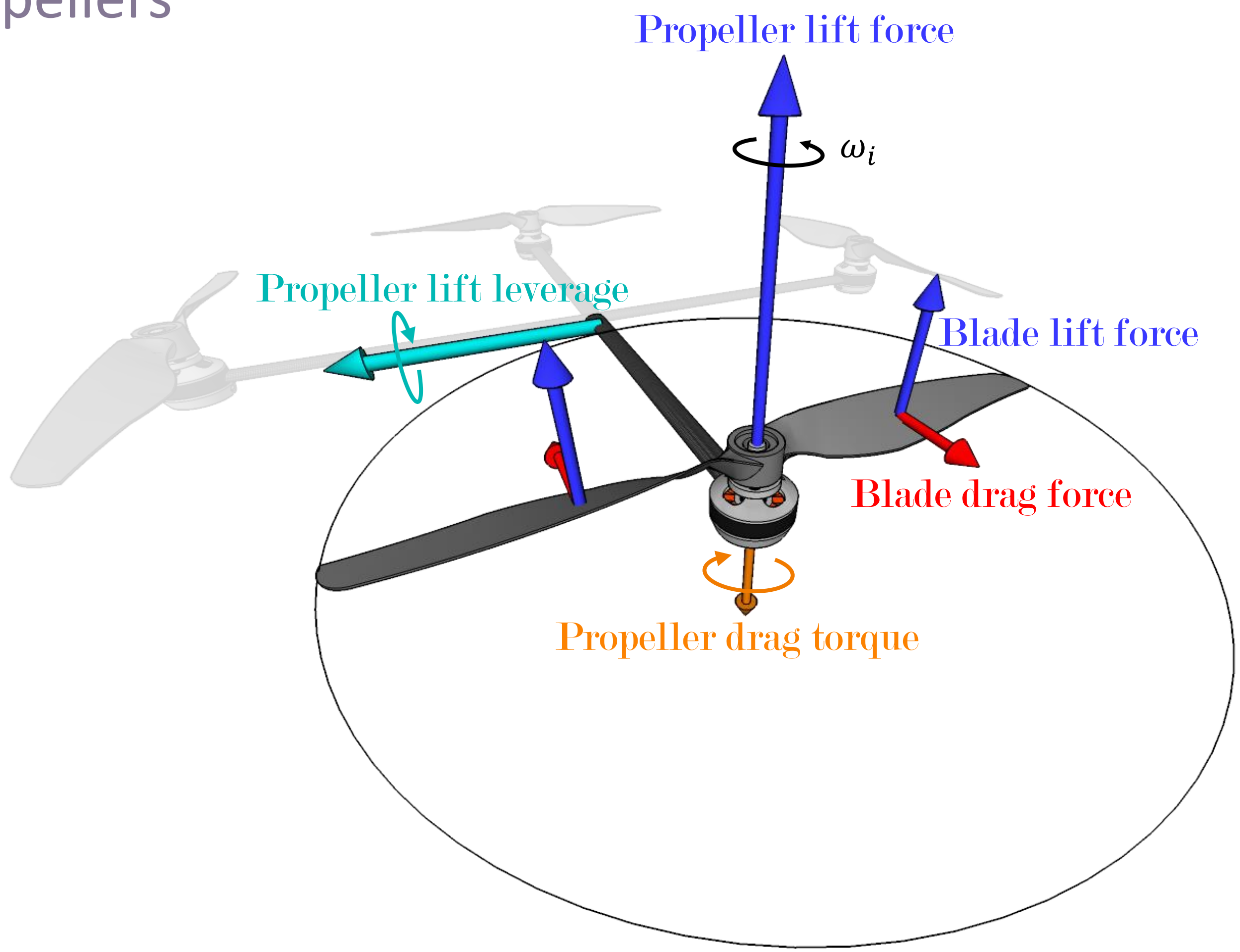
Rigid body dynamics

Attitude representation – Comparison

Euler angles	Unit quaternions	Rotation matrices
3 parameters	4 parameters	9 parameters
No constraint	1 constraint	6 constraints
Non unique	Non unique	Unique
Intuitive	Not intuitive	Not intuitive
Singular	Regular	Regular
Trigonometric operations	Matrix operations	Matrix operations
(unique with additional constraints)	*(unique with additional constraints)*	

Mechanical actions

Propellers



Lift force

$$\mathbf{t}_i = -a\omega_i^2\, \mathbf{z}_{\mathcal{D}}$$

Drag torque

$$\mathbf{d}_i = \pm b\omega_i^2\, \mathbf{z}_{\mathcal{D}}$$

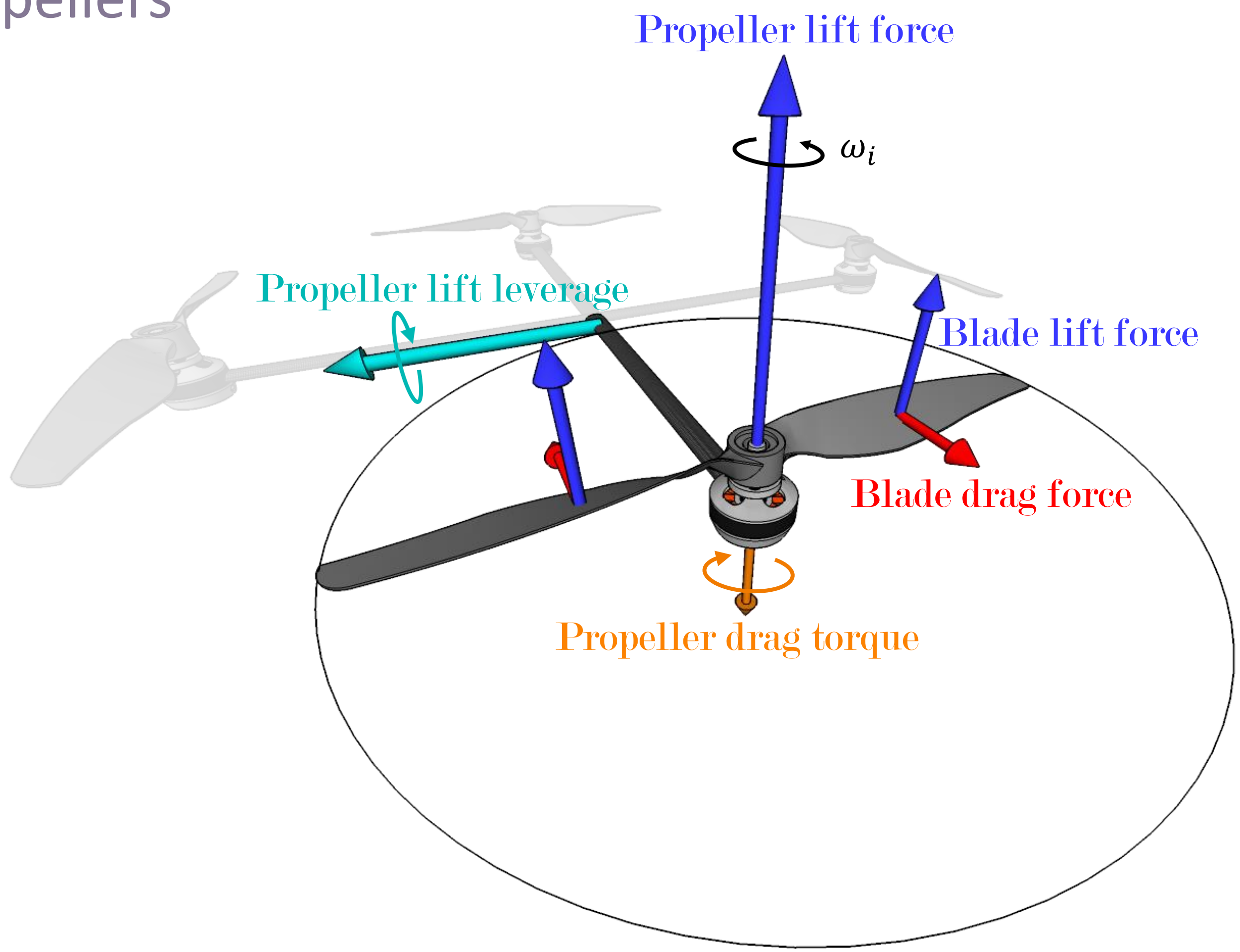
Lift leverage

$$\boldsymbol{l}_i = \overrightarrow{GP_i} \times \mathbf{t}_i$$

$$\boldsymbol{l}_i = \pm al\omega_i^2\, \mathbf{x}_{\mathcal{D}} \pm al\omega_i^2\, \mathbf{y}_{\mathcal{D}}$$

Mechanical actions

Propellers



Total lift force

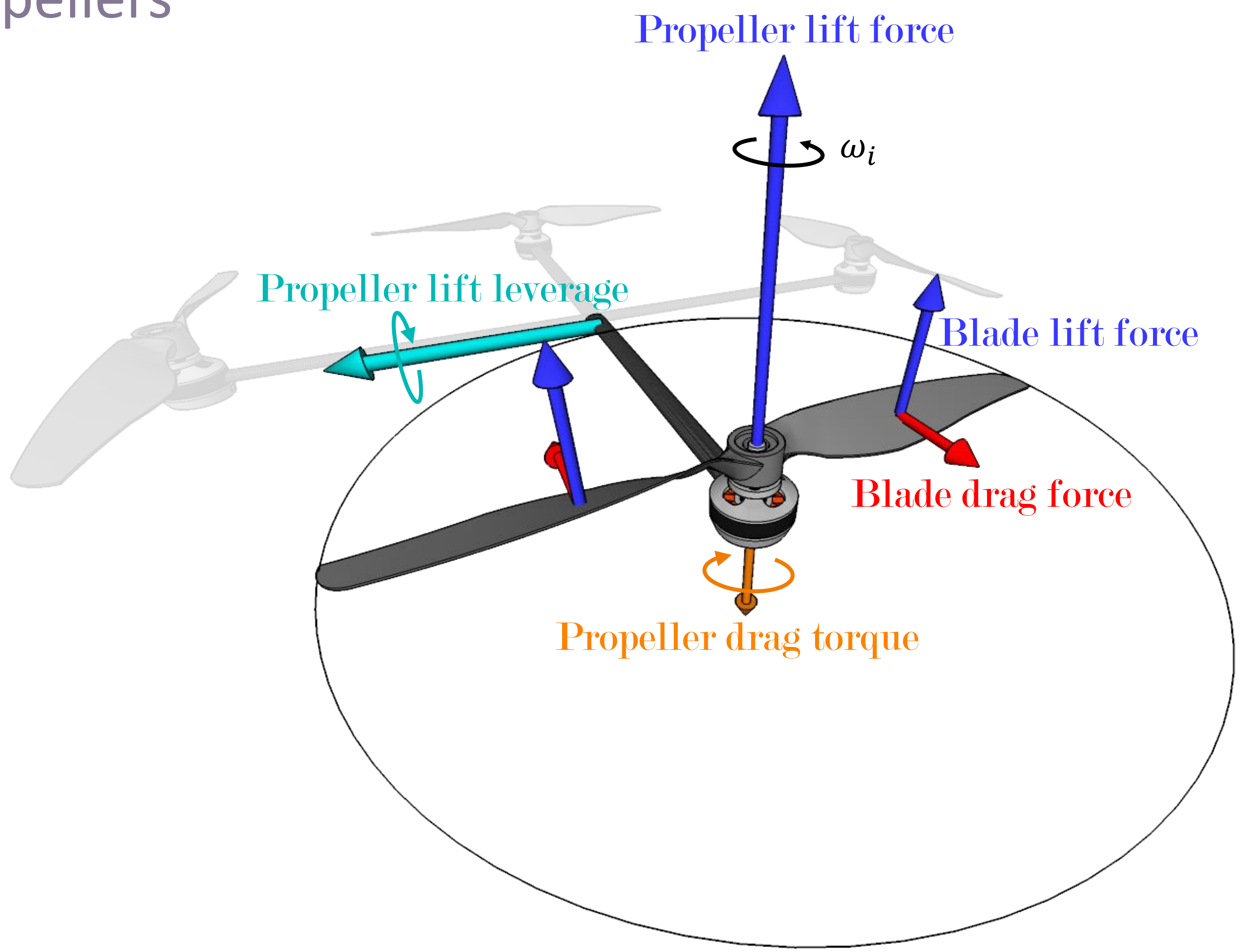
$$\mathbf{f} = \sum_{i=1}^{4} -a\omega_i^2 \, \mathbf{z}_{\mathcal{D}} = -f\mathbf{z}_{\mathcal{D}}$$

Total torque

$$\boldsymbol{\tau} = \sum_{i=1}^{4} \mathbf{d}_i + \boldsymbol{l}_i$$

Mechanical actions

Propellers



$$\begin{pmatrix} f \\ \tau_x \\ \tau_y \\ \tau_z \end{pmatrix} = \underbrace{\begin{pmatrix} a & a & a & a \\ al & -al & -al & al \\ al & al & -al & -al \\ b & -b & b & -b \end{pmatrix}}_{\mathbf{M}} \begin{pmatrix} \omega_1^2 \\ \omega_2^2 \\ \omega_3^2 \\ \omega_4^2 \end{pmatrix}$$

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \end{pmatrix} = \sqrt{\mathbf{M}^{-1} \begin{pmatrix} f \\ \tau_x \\ \tau_y \\ \tau_z \end{pmatrix}}$$

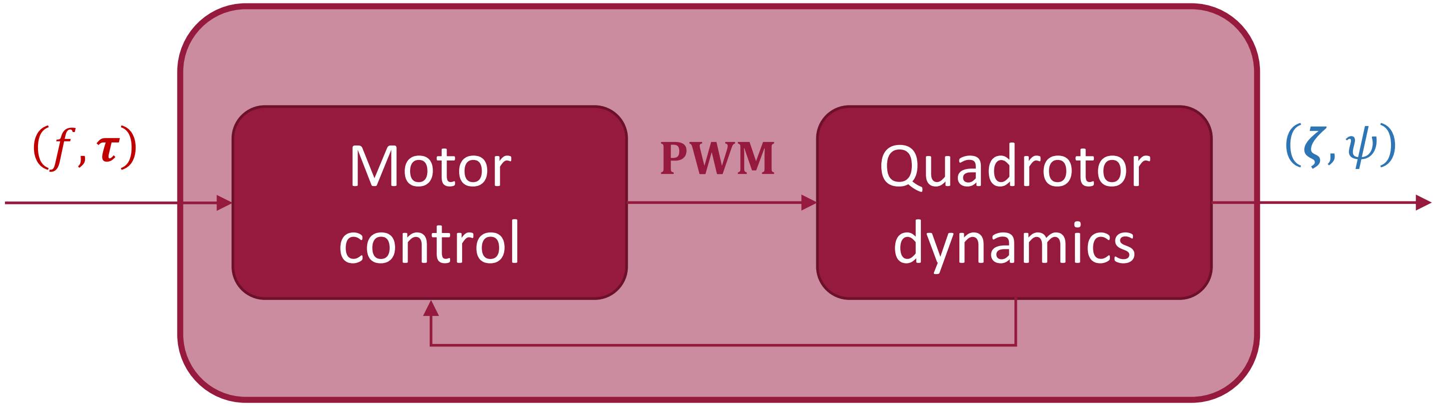
Simplified nonlinear model

Thrust/torque model

Inputs
- 1D thrust f
- 3D torque $\boldsymbol{\tau} = (\tau_x, \tau_y, \tau_z)$

State
- 3D position $\boldsymbol{\zeta}$
- 3D velocity $\dot{\boldsymbol{\zeta}}$
- 3D attitude $\mathbf{R}$
- 3D angular velocity $\boldsymbol{\Omega}$

Outputs
- 3D position $\boldsymbol{\zeta}$
- 1D heading ψ



Quadrotor model

Simplified nonlinear model

Rotation matrix

Inputs
- 1D thrust f
- 3D torque $\boldsymbol{\tau} = (\tau_x, \tau_y, \tau_z)$

State
- 3D position $\boldsymbol{\zeta}$
- 3D velocity $\dot{\boldsymbol{\zeta}}$
- 3D attitude $\mathbf{R}$
- 3D angular velocity $\boldsymbol{\Omega}$

Outputs
- 3D position $\boldsymbol{\zeta}$
- 1D heading ψ

Acceleration $$m\,\ddot{\zeta} = -f\,\mathbf{R}\,\mathbf{z}_{\mathcal{W}} + mg\,\mathbf{z}_{\mathcal{W}}$$

Rotation $$\dot{\mathbf{R}} = \mathbf{R}\widehat{\boldsymbol{\Omega}}$$

Angular acceleration $$\mathbf{J}\,\dot{\boldsymbol{\Omega}} = \tau_x\,\mathbf{x}_{\mathcal{D}} + \tau_y\,\mathbf{y}_{\mathcal{D}} + \tau_z\,\mathbf{z}_{\mathcal{D}} - \widehat{\boldsymbol{\Omega}}\mathbf{J}\,\boldsymbol{\Omega}$$

Heading $$\psi = \arctan_2(R_{2,1}, R_{1,1})$$

Simplified nonlinear model

Quaternion

Inputs
- 1D thrust f
- 3D torque $\boldsymbol{\tau} = (\tau_x, \tau_y, \tau_z)$

State
- 3D position $\boldsymbol{\zeta}$
- 3D velocity $\dot{\boldsymbol{\zeta}}$
- 3D attitude $\boldsymbol{q}$
- 3D angular velocity $\boldsymbol{\Omega}$

Outputs
- 3D position $\boldsymbol{\zeta}$
- 1D heading ψ

Acceleration

$$m\,\ddot{\boldsymbol{\zeta}} = -f\,\mathbf{R}(\boldsymbol{q})\,\mathbf{z}_{\mathcal{W}} + mg\,\mathbf{z}_{\mathcal{W}}$$

Rotation

$$\dot{\boldsymbol{q}} = \frac{1}{2}\boldsymbol{q}\boldsymbol{\Omega}_{\mathbb{H}}$$

Angular acceleration

$$\mathbf{J}\,\dot{\boldsymbol{\Omega}} = \tau_x\,\mathbf{x}_{\mathcal{D}} + \tau_y\,\mathbf{y}_{\mathcal{D}} + \tau_z\,\mathbf{z}_{\mathcal{D}} - \widehat{\boldsymbol{\Omega}}\,\mathbf{J}\,\boldsymbol{\Omega}$$

Heading

$$\psi = \arctan_2\left(R_{2,1}, R_{1,1}\right)$$

Integrator model

Simple PD attitude control for cascade control

Attitude error

$$\mathbf{q}_\varepsilon = \mathbf{q}^* \mathbf{q}_{\mathrm{ref}}$$

$$\mathbf{q}_\varepsilon = c_{\alpha_\varepsilon/2} + s_{\alpha_\varepsilon/2}\, \mathbf{v}_\varepsilon$$

$$\boldsymbol{\varepsilon}_q = \alpha_\varepsilon \mathbf{v}_\varepsilon$$

Angular velocity error

$$\boldsymbol{\varepsilon}_\Omega = \boldsymbol{\Omega}_{\mathrm{ref}} - \boldsymbol{\Omega}$$

PD control law

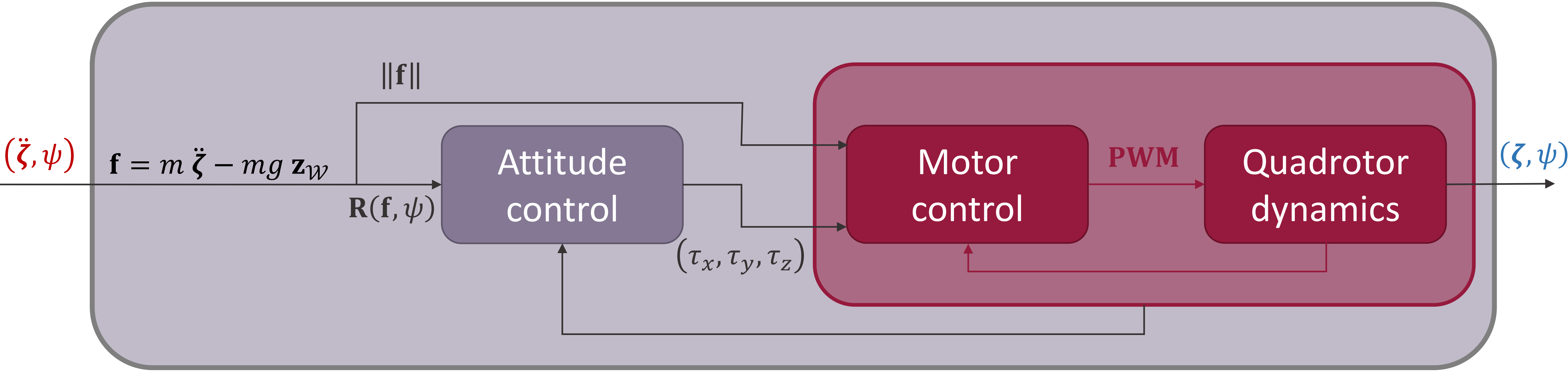
$$\boldsymbol{\tau} = k_q \boldsymbol{\varepsilon}_q + k_\Omega \boldsymbol{\varepsilon}_\Omega$$

- ❌ Unwinding
- ❌ Simplistic

Integrator model

Double integrator model

Example of cascade control



Quadrotor model

Integrator model

Simple PID position control model

Position error

$$\boldsymbol{\varepsilon}_p = \zeta_{ref} - \zeta$$

Velocity error

$$\boldsymbol{\varepsilon}_v = \dot{\zeta}_{ref} - \dot{\zeta}$$

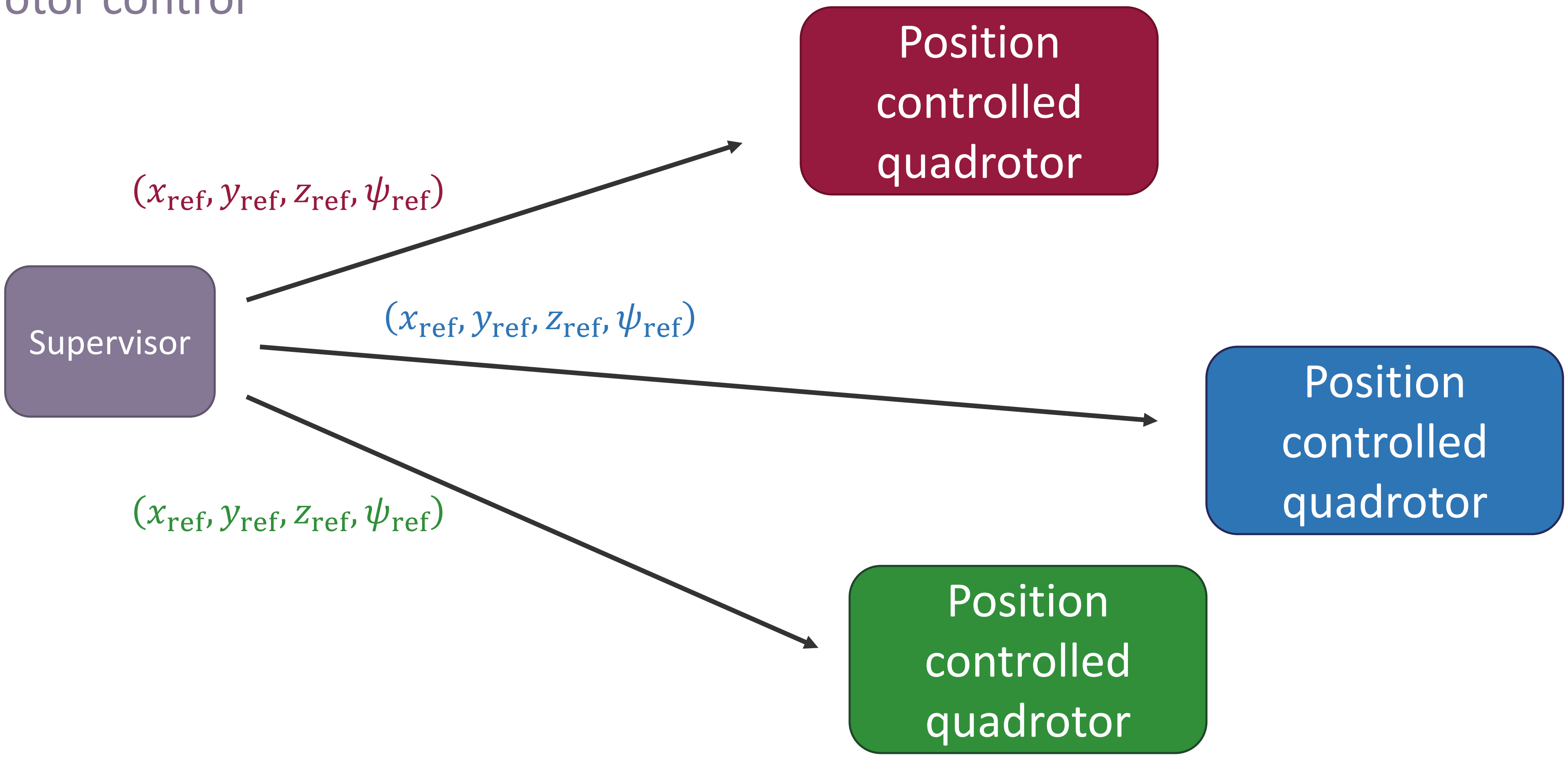
PID control law

$$\mathbf{f} = k_p \boldsymbol{\varepsilon}_p + k_v \boldsymbol{\varepsilon}_v + k_i \int \boldsymbol{\varepsilon}_p$$

$$(x_{\mathrm{ref}}, y_{\mathrm{ref}}, z_{\mathrm{ref}}, \psi_{\mathrm{ref}}) \rightarrow \boxed{\text{Position controlled quadrotor}} \rightarrow (x, y, z, \psi)$$

Integrator model

Multi quadrotor control



Trajectory feasibility

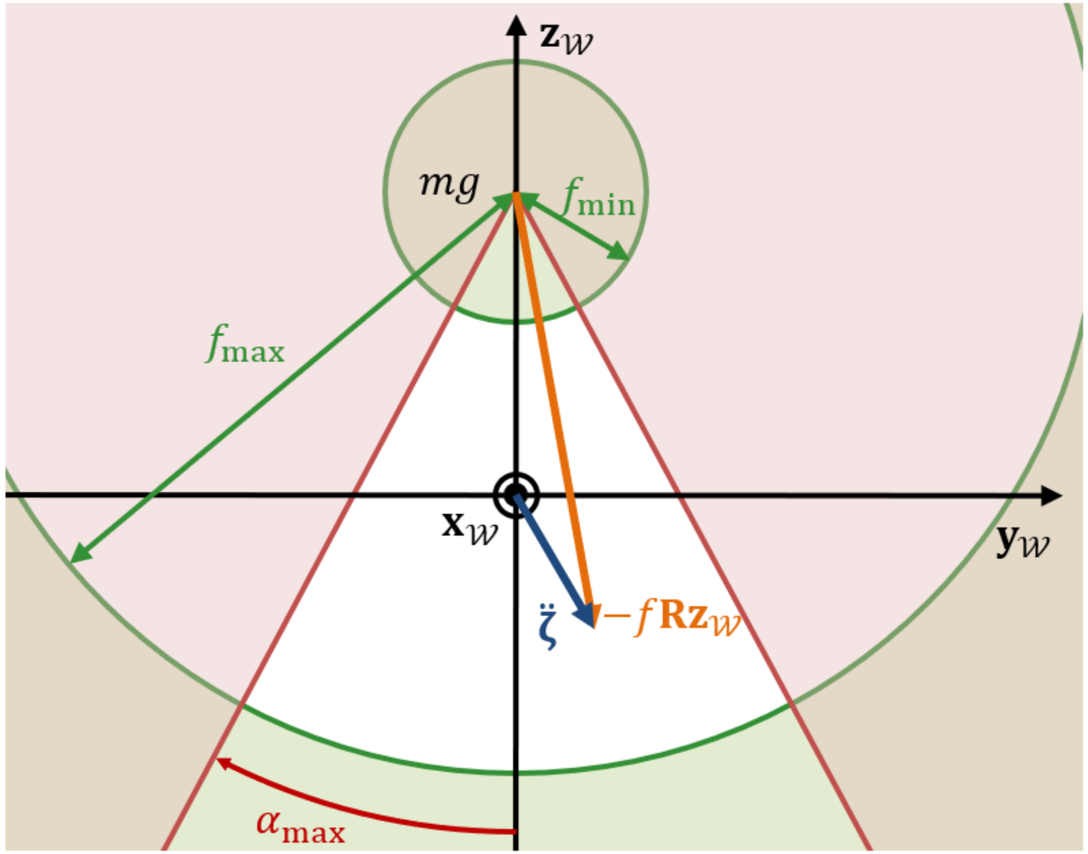
Thrust

Thrust magnitude constraint

$$f_{\min} \leq f \leq f_{\max}$$

Thrust angle constraint

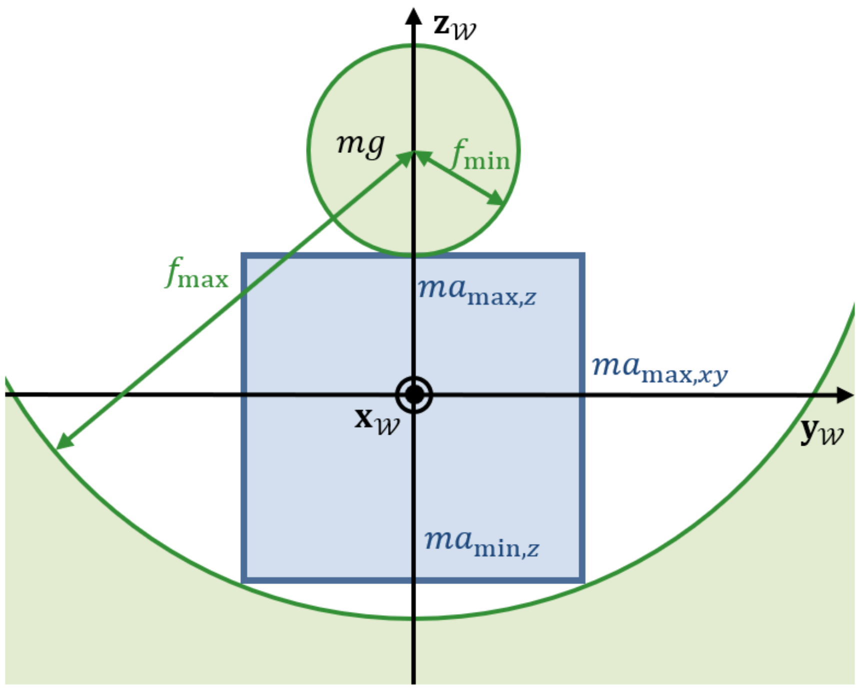
$$\alpha \leq \alpha_{\max}$$



Trajectory feasibility

Trajectory feasibility - Thrust

M.W. Mueller, **R. D'Andrea** (2013), *A model predictive controller for quadcopter state interception*, European Control Conference (ECC), Zürich, Switzerland



$$\begin{cases} |\ddot{\zeta}_x| \le a_{\max,xy} \\ |\ddot{\zeta}_y| \le a_{\max,xy} \\ a_{\min,z} \le \ddot{\zeta}_z \le a_{\max,z} \end{cases} \Rightarrow f_{\min} \le f \le f_{\max}$$

Example

$$a_{\max,z} = g - \frac{f_{\min}}{m}$$

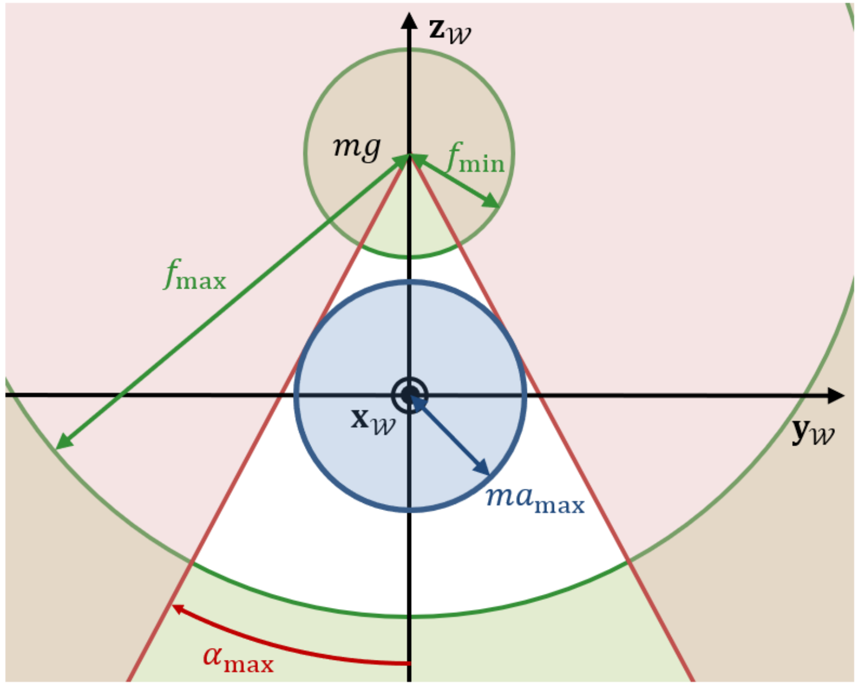
$$a_{\min,z} = \lambda\left(g - \frac{f_{\max}}{m}\right)$$

$$a_{\max,xy} = \sqrt{\frac{1}{2}\frac{f_{\max}^2}{m^2}(1-\lambda^2) + \lambda g \frac{f_{\max}}{m} - \frac{1}{2}\lambda^2 g^2}$$

Trajectory feasibility

Trajectory feasibility - Thrust

G. Rousseau, *Optimal trajectory planning and predictive control for cinematographic flight plans with quadrotors*, PhD thesis, Université Paris Saclay, 2019, Saclay, France



$$\|\ddot{\zeta}\|_2 \leq a_{\max} \triangleq \min\left\{g - \frac{f_{\min}}{m}, \frac{f_{\max}}{m} - g, g\, s_{\alpha_{\max}}\right\} \Rightarrow \begin{cases} f \geq f_{\min} \\ f \leq f_{\max} \\ \alpha \leq \alpha_{\max} \end{cases}$$