C 1.3 Quadrotors modeling - Supplement

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Linear Euler angles model

Equations of motion in body frame

Acceleration in body frame

Drone velocity relatively to NED, in drone frame

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \mathbf{R} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}$$

Drone acceleration relatively to NED, in drone frame

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \mathbf{R} \left(\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} + \begin{pmatrix} qw - rv \\ ru - pw \\ pv - qu \end{pmatrix} \right)$$

Equation of motion in body frame

$$m \, \ddot{\boldsymbol{\zeta}} = -f \, \mathbf{R} \, \mathbf{z}_{\mathcal{W}} + mg \, \mathbf{z}_{\mathcal{W}}$$

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{pmatrix} = \mathbf{R}^{\top} \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ f/m \end{pmatrix} - \begin{pmatrix} qw - rv \\ ru - pw \\ pv - qu \end{pmatrix}$$

Linear Euler angles model

Linearization near hovering

Small angles approx.

$$\mathbf{R} = \begin{pmatrix} c_\theta c_\psi & s_\varphi s_\theta c_\psi - c_\varphi s_\psi & c_\varphi s_\theta c_\psi + s_\varphi s_\psi \\ c_\theta s_\psi & s_\varphi s_\theta s_\psi + c_\varphi c_\psi & c_\varphi s_\theta s_\psi - s_\varphi c_\psi \\ -s_\theta & s_\varphi c_\theta & c_\varphi c_\theta \end{pmatrix} \Rightarrow \mathbf{R} \approx \begin{pmatrix} c_\psi & -s_\psi & \theta c_\psi + \varphi s_\psi \\ s_\psi & c_\psi & \theta s_\psi - \varphi c_\psi \\ -\theta & \varphi & 1 \end{pmatrix}$$

$$\Rightarrow \mathbf{R}^\top \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix} \approx \begin{pmatrix} -\theta g \\ \varphi g \\ g \end{pmatrix}$$

Linear Euler angles model

Linearization near hovering

Small angular velocity approx.

$$\begin{pmatrix}\dot\varphi\\ \dot\theta\\ \dot\psi\end{pmatrix} = \begin{pmatrix}1 & s_\varphi t_\theta & c_\varphi t_\theta\\ 0 & c_\varphi & -s_\varphi\\ 0 & s_\varphi/c_\theta & c_\varphi/c_\theta\end{pmatrix}\begin{pmatrix}p\\ q\\ r\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}\dot\varphi\\ \dot\theta\\ \dot\psi\end{pmatrix} \approx \begin{pmatrix}p\\ q\\ r\end{pmatrix} \Rightarrow \begin{pmatrix}\ddot\varphi\\ \ddot\theta\\ \ddot\psi\end{pmatrix} \approx \begin{pmatrix}\dot p\\ \dot q\\ \dot r\end{pmatrix}$$

$$\mathbf{J}\,\dot{\boldsymbol{\Omega}} = \boldsymbol{\tau} - \widehat{\boldsymbol{\Omega}}\,\mathbf{J}\,\boldsymbol{\Omega} \Rightarrow \mathbf{J}\,\dot{\boldsymbol{\Omega}} \approx \boldsymbol{\tau}$$

Diagonal inertia approx.

$$\mathbf{J}\,\dot{\boldsymbol{\Omega}} = \tau_x\,\mathbf{x}_\mathcal{D} + \tau_y\,\mathbf{y}_\mathcal{D} + \tau_z\,\mathbf{z}_\mathcal{D} \cancel{- \widehat{\boldsymbol{\Omega}}\mathbf{J}\,\boldsymbol{\Omega}}$$

$$\mathbf{J} = \begin{pmatrix}J_x & & \\ & J_y & \\ & & J_z\end{pmatrix} \Rightarrow \begin{pmatrix}\ddot\varphi\\ \ddot\theta\\ \ddot\psi\end{pmatrix} \approx \begin{pmatrix}\tau_x/J_x\\ \tau_x/J_y\\ \tau_x/J_z\end{pmatrix}$$

Linear Euler angles model

Linearization near hovering

Small linear speed

$$\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{pmatrix} = \mathbf{R}\left(\begin{pmatrix}\dot{u}\\ \dot{v}\\ \dot{w}\end{pmatrix} + \begin{pmatrix}qw - rv\\ ru - pw\\ pv - qu\end{pmatrix}\right) \Rightarrow \begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{pmatrix} \approx \mathbf{R}\begin{pmatrix}\dot{u}\\ \dot{v}\\ \dot{w}\end{pmatrix}$$

Linear model

$$\begin{pmatrix}\dot{u}\\ \dot{v}\\ \dot{w}\end{pmatrix} \approx \begin{pmatrix}-\theta g\\ \varphi g\\ -f/m\end{pmatrix} + \begin{pmatrix}0\\ 0\\ g\end{pmatrix}$$

$$\begin{pmatrix}\ddot{\varphi}\\ \ddot{\theta}\\ \ddot{\psi}\end{pmatrix} \approx \begin{pmatrix}\tau_x/J_x\\ \tau_x/J_y\\ \tau_x/J_z\end{pmatrix}$$

Linear Euler angles model

Linear models

$$\Delta f = mg - f$$

Manual piloting (racing)

$$\dot{w}(s) = \frac{1}{m}\Delta f(s)$$
$$p(s) = \frac{1}{J_x s}\tau_x(s)$$
$$q(s) = \frac{1}{J_y\, s}\tau_y(s)$$
$$r(s) = \frac{1}{J_z\, s}\tau_z(s)$$

Manual piloting (video)

$$w(s) = \frac{1}{m\, s}\Delta f(s)$$
$$\varphi(s) = \frac{1}{J_x\, s^2}\tau_x(s)$$
$$\theta(s) = \frac{1}{J_y\, s^2}\tau_y(s)$$
$$\dot{\psi}(s) = \frac{1}{J_z\, s}\tau_z(s)$$

Body velocity control + ψ

$$u(s) = \frac{-g}{s}\theta(s)$$
$$v(s) = \frac{g}{s}\varphi(s)$$
$$w(s) = \frac{1}{m\, s}\Delta f(s)$$
$$\psi(s) = \frac{1}{J_z\, s^2}\tau_z(s)$$

NED Position control + $\psi = 0$

$$x(s) = \frac{-g}{s^2}\theta(s)$$
$$y(s) = \frac{g}{s^2}\varphi(s)$$
$$z(s) = \frac{1}{m\, s^2}\Delta f(s)$$

Trajectory feasibility

Triple integrator model – Trajectory feasibility – Angular velocity

M.W. Mueller, **R. D’Andrea** (2013), *A model predictive controller for quadcopter state interception*, European Control Conference (ECC), Zürich, Switzerland

Angular velocity bound

$$\|\boldsymbol{\Omega}\| \leq \Omega_{\max}$$

Time derivative of acceleration equation

$$m\,\ddot{\boldsymbol{\zeta}} = -f\,\mathbf{R}\,\mathbf{z}_{\mathcal{W}} + mg\,\mathbf{z}_{\mathcal{W}}$$

$$\dot{\mathbf{R}} = \mathbf{R}\widehat{\boldsymbol{\Omega}}$$

$$\Rightarrow \quad \widehat{\boldsymbol{\Omega}}\,\mathbf{z}_{\mathcal{W}} = -\frac{\dot{f}}{f}\mathbf{z}_{\mathcal{W}} - \frac{m}{f}\,\mathbf{R}^{\top}\,\boldsymbol{\zeta}^{(3)}$$

$$\begin{pmatrix} q \\ p \\ 0 \end{pmatrix} = -\frac{\dot{f}}{f}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \frac{m}{f}\begin{pmatrix} \mathbf{R}^{\top}\,\boldsymbol{\zeta}^{(3)} \cdot \mathbf{x}_{\mathcal{W}} \\ \mathbf{R}^{\top}\,\boldsymbol{\zeta}^{(3)} \cdot \mathbf{y}_{\mathcal{W}} \\ \mathbf{R}^{\top}\,\boldsymbol{\zeta}^{(3)} \cdot \mathbf{z}_{\mathcal{W}} \end{pmatrix}$$

$$\Omega_{xy} \leq \frac{m}{f}\left\|\boldsymbol{\zeta}^{(3)}\right\|$$

If we impose $r = 0$

$$\|\boldsymbol{\Omega}\| \leq \frac{m}{f}\left\|\boldsymbol{\zeta}^{(3)}\right\|$$

$$\left\|\boldsymbol{\zeta}^{(3)}\right\| \leq j_{\max} \triangleq \frac{f_{\min}}{m}\Omega_{\max} \Rightarrow \|\boldsymbol{\Omega}\| \leq \Omega_{\max}$$

Nonlinear control

Nonlinear attitude control on SO(3)

T. Lee, M. Leok and N. H. McClamroch (2010), *Geometric tracking control of a quadrotor UAV on SE(3)*, Conference on Decision and Control (CDC), Atlanta, GA, USA

Attitude error

$$\boldsymbol{\varepsilon}_R = \frac{1}{2}\left(\mathbf{R}^\intercal \mathbf{R}_{\mathrm{ref}} - \mathbf{R}_{\mathrm{ref}}^\intercal \mathbf{R}\right)^\vee \qquad \hat{\mathbf{u}}^\vee = \mathbf{u}$$

Angular velocity error

$$\boldsymbol{\varepsilon}_\Omega = \mathbf{R}^\intercal \mathbf{R}_{\mathrm{ref}}\ \boldsymbol{\Omega}_{\mathrm{ref}} - \boldsymbol{\Omega}$$

Feedforward

$$\boldsymbol{\tau}_J = \widehat{\boldsymbol{\Omega}}\mathbf{J}\,\boldsymbol{\Omega}$$

$$\boldsymbol{\tau}_L = \mathbf{J}\left(\mathbf{R}^\intercal \mathbf{R}_{\mathrm{ref}} \dot{\boldsymbol{\Omega}}_{\mathrm{ref}} - \widehat{\boldsymbol{\Omega}}\mathbf{R}^\intercal \mathbf{R}_{\mathrm{ref}}\ \boldsymbol{\Omega}_{\mathrm{ref}}\right)$$

Control law

$$\boldsymbol{\tau} = k_R \boldsymbol{\varepsilon}_R + k_\Omega \boldsymbol{\varepsilon}_\Omega + \boldsymbol{\tau}_J + \boldsymbol{\tau}_L$$

Nonlinear control

Nonlinear position control on SO(3)

T. Lee, M. Leok and N. H. McClamroch (2010), *Geometric tracking control of a quadrotor UAV on SE(3)*, Conference on Decision and Control (CDC), Atlanta, GA, USA

Position error

$$\boldsymbol{\varepsilon}_p = \zeta_{ref} - \zeta$$

Velocity error

$$\boldsymbol{\varepsilon}_v = \dot{\zeta}_{ref} - \dot{\zeta}$$

Feedforward

$$\mathbf{f}_g = -mg\,\mathbf{z}_{\mathcal{W}}$$

$$\mathbf{f}_a = m\,\ddot{\zeta}_{\mathrm{ref}}$$

Control law

$$\mathbf{t} = k_p \boldsymbol{\varepsilon}_p + k_v \boldsymbol{\varepsilon}_v + \mathbf{f}_g + \mathbf{f}_a$$

$$f = -\mathbf{t} \cdot \mathbf{R}\,\mathbf{z}_{\mathcal{W}}$$

Attitude reference

$$\mathbf{x}_\psi = c_\psi \mathbf{x}_{\mathcal{W}} + s_\psi \mathbf{y}_{\mathcal{W}}$$

$$\mathbf{z}_{\mathcal{R}} = \frac{-\mathbf{t}}{\|\mathbf{t}\|}$$

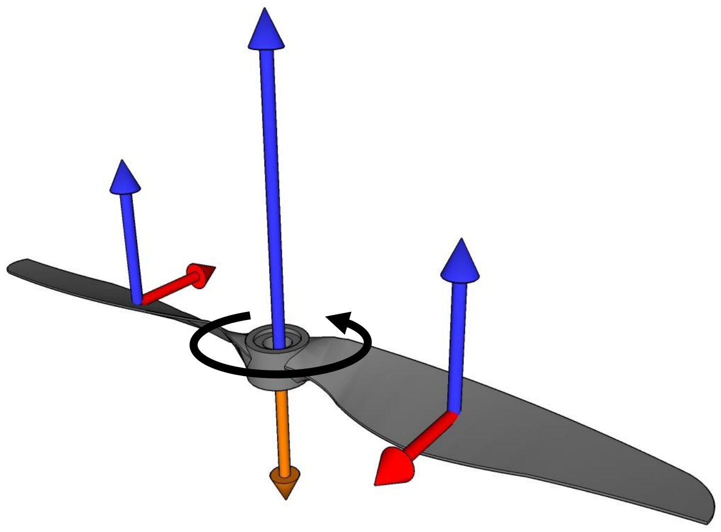
$$\mathbf{y}_{\mathcal{R}} = \frac{\mathbf{z}_{\mathcal{R}} \times \mathbf{x}_\psi}{\|\mathbf{z}_{\mathcal{R}} \times \mathbf{x}_\psi\|}$$

$$\mathbf{x}_{\mathcal{R}} = \mathbf{y}_{\mathcal{R}} \times \mathbf{z}_{\mathcal{R}}$$

$$\mathbf{R}_{\mathrm{ref}} = (\mathbf{x}_{\mathcal{R}} \quad \mathbf{y}_{\mathcal{R}} \quad \mathbf{z}_{\mathcal{R}})$$

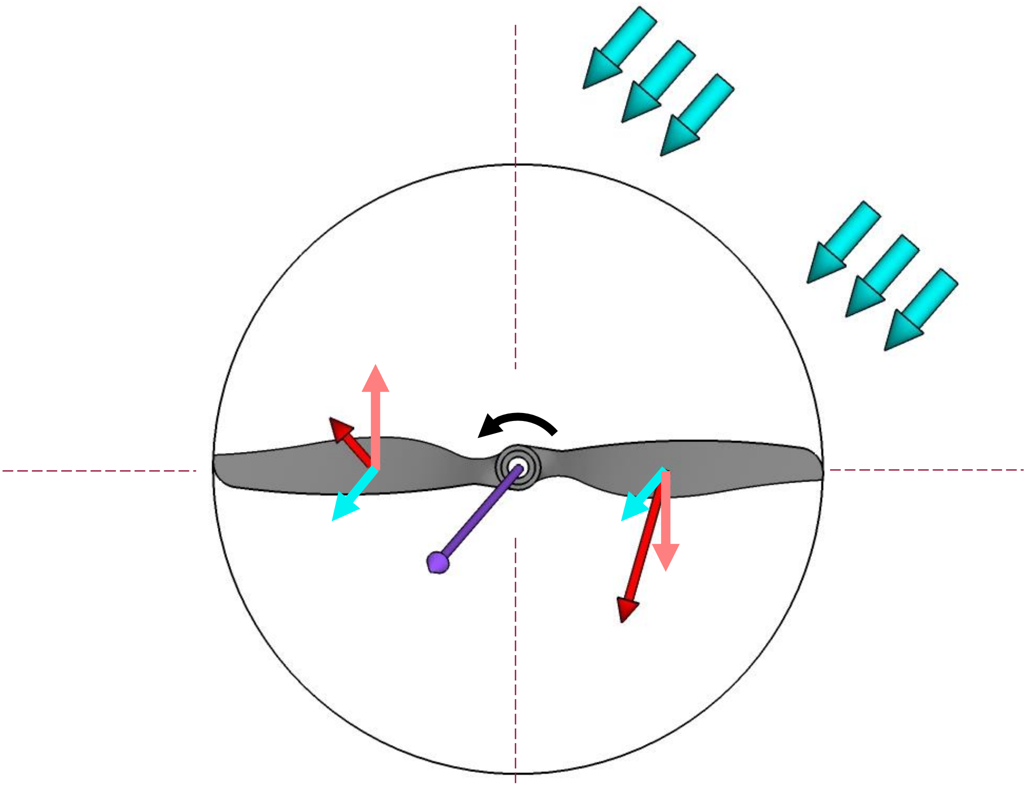
Quadrotor model

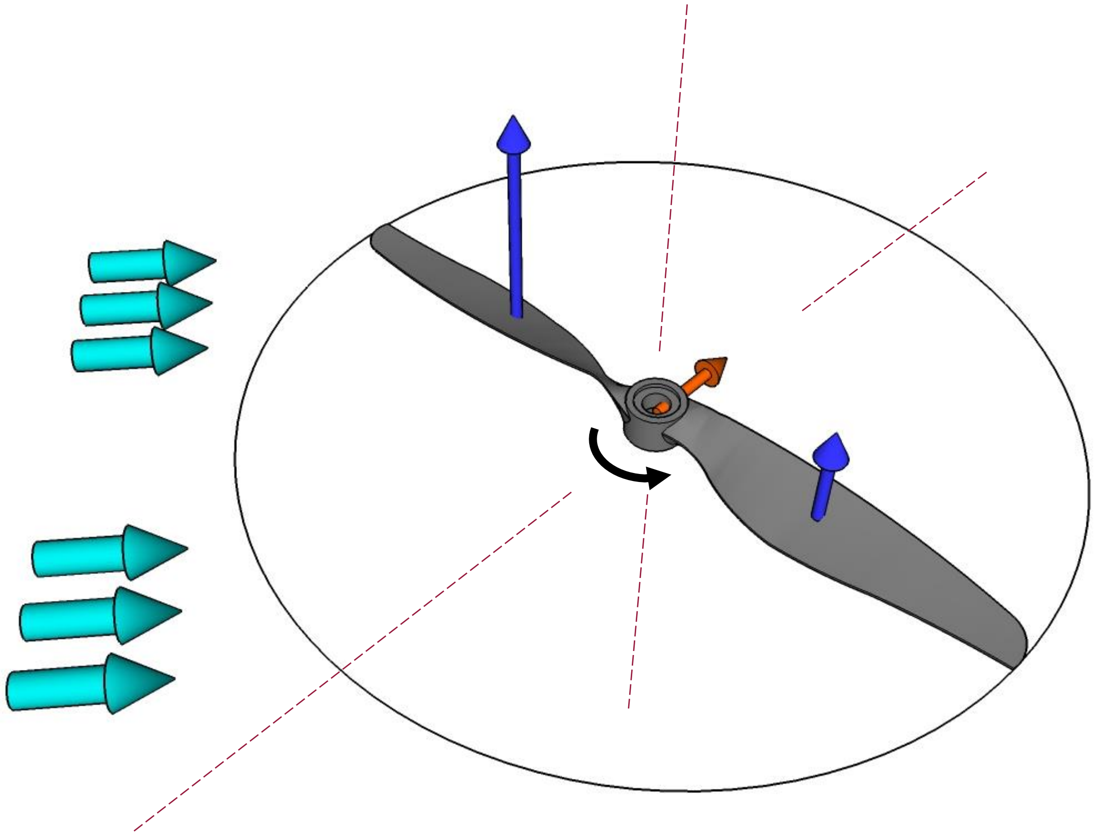
Aerodynamics disturbances



Quadrotor model

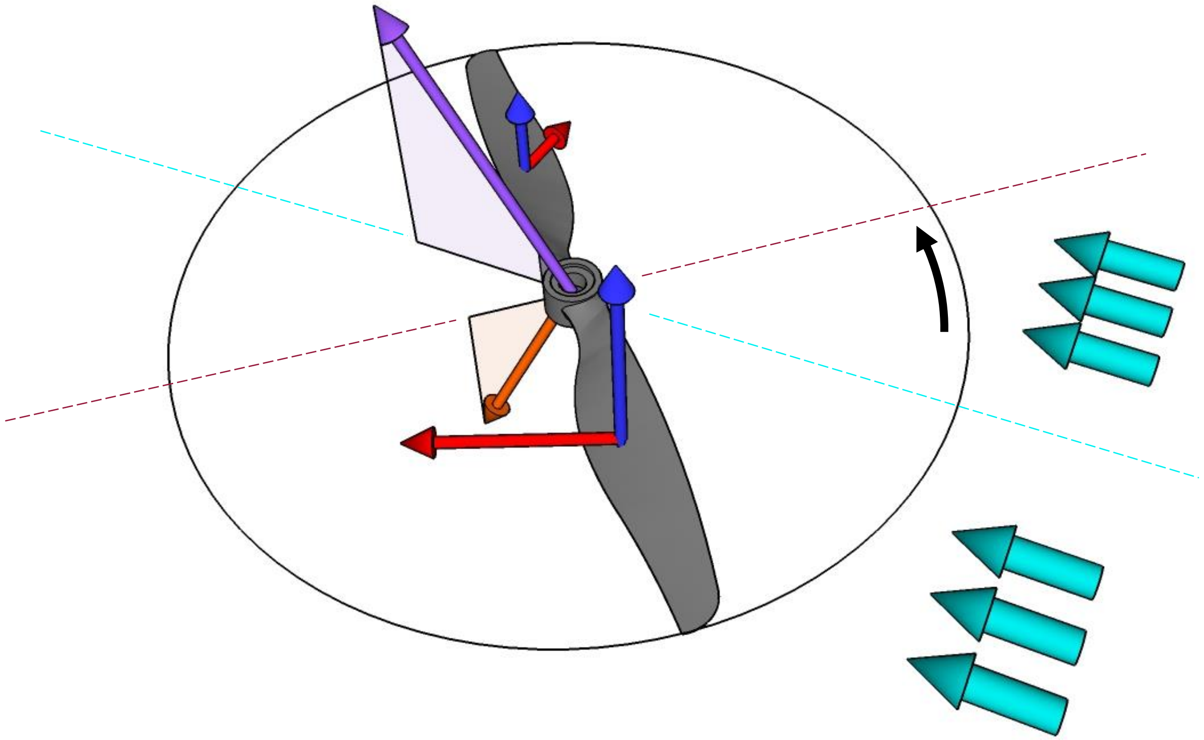
Aerodynamics disturbances - Dissymmetry of lift





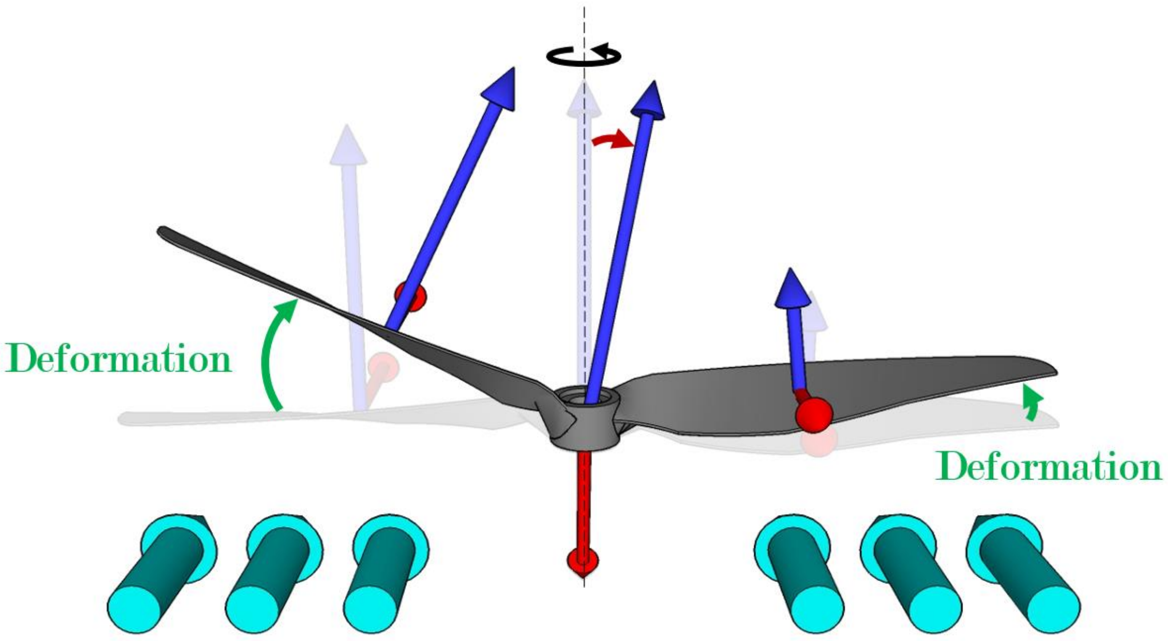
Quadrotor model

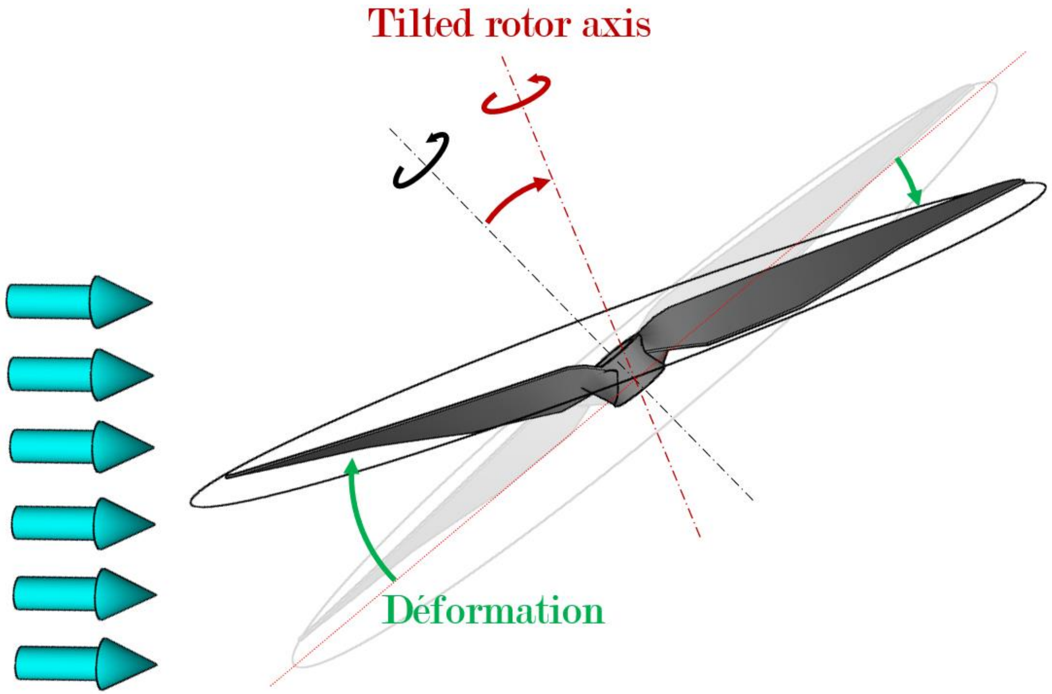
Aerodynamics disturbances - Dissymmetry of lift



Quadrotor model

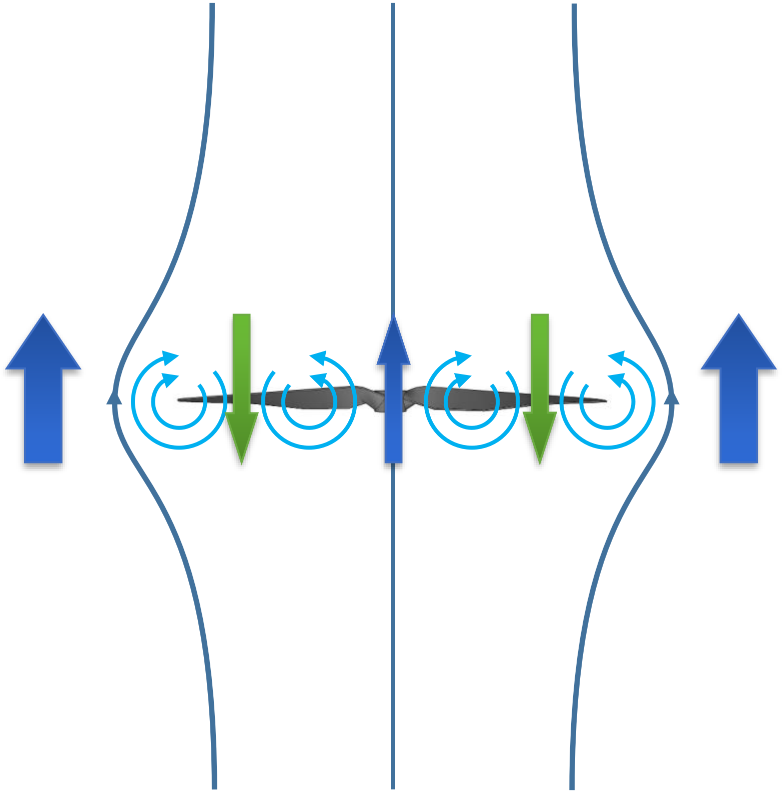
Aerodynamics disturbances – Blade flapping





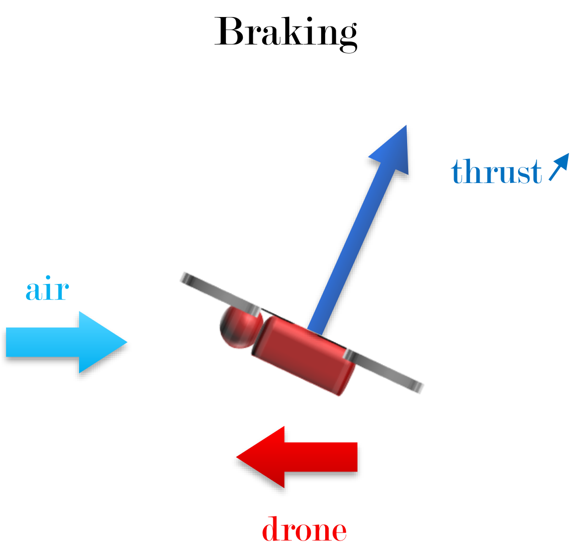
Quadrotor model

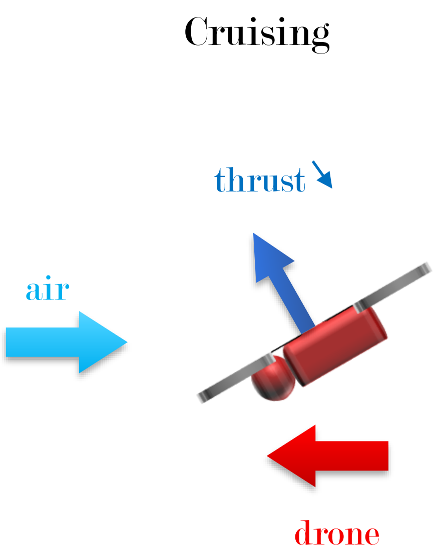
Aerodynamics disturbances – Vortex ring state



Quadrotor model

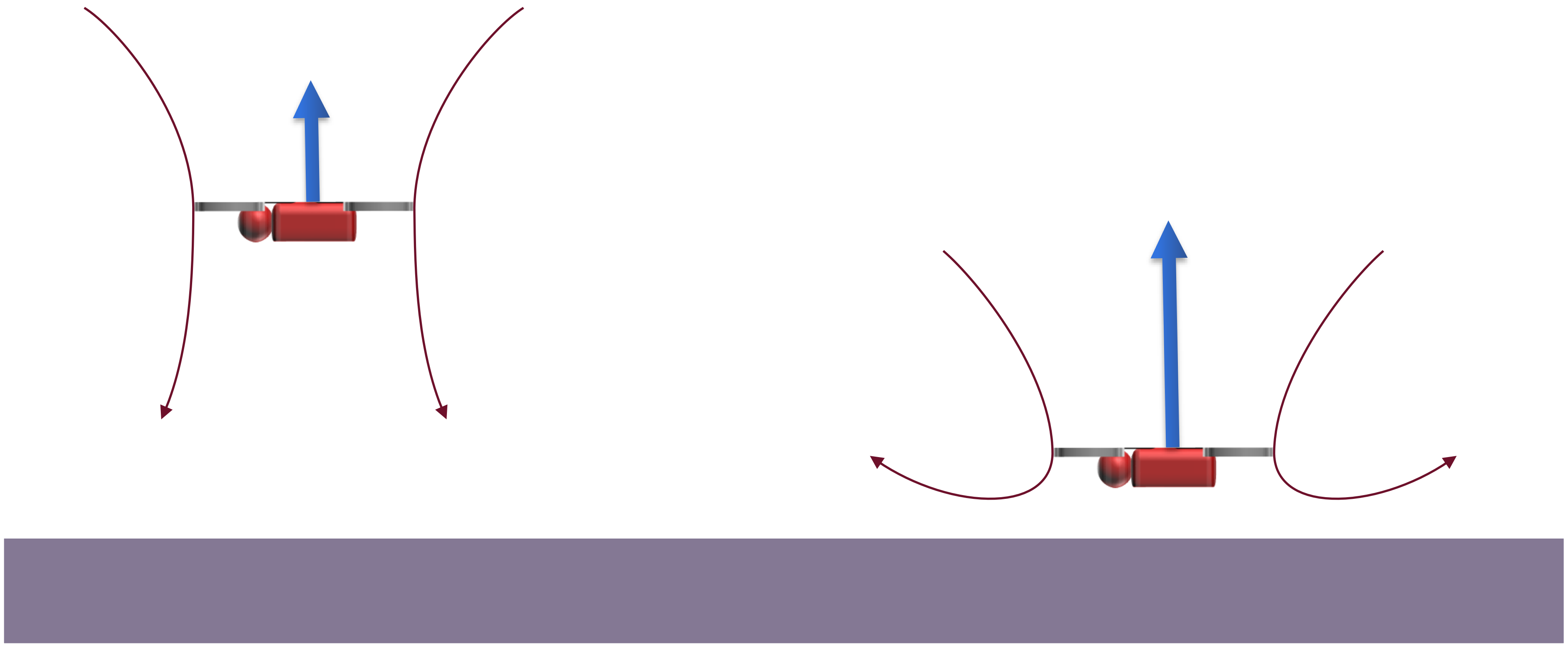
Aerodynamics disturbances – Angle of attack





Quadrotor model

Aerodynamics disturbances – Ground effect



Quadrotor model

Aerodynamics disturbances – Body drag