# The Real Numbers: Cantor

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**Problem (Week 2, 1.3.7).** Prove that if $a$ is an upper bound for $A$, and if $a$ is also an element of $A$, then it must be that $a = \sup A$.

**Solution.**
Let $s = supA$. $a$ is an upper bound for A and by definition 1.3.2 $(ii)$ [1] it follows that $s \leq a$. However, since s is also an upper bound, definition 1.3.1 [1] implies that $x \leq s\ \forall x \in A$. We know $a \in A$, then $a \leq s$. The two inequalities are satisfied only if $s = a$. Hence $a = supA$.

□

**Problem (Week 3, 1.6.1).** Show that $(0, 1)$ is uncountable if and only if $\mathbb{R}$ is uncountable.

**Solution.**
First I will show that the two sets have the same cardinality - $(0, 1) \sim \mathbb{R}$. By definition 1.5.2 [1] the set $(0, 1)$ has the same cardinality as $\mathbb{R}$ if there exists a mapping $f : (0, 1) \to \mathbb{R}$ that is bijective. By definition 1.5.1 [1] the function $f : (0, 1) \to \mathbb{R}$ is a bijection if $(i)$ for any two elements $a_1, a_2 \in (0, 1)$ implies $f(a_1) \neq f(a_2)$ in $\mathbb{R}$ (injection) and $(ii)$ given any $b \in \mathbb{R}$, $\exists a \in (0, 1)$ s.t. $f(a) = b$ (surjection). One such function is $f(x) = \frac{2x-1}{x(x-1)}$ because we want two vertical asymptotes at 0 and 1 and symmetry at the middle of the interval which means root at 0.5.

(i) Injection: Assume that $f(x_1) = f(x_2)$:

$$\frac{2x_1 - 1}{x_1(x_1 - 1)} = \frac{2x_2 - 1}{x_2(x_2 - 1)}.$$

This is equivalent to:

$$x_2(2x_1 - 1)(x_2 - 1) = x_1(x_1 - 1)(2x_2 - 1)$$

$$2x_2^2x_1 - 2x_1x_2 - x_2^2 + x_2 = 2x_1^2x_2 - 2x_1x_2 - x_1^2 + x_1$$
$$2x_2^2x_1 - 2x_1x_2 - x_2^2 + x_2 = 2x_1^2x_2 - 2x_1x_2 - x_1^2 + x_1$$
$$2x_2^2x_1 - x_2^2 + x_2 = 2x_1^2x_2 - x_1^2 + x_1$$

$$2x_2^2x_1 - 2x_1^2x_2 + x_2 - x_1 = x_2^2 - x_1^2$$
$$2x_2x_1(x_2 - x_1) + (x_2 - x_1) - (x_2 - x_1)(x_2 + x_1) = 0$$
$$(x_2 - x_1)(2x_2x_1 + 1 - x_2 - x_1) = 0$$

Here, one obvious solution is $x_2 = x_1$. To prove the injection, we have to show that there are no solutions that satisfy the following for values of $x_1, x_2 \in (0, 1)$:

$$2x_1x_2 + 1 - x_1 - x_2 = 0.$$

This is equivalent to:

$$x_2 = \frac{x_1 - 1}{2x_1 - 1}.$$

Let's look at the following cases and show that if $x_1 \in (0, 1)$ it is impossible for $x_2$ to be within that interval as well.

Case 1: $\frac{1}{2} < x_1 < 1$. Thus, $-\frac{1}{2} < x_1 - 1 < 0$. $1 < 2x_1 < 2$ and $1 - 1 < 2x_1 - 1 < 2 - 1$. Hence, the numerator is strictly negative and the denominator is strictly positive. This means $x_2$ must be negative and it is not between $(0, 1)$.

Case 2: $0 < x_1 < \frac{1}{2}$. Now let's show that $\frac{x_1-1}{2x_1-1}$ is above 1 for every $0 < x_1 < \frac{1}{2}$:

$$\frac{x_1 - 1}{2x_1 - 1} > 1$$

$$\frac{x_1 - 1}{2x_1 - 1} - \frac{2x_1 - 1}{2x_1 - 1} > 0$$
$$\frac{x_1 - 1 - (2x_1 - 1)}{2x_1 - 1} > 0$$
$$\frac{-x_1}{2x_1 - 1} > 0$$
$$\frac{x_1}{2x_1 - 1} < 0$$

Solving the inequality by the sign-line method, it is satisfied only for $x_1 \in (0, \frac{1}{2})$. This means that the $x_2 > 1$. Hence, we have shown that they are no solutions for the equation $2x_1x_2 + 1 - x_1 - x_2 = 0$ for $x_1, x_2 \in (0, 1)$. Thus, the only possible solution for $f(x_1) = f(x_2)$ for values of $x_1, x_2 \in (0, 1)$ is when $x_1 = x_2$. This concludes the proof of $f$ being an injective function.

(ii) Surjection:

To show the onto property of $f$, let $y \in \mathbb{R}$. We claim there is some $x \in (0, 1)$ such that

$$f(x) \;=\; \frac{2x - 1}{x(x - 1)} \;=\; y.$$

Rearranging the equation:

$$y \;=\; \frac{2x - 1}{x(x - 1)} \quad \iff \quad y\,x(x - 1) \;=\; 2x - 1.$$

Distributing the left-hand side gives:

$$y\,x^2 - y\,x \;=\; 2x - 1 \quad\Longleftrightarrow\quad y\,x^2 \;-\; (y+2)\,x \;+\; 1 \;=\; 0.$$

Define the continuous polynomial function:

$$G(x) \;=\; y\,x^2 - (y+2)\,x + 1.$$

We now check $G$ at $x = 0$ and $x = 1$:

$$G(0) = 1, \qquad G(1) = y \cdot 1^2 - (y+2)\cdot 1 + 1 = y - y - 2 + 1 = -1.$$

Since $G(0) = 1 > 0$ and $G(1) = -1 < 0$ and $G$ is polynomial so it must be continuous, the Intermediate Value Theorem guarantees the existence of some $c \in (0, 1)$ such that $G(c) = 0$. In other words,

$$G(c) \;=\; y\,c^2 - (y+2)\,c + 1 \;=\; 0,$$

so

$$y \;=\; \frac{2c-1}{c(c-1)} \;=\; f(c).$$

Hence for any $y \in \mathbb{R}$, we found $c \in (0, 1)$ with $f(c) = y$, proving that $f$ is onto $\mathbb{R}$. We have shown that there exists a bijective mapping $f : (0, 1) \to \mathbb{R}$. This means that $(0, 1) \sim \mathbb{R}$. Then if $(0, 1)$ is uncountable, it follows that $\mathbb{R}$ is uncountable. I show in 1.5.5.$b$) that $(0, 1) \sim \mathbb{R}$ is equivalent to $\mathbb{R} \sim (0, 1)$ and thus, if $\mathbb{R}$ is uncountable, then $(0, 1)$ is uncountable.

$\square$

**Problem (1.5.5).**

(a) Why is $A \sim A$ for every set $A$?

(b) Given sets $A$ and $B$, explain why $A \sim B$ is equivalent to asserting $B \sim A$.

(c) For three sets $A$, $B$, and $C$, show that $A \sim B$ and $B \sim C$ implies $A \sim C$.

**Solution.**
**(a)** From Definition 1.5.2 [1], two sets $A$ and $B$ have the same cardinality if there is a function $f : A \to B$ that is both 1–1 (injective) and onto (subjective). An obvious choice is the *identity* function

$$f(x) \;=\; x \quad \text{for all } x \in A.$$

- Injective: Suppose $f(x_1) = f(x_2)$. Then $x_1 = x_2$, so $f$ is injective.
- Surjective: For each $y \in A$, we can find some $x \in A$ such that $f(x) = y$. But if we take $x = y$, then $f(x) = x = y$. Thus $f$ is onto.

Hence $f$ is a bijection and $A \sim A$.

**(b)** To show that both statements are equivalent, we need to show both directions: if $A \sim B$, then $B \sim A$ and the opposite. Let's first address the forward direction. If $A \sim B$, then $\exists f : A \to B$ and f is bijective. This implies that the inverse function $f^{-1} : B \to A$ is well defined. To show its injectivity, we assume $f^{-1}(y_1) = f^{-1}(y_2)$ for some $y_1, y_2 \in B$. We can represent it as: $x_1 = f^{-1}(y_1) = f^{-1}(y_2) = x_2$. Applying $f$ to both sides: $f(x_1) = f(x_2)$. We have defined $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. Thus, $f(x_1) = y_1$ and $f(x_2) = y_2$. We have shown that if $f^{-1}(y_1) = f^{-1}(y_2)$, then $y_1 = y_2$. Therefore, $f^{-1}$ is injective. Since $f$ is onto, then $\forall y \in B$, $\exists x \in A$: $f(x) = y$. Applying the inverse, we get $f^{-1}(f(x)) = x, \forall x \in A$. This means that $\forall x \in A$, we can find $y \in B$. s.t. $f^{-1}(f(x)) = f^{-1}(y) = x$. Hence, $f^{-1}$ is surjective. $f^{-1}$ is both onto and 1-1, then it is bijective. It follows if $A \sim B$, then $B \sim A$. For the reverse direction, if $B \sim A$, then $A \sim B$. In broader terms we have demonstrated that if $f : A \to B$ is bijective, then the inverse $f^{-1} : B \to A$ is also bijective. Let $g : B \to A$. By the same argument as before, the inverse function $g^{-1} : A \to B$ is bijective. Thus, if $B \sim A$, then $A \sim B$. Thus, the $\sim$ relation is symmetric.

**(c)** We need to show that there is a bijective function between $A$ and $C$. Let $g : A \to B$ be a bijection and $f : B \to C$ be a bijection. Then the composition is another function: $h = g \circ f : A \to C$. To show injectivity of h, assume $h(x_1) = h(x_2)$ for some $x_1, x_2 \in A$. However, $h(x) = f(g(x))\ \forall x \in A$. Then, $h(x_1) = f(g(x_1)) = h(x_2) = f(g(x_2))$. $f : A \to B$ is bijective and thus, injective. This implies if $f(g(x_1)) = f(g(x_2))$, then $g(x_1) = g(x_2)$. Analogically, if $g(x_1) = g(x_2)$, $x_1 = x_2$. Thus, we showed directly given $h(x_1) = h(x_2)$, $x_1 = x_2$. The function $h : A \to C$ is injective. Since $f$ and $g$ are both surjective, then $\forall y \in B$, $\exists x \in A$, s.t. $g(x) = y$ and $\forall z \in C$, $\exists y \in B$, s.t. $f(y) = z$. That is, t$\forall z \in C$, $\exists x \in A$, s.t. $f(g(x)) = z$. Hence, $h$ is both surjective and injective, thus bijective. Therefore, $A$ and $C$ have the same cardinality. $\square$

**Problem (1.5.6).**

(a) Give an example of a countable collection of disjoint open intervals.

(b) Give an example of an uncountable collection of disjoint open intervals or argue that no such collection exists.

**Solution.**

(a) We can construct a countable collection of disjoint open intervals by taking the intervals between every two consecutive natural numbers:

$$(0, 1), (1, 2), ..., (n - 1, n), ...$$

$$I_n = (n - 1, n)\ \forall n \in \mathbb{N}.$$

It is obvious that these intervals are non-intersecting since each interval is separated by the integers on the number line. Now, I will demonstrate formally why this collection is countable. Define $f : \mathbb{N} \to \mathcal{S}$ by:

$$f(n) = I_n.$$

- Injectivity: If $f(n) = f(m)$, then $(n-1, n) = (m-1, m)$, so $n = m$.
- Surjectivity: For every $I_k \in \mathcal{S}$, $f(k) = I_k$.

Since $f$ bijectively maps $\mathbb{N}$ to $\mathcal{S}$: $\mathbb{N} \sim \mathcal{S}$, and from 1.5.5 we know this is equivalent to $\mathcal{S} \sim \mathbb{N}$, then $\mathcal{S}$ is countable.

(b) Suppose such set $\mathcal{T}$ exists and $I_n$ is one element of $\mathcal{T}$. Theorem 1.4.3. [1] states that for every two real numbers $a$ and $b$ with $a < b$, there exists a rational number $r$ satisfying $a < r < b$. That is, $\exists r \in \mathbb{Q}$ s.t. $r \in I_n$. Since every interval $I_n$ in $\mathcal{T}$ contains at least one rational number, we can define a function:

$$f : \mathcal{T} \to \mathbb{Q}, f(I_n) = r_n,$$

where $r_n \in \mathbb{Q}$. The function is injective because no two disjoint intervals can contain the same rational number. Since $\mathbb{Q}$ is countable, the image $f(\mathcal{T})$ is a subset of $\mathbb{Q}$, then according to Theorem 1.5.7 [1] $f(\mathcal{T})$ is also countable. By the definition of an injective function, $\mathcal{T}$ must also be countable. This contradicts the initial supposition that $\mathcal{T}$ is uncountable. Therefore, there is no uncountable collection of disjoint open intervals in $\mathbb{R}$.

$\square$

**Problem (1.6.7).** Return to the particular functions constructed in Exercise 1.6.6 and construct the subset $B$ that results using the preceding rule. In each case, note that $B$ is not in the range of the function used.

**Solution.**
Looking at the functions from 1.6.6.$a$) we have the set $A = \{a, b, c\}$ and the power set $P(A)$. The two injective mappings are $f : A \to P(A)$ for which $f(a) = \{a\}, f(b) = \{b\}, f(c) = \{c\}$ and $g : A \to P(A)$ for which $g(a) = \{b, c\}, g(b) = \{a, c\}, g(c) = \{a, b\}$. Constructing the set $B = \{a \in A : a \notin f(a)\}$ for $f$ would be $B = \emptyset$, because $a \in f(a) = \{a\}$ and the same for $b$ and $c$. However, looking at the mapping $g$, $a \notin g(a) = \{b, c\}$, $b \notin g(b) = \{a, c\}$, $c \notin g(c) = \{a, b\}$. Hence, $B = \{a, b, c\}$. In 1.6.6.$b$) we are given the set $C = \{1, 2, 3, 4\}$ and an example of a 1-1 map $h : C \to P(C)$ is:

$$h(1) = \{1\}, h(2) = \{1, 3\}, h(3) = \{4\}, h(4) = \{1, 2, 3, 4\}.$$

The set $B$ in this case is $B = \{2, 3\}$, because $2 \notin h(2)$ and $3 \notin h(3)$. $\square$

**Problem (1.6.8).**

(a) First, show that the case $a' \in B$ leads to a contradiction.

(b) Now, finish the argument by showing that the case $a' \notin B$ is equally unacceptable.

**Solution.**

(a) Since we have assumed that our function $f : A \to P(A)$ is onto and $B = \{a \in A : a \notin f(a)\}$, then $B \subset A$. There must be some $a' \in A$, $s.t.$ $f(a') = B$. If $a' \in B$, then by definition of $B$, $a' \notin f(x) = B$. We get the contradiction that $a'$ is not an element of $B$, which is a contradiction.

(b) Assume $a' \notin B$, then $a' \in f(x) = B$, then we get the contradiction that $a'$ must be an element of $B$ which contradicts the supposition. Therefore, $f : A \to P(A)$ is not surjective.

$\square$

# References


[1] S. Abbott. *Understanding Analysis*. Springer, 2015. ISBN: 9781493927135. URL: `https://books.google.com/books?id=R2DkzQEACAAJ`.