NOTES FROM NSF-MSGI INTERNSHIP

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1. Introduction

We will explore the connection between statistical mechanics models and quantum Hamiltonians, primarily via examples.

More stuff here later. Basic objects/definitions.

2. Ising Models

The Ising models are a family of statistical mechanics models with nearest-neighbor interaction. Given any lattice $\mathcal{L} \subset \mathbb{Z}^d$ with N total points, define a *spin* at each site of the lattice via a variable $s \in \Omega_0 = \{\pm 1\}$. A *configuration* is a collection $\{s\} = \{s_\ell\}_{\ell \in \mathcal{L}} \in \Omega_0^N$. For convenience, we will usually work with cubic lattices of the form $\mathcal{L} = \{-n, -n+1, \dots, n-1, n\}^d$ or $\mathcal{L} = \{1 \dots, n\}^d$ and impose periodic boundary conditions. The *action* associated to this model is the function

$$\mathcal{S}(\{s\}, \tau) = -\beta_\tau(\tau) \sum_{i \sim j} s_i s_j - \beta(\tau) h \sum_i s_i,$$

where τ is the lattice spacing in the "time" direction, h is the strength of an external magnetic field, and the first sum is over nearest neighbor sites of the lattice. The coefficients β_τ and β may depend on the time direction lattice spacing and are otherwise chosen to reflect some physical situation (e.g. they may be taken to be proportional to the "inverse temperature" $\frac{1}{kT}$ where k is Boltzmann's constant).

We will be interested in the *partition function* Z defined as

$$Z = \sum_{\{s\} \in \Omega_0^N} e^{-\mathcal{S}(\{s\}, \tau)}.$$

2.1. 1+0 Ising. In the 1+0 Ising model, we take a 1-dimensional statistical mechanics system and relate it to a 0-dimensional (point) quantum Hamiltonian. See [1] and [6]. The definitions above simplify to the following. The lattice is now a set of N points $x_0, \dots, x_{N-1}$ on a circle so that $s_0 = s_N$. The action is given by

$$\mathcal{S}(\{s\}, \tau) = -\beta_\tau(\tau) \sum_{i=0}^{N-1} s_i s_{i+1} - \beta(\tau) h \sum_{i=0}^{N-1} s_i.$$

It will be helpful to rewrite this expression to be symmetric and expressed in terms of sums and differences of the s_i. Up to a constant factor of $-\beta_\tau N$, we find

$$\mathcal{S}(\{s\},\tau) = \frac{1}{2}\beta_\tau(\tau)\sum_{i=0}^{N-1}(s_i - s_{i+1})^2 - \frac{1}{2}\beta(\tau)h\sum_{i=0}^{N-1}(s_i + s_{i+1}).$$

Note that we needed periodicity in order to rewrite the second term in this way. Further, we now have $\mathcal{S} = 0$ when all of the spins are identical and $h = 0$.

Now the partition function becomes

$$\begin{aligned} Z &= \sum_{\{s\}\in\Omega_0^N} e^{\frac{1}{2}\beta_\tau(\tau)\sum_{i=0}^{N-1}(s_i-s_{i+1})^2 - \frac{1}{2}\beta(\tau)h\sum_{i=0}^{N-1}(s_i+s_{i+1})} \\ &= \sum_{s_0\in\{\pm1\}}\cdots\sum_{s_{N-1}\in\{\pm1\}} e^{\frac{1}{2}\beta_\tau(s_0-s_1)^2}e^{-\frac{1}{2}\beta h(s_0+s_1)}\cdot\ldots\cdot e^{\frac{1}{2}\beta_\tau(s_{N-1}-s_0)^2}e^{-\frac{1}{2}\beta h(s_{N-1}+s_0)}. \end{aligned}$$

However, each factor in the product depends only on what the values of $s_i - s_{i+1}$ and $s_i + s_{i+1}$ are. We can construct the *transfer matrix*, T, to record this data, indexed by the possible spins at each site. So we have

$$T_{-1,-1} = e^{-\beta h}, \quad T_{-1,1} = T_{1,-1} = e^{-2\beta_\tau}, \quad \text{and } T_{1,1} = e^{\beta h}.$$

Thus

$$T = \begin{bmatrix} e^{-\beta h} & e^{-2\beta_\tau} \\ e^{-2\beta_\tau} & e^{\beta h} \end{bmatrix}.$$

Now we can write $Z = \operatorname{Tr} T^N$.

Our goal is to choose functions $\beta_\tau(\tau)$ and $\beta(\tau)$ so that the transfer matrix operator T has the form $T = e^{-\tau H} \approx I - \tau H$ for some quantum Hamiltonian H (independent of τ) acting on a 2-dimensional vector space. This is the *τ-continuum Hamiltonian* for the model. Here the idea is that statistical mechanics properties of the lattice action system (e.g. magnetization per site, average magnetization, two-point correlations, correlation length) will map to properties of the operator H related to its eigenvalues and eigenvectors.

For example, we could choose $\beta_\tau(\tau) = -\frac{1}{2}\log\tau$ and $\beta(\tau) = \tau$. Then we have

$$T = \begin{bmatrix} e^{-\tau h} & \tau \\ \tau & e^{\tau h} \end{bmatrix} \approx I_2 - \tau\begin{bmatrix} h & -1 \\ -1 & -h \end{bmatrix}.$$

From this it follows that

$$H = \begin{bmatrix} h & -1 \\ -1 & -h \end{bmatrix} = -\sigma_1 + h\sigma_3,$$

where σ_1, σ_2, and σ_3 are the Pauli matrices.

More generally, we can ask what constraints β and β_τ must satisfy to be able to do this. Analyzing each of the four matrix entries shows that for small τ we need

$$e^{-\beta h} \approx 1 - \tau H_{0,0}, \quad e^{-2\beta_\tau} \approx -\tau H_{1,0} = -\tau H_{0,1}, \quad \text{and } e^{\beta h} \approx 1 - \tau H_{1,1}.$$

Since $H_{1,0}$ and $H_{0,1}$ are constant with respect to τ we need $-2\beta_\tau \approx \log(\lambda\tau)$ for small τ and some $\lambda \in \mathbb{R}_{>0}$. Similarly, if β is small when τ is small we have $e^{-\beta h} \approx$

$1 - \beta h \approx 1 - \tau H_{0,0}$ and $e^{\beta h} \approx 1 + \beta h \approx 1 - \tau H_{1,1}$. So to first order we have $\beta \approx \tau$ and we see $H_{0,0} = -H_{1,1} = \mu h$ for some $\mu \in \mathbb{R} \setminus \{0\}$.

Putting all of this together, we see that the general τ-continuum Hamiltonian for this model will have the form

$$H = \begin{bmatrix} \mu h & -\lambda \\ -\lambda & -\mu h \end{bmatrix} = -\lambda \sigma_1 + \mu h \sigma_3.$$

Now we could also attempt to find a second order approximation of T so that $T \approx I - \tau H + \frac{1}{2}\tau^2 H^2$ for an operator H not depending on τ. However, following the approach above will show us that there is no easy solution in this case. To get a higher order approximation (or the actual matrix logarithm solving $T = e^{-\tau H}$ for H), we may need to allow H to depend on τ.

In the second order case, we need

$$\begin{aligned} e^{-\beta h} &\approx 1 - \tau H_{0,0} + \frac{\tau^2}{2}(H_{0,0}^2 + H_{0,1}^2), \\ e^{-2\beta_\tau} &\approx -\tau H_{0,1} + \frac{\tau^2}{2}(H_{0,0}H_{0,1} + H_{0,1}H_{1,1}), \\ e^{\beta h} &\approx 1 - \tau H_{1,1} + \frac{\tau^2}{2}(H_{0,1}^2 + H_{0,0}^2) \end{aligned}$$

for small τ, where we have used the fact that H should be real and symmetric and so $H_{0,1} = H_{1,0}$. From the first and third equations, we see that $H_{0,0}$ and $H_{1,1}$ must depend on h. But the second equation has no dependence on h, so the dependence of $H_{0,1}$ on h must be inversely related to the $H_{0,0}$ and $H_{1,1}$ dependence. On the other hand adding the first and third equations yields

$$\cosh(\beta h) \approx 1 - \frac{\tau}{2}(H_{0,0} + H_{1,1}) + \frac{\tau^2}{4}(H_{0,0}^2 + 2H_{0,1}^2 + H_{1,1}^2),$$

which naturally suggests taking $\beta = \mu\tau$, $H_{0,0} = -H_{1,1} = \mu h$, and $H_{0,1} = H1, 0 = 0$. But this is a contradiction.

2.2. 1+1 Ising Model. We may analyze the $1+1$ Ising model similarly to the $1+0$ model. We are going to relate a statistical mechanics system on a 2-dimensional lattice (with one temporal and one spatial dimension) to a quantum mechanical Hamiltonian on a 1-dimensional system of interacting spins. See [1] and [6].

As in the 1+0 Ising model, we will work with a lattice with periodic boundary conditions. Suppose $\mathcal{L}$ is a $N_x \times N_\tau$ lattice of points in $\mathbb{Z}^2$ with $\vec{\tau}$ a unit vector in the time direction and $\vec{x}$ a unit vector in the spatial direction. Then the action for the model is

$$\mathcal{S} = -\sum_{\ell} \beta_\tau s_\ell s_{\ell+\vec{\tau}} + \beta s_\ell s_{\ell+\vec{x}},$$

where the sum is over all lattice points $\ell \in \mathcal{L}$.

We want to write $\mathcal{S} = \sum_{j=1}^{N_\tau} L(j, j+1)$ for some $L(j, j+1)$ that describes the interaction between the spatial rows j and $j+1$. To do this, we first rewrite

$$\mathcal{S} = \sum_{\ell} \frac{\beta_\tau}{2}(s_\ell - s_{\ell+\vec{\tau}})^2 - \frac{1}{2}\beta(s_\ell s_{\ell+\vec{x}} + s_{\ell+\vec{\tau}} s_{\ell+\vec{\tau}+\vec{x}}),$$

using the periodic boundary conditions. Note that the new $\mathcal{S}$ differs from the previous one by a normalization constant so that the first term is 0 when all of the spins are aligned, rather than being $-N\beta_\tau$. We can now define

$$L(j, j+1) := \sum_{\ell=1}^{N_x} \frac{\beta_\tau}{2}(s_\ell - \tilde{s}_\ell)^2 - \frac{\beta}{2}(s_\ell s_{\ell+1} + \tilde{s}_\ell \tilde{s}_{\ell+1}),$$

where the sum is over the N_x indices in the spatial row, $\{s\}$ is the configuration of row j and $\{\tilde{s}\}$ is the configuration of row $j+1$.

Then the partition function is

$$Z = \sum_{\{s\}} e^{-L(1,2)} e^{-L(2,3)} \dots e^{-L(N_\tau,1)}.$$

As in the previous section, we express Z as the trace of the N_τ^{th} power of a transfer matrix $\hat{T}$ describing the transition between rows. Since there are 2^{N_x} configurations for each row, $\hat{T}$ will be a $2^{N_x} \times 2^{N_x}$ matrix. The elements of $\hat{T}$ can be organized by the number of spin flips between configurations, since these determine the value of the first term of $L(j, j+1)$. We want to find β, β_τ so that for τ near 0, $\hat{T} \approx 1 - \tau \hat{H}$.

$$\begin{aligned}
\hat{T}|_{0\,\text{flips}} &= e^{\beta \sum_{\ell=1}^{N_x} s_\ell s_{\ell+1}}, \\
\hat{T}|_{1\,\text{flip}} &= e^{-2\beta_\tau} e^{\frac{\beta}{2}\sum_{\ell=1}^{N_x}(s_\ell s_{\ell+1} + \tilde{s}_\ell \tilde{s}_{\ell+1})}, \\
&\vdots \qquad\qquad \vdots \\
\hat{T}|_{k\,\text{flips}} &= e^{-2k\beta_\tau} e^{\frac{\beta}{2}\sum_{\ell=1}^{N_x}(s_\ell s_{\ell+1} + \tilde{s}_\ell \tilde{s}_{\ell+1})}
\end{aligned}$$

The nicest solution to this is to choose $\beta = \lambda\tau$ and $\tau = e^{-2\beta_\tau}$ for some $\lambda \in \mathbb{R}_{>0}$. Then to first order approximation in τ,

$$\begin{aligned}
\hat{T}|_{0\,\text{flips}} &\approx 1 - \mu\lambda\tau, \\
\hat{T}|_{1\,\text{flip}} &\approx \tau(1 + \kappa\lambda\tau) \approx \tau, \\
&\vdots \qquad\qquad \vdots \\
\hat{T}|_{k\,\text{flips}} &\approx \tau^k(1 + \kappa\lambda\tau) \approx 0,
\end{aligned}$$

where $\mu = \sum_{\ell=1}^{N_x} s_\ell s_{\ell+1}$ and $\kappa = \frac{1}{2}\sum_{\ell=1}^{N_x}(s_\ell s_{\ell+1} + \tilde{s}_\ell \tilde{s}_{\ell+1})$ are independent of τ.

We can now express $\hat{T}$ as $I - \tau\hat{H}$ where the Hamiltonian $\hat{H}$ is given in terms of the Pauli operators at each site ℓ, $\hat{\sigma}_1(\ell)$ and $\hat{\sigma}_3(\ell)$:

$$\hat{H} = -\lambda \sum_{\ell=1}^{N_x} \hat{\sigma}_3(\ell)\hat{\sigma}_3(\ell+1) - \sum_{\ell=1}^{N_x} \hat{\sigma}_1(\ell).$$

2.3. Infinite Lattice Ising Models. The Ising model can be generalized to an *infinite volume model* by letting the lattice $\mathcal{L}$ grow to $\mathbb{Z}^d$ appropriately. We'll briefly sketch the ideas here - a detailed exposition can be found in [2] in chapters 3 and 6.

3. O(N) Model

These models are extensively treated in [1],[4],[6]. As in the Ising model, we have a lattice $\mathcal{L} \subset \mathbb{Z}^d$ of M points, usually cubic. The spins at each site i of the lattice are now $\vec{x_i} \in \Omega_0 = S^{N-1} = \{\vec{x} \in \mathbb{R}^N \mid \|\vec{x}\| = 1\}$. The action is

$$\mathcal{S} = \sum_{i,j} -J_{i,j}\vec{x_i} \cdot \vec{x_j},$$

where the sum is over nearest-neighbor pairs in $\mathcal{L}$. Frequently we will take the interaction constants $J_{i,j}$ to depend only on which coordinate of the lattice the points differ in. As in the earlier examples, it may be helpful to renormalize the action so that when all the $\vec{x_i}$ are equal we get $\mathcal{S} = 0$. This results in a renormalized action

$$\mathcal{S} = \sum_{i,j} \frac{1}{2} J_{i,j}(\vec{x_i} - \vec{x_j})^2.$$

As before, we are interested in thinking of our d-dimensional lattice as having 1 time dimension and $d-1$ spatial dimensions and finding a τ-continuum quantum mechanical Hamiltonian H that corresponds to the action as the lattice spacing τ goes to 0. Since the spin configuration space is now continuous, the partition function will now be an integral rather than a summation:

$$Z = \int_{(S^N)^M} d\vec{x_1} \dots d\vec{x_M} e^{-\beta \sum_{i,j} \frac{1}{2} J_{i,j}(\vec{x_i} - \vec{x_j})^2}.$$

3.1. O(2) model on a 1-dimensional lattice. First we consider the $O(2)$ model on a 1-dimensional lattice of N points with periodic boundary conditions. When our configuration space is S^1, we can parameterize the spins $\vec{x_i} = (\cos(\theta_i), \sin(\theta_i))$ for $\theta \in [0, 2\pi)$. Under this parameterization, the action becomes

$$\mathcal{S} = -J \sum_{j=1}^{N} \cos(\theta_j)\cos(\theta_{j+1}) + \sin(\theta_j)\sin(\theta_{j+1}) = -J \sum_{j=1}^{N} \cos(\theta_j - \theta_{j+1}).$$

The corresponding partition function is then

$$Z = \int_{[0,2\pi]^N} \left(\prod_{j=1}^{N} d\theta_j \right) e^{\sum_{j=1}^{N} \beta J \cos(\theta_j - \theta_{j+1})},$$

where again $\beta = \frac{1}{kT}$ is the inverse temperature.

We now wish find an analogue of the transfer matrix from the analysis of the Ising model.

4. Spherical Model

5. Lattice Gauge Theories

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