NOTES FROM NSF-MSGI INTERNSHIP

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1. Introduction

We will explore the connection between statistical mechanics models and quantum Hamiltonians, primarily via examples.

More stuff here later. Basic objects/definitions.

2. Ising Models

The Ising models are a family of statistical mechanics models with nearest-neighbor interaction. Given any lattice $\mathcal{L} \subset \mathbb{Z}^d$ with N total points, define a *spin* at each site of the lattice via a variable $s \in \Omega_0 = \{\pm 1\}$. A *configuration* is a collection $\{s\} = \{s_\ell\}_{\ell \in \mathcal{L}} \in \Omega_0^N$. For convenience, we will usually work with cubic lattices of the form $\mathcal{L} = \{-n, -n+1, \ldots, n-1, n\}^d$ or $\mathcal{L} = \{1 \ldots, n\}^d$ and impose periodic boundary conditions. The *action* associated to this model is the function

$$\mathcal{S}(\{s\}, \tau, h) = -\beta_\tau(\tau) \sum_{i \sim j} s_i s_j - \beta(\tau) h \sum_i s_i,$$

where τ is the lattice spacing in the "time" direction, h is the strength of an external magnetic field, and the first sum is over nearest neighbor sites of the lattice. The coefficients β_τ and β may depend on the time direction lattice spacing and are otherwise chosen to reflect some physical situation (e.g. they may be taken to be proportional to the "inverse temperature" $\frac{1}{kT}$ where k is Boltzmann's constant).

We will be interested in the *partition function* Z defined as

$$Z(\tau, h) = \sum_{\{s\} \in \Omega_0^N} e^{-\mathcal{S}(\{s\}, \tau, h)}.$$

It turns out that we can use statistical mechanics to show that many other interesting physical quantities can be derived from the partition function [2, Ch. 3]. The *pressure in* $\mathcal{L}$ of the model is defined to be

$$\psi_{\mathcal{L}}(\tau, h) := \frac{1}{N} \log Z(\tau, h).$$

The *magnetization density in* $\mathcal{L}$ is by definition

$$m_{\mathcal{L}} := \frac{1}{N} \sum_{\ell \in \mathcal{L}} s_\ell.$$

The expected magnetization density, $\langle m_{\mathcal{L}} \rangle$, is related to the pressure and partition function:

$$\langle m_{\mathcal{L}} \rangle(\tau, h) = \frac{\partial \psi_{\mathcal{L}}}{\partial h}(\tau, h) = \frac{\partial}{\partial h}\left(\frac{1}{N} \log Z(\tau, h)\right).$$

2.1. Infinite Lattice Ising Models. The Ising model can be generalized to an *infinite volume model* by letting the lattice $\mathcal{L}$ grow to $\mathbb{Z}^d$ appropriately. We'll briefly sketch the ideas here - a detailed exposition can be found in [2] in chapters 3 and 6. In general, we say a sequence of lattices $\{\mathcal{L}_n\}$ converges to $\mathbb{Z}^d$, denoted $\mathcal{L}_n \Uparrow \mathbb{Z}^d$, if

(1) $\mathcal{L}_n$ is increasing, i.e. $\mathcal{L}_n \subset \mathcal{L}_{n+1}$ for all n,
(2) $\bigcup_n \mathcal{L}_n = \mathbb{Z}^d$,
(3) $\lim_{n\to\infty} \frac{|\partial \mathcal{L}_n|}{|\mathcal{L}_n|} = 0$, where $\partial\mathcal{L}_n$ is the boundary of the lattice, i.e.

$$\partial\mathcal{L}_n := \{\ell \in \mathcal{L}_n \mid \ell \sim m \,\text{for some}\; m \in \mathbb{Z}^d \setminus \mathcal{L}_n\}.$$

It turns out that the pressure $\psi_{\mathcal{L}}$ defined above is convex as a function of h and that there is a well-defined limit

$$\psi(\tau, h) := \lim_{\mathcal{L}\Uparrow\mathbb{Z}^d} \psi_{\mathcal{L}}(\tau, h).$$

Because ψ is a convex function of h, the *average magnetization density* given by

$$m(\tau, h) = \lim_{\mathcal{L}\Uparrow\mathbb{Z}^d} \langle m_{\mathcal{L}} \rangle(\tau, h),$$

exists for all but a countable set of $h \in \mathbb{R}$. The points at which the average magnetization density fails to exist (always because the left and right derivatives do not equal each other) are called *first-order phase transitions.*

2.2. 1+0 Ising. In the 1+0 Ising model, we take a 1-dimensional statistical mechanics system and relate it to a 0-dimensional (point) quantum Hamiltonian. See [1] and [6]. The definitions above simplify to the following. The lattice is now a set of N points $x_0, \dots, x_{N-1}$ on a circle so that $s_0 = s_N$. The action is given by

$$\mathcal{S}(\{s\}, \tau, h) = -\beta_\tau(\tau) \sum_{i=0}^{N-1} s_i s_{i+1} - \beta(\tau) h \sum_{i=0}^{N-1} s_i.$$

It will be helpful to rewrite this expression to be symmetric and expressed in terms of sums and differences of the s_i. Up to a constant factor of $-\beta_\tau N$, we find

$$\mathcal{S}(\{s\}, \tau) = \frac{1}{2}\beta_\tau(\tau) \sum_{i=0}^{N-1} (s_i - s_{i+1})^2 - \frac{1}{2}\beta(\tau) h \sum_{i=0}^{N-1} (s_i + s_{i+1}).$$

Note that we needed periodicity in order to rewrite the second term in this way. Further, we now have $\mathcal{S} = 0$ when all of the spins are identical and $h = 0$.

Now the partition function becomes

$$\begin{aligned} Z &= \sum_{\{s\}\in\Omega_0^N} e^{\frac{1}{2}\beta_\tau(\tau)\sum_{i=0}^{N-1}(s_i - s_{i+1})^2 - \frac{1}{2}\beta(\tau) h \sum_{i=0}^{N-1}(s_i+s_{i+1})} \\ &= \sum_{s_0\in\{\pm1\}} \cdots \sum_{s_{N-1}\in\{\pm1\}} e^{\frac{1}{2}\beta_\tau(s_0-s_1)^2} e^{-\frac{1}{2}\beta h(s_0+s_1)} \cdot \ldots \cdot e^{\frac{1}{2}\beta_\tau(s_{N-1}-s_0)^2} e^{-\frac{1}{2}\beta h(s_{N-1}+s_0)}. \end{aligned}$$

However, each factor in the product depends only on what the values of $s_i - s_{i+1}$ and $s_i + s_{i+1}$ are. We can construct the *transfer matrix*, T, to record this data, indexed by the possible spins at each site. So we have

$$T_{-1,-1} = e^{-\beta h}, \quad T_{-1,1} = T_{1,-1} = e^{-2\beta_\tau}, \quad \text{and}\; T_{1,1} = e^{\beta h}.$$

Thus

$$T = \begin{bmatrix} e^{-\beta h} & e^{-2\beta_\tau} \\ e^{-2\beta_\tau} & e^{\beta h} \end{bmatrix}.$$

Now we can write $Z = \operatorname{Tr} T^N$. From this result, we can derive the $N \to \infty$ limit of the quantities of interest above, in particular the average magnetization. Since T is diagonalizable, we can write $T = SDS^{-1}$ and so

$$\operatorname{Tr} T^N = \operatorname{Tr}\big(SD^N S^{-1}\big) = \operatorname{Tr}\big(D^N\big) = \lambda_0^N + \lambda_1^N,$$

where λ_0, λ_1 are the eigenvalues of T.

Via a bit of algebra, we see that

$$\lambda_0 = \cosh(\beta h) + \sqrt{e^{-4\beta_\tau} + \sinh^2(\beta h)}$$
$$\lambda_1 = \cosh(\beta h) - \sqrt{e^{-4\beta_\tau} + \sinh^2(\beta h)}$$

Later, we will examine the Hamiltonian when $\beta_\tau = -\frac{1}{2}\log\tau$ and $\beta = \tau$. In this case,

$$\lambda_0 = \cosh(\tau h) + \sqrt{\tau^2 + \sinh^2(\tau h)}$$
$$\lambda_1 = \cosh(\tau h) - \sqrt{\tau^2 + \sinh^2(\tau h)}$$

Now the average magnetization is

$$\begin{aligned} m(\tau, h) &= \lim_{N\to\infty} \frac{\partial}{\partial h}\left(\frac{1}{N}\log\big(\lambda_0^N + \lambda_1^N\big)\right) \\ &= \frac{\partial}{\partial h}\lim_{N\to\infty}\left(\frac{1}{N}\left(\log\big(\lambda_0^N\big) + \log\left(1 + \left(\frac{\lambda_1}{\lambda_0}\right)^N\right)\right)\right) \\ &= \frac{\partial}{\partial h}\log(\lambda_0), \end{aligned}$$

since $|\lambda_1/\lambda_0| < 1$. In our example case,

$$\begin{aligned} m(\tau, h) &= \frac{1}{\cosh(\tau h) + \sqrt{\tau^2 + \sinh^2(\tau h)}}\left(\tau\sinh(\tau h) + \frac{\tau\sinh(\tau h)\cosh(\tau h)}{\sqrt{\tau^2 + \sinh^2(\tau h)}}\right) \\ &= \frac{\tau\sinh(\tau h)}{\sqrt{\tau^2 + \sinh(\tau h)}}. \end{aligned}$$

2.2.1. *Deriving the quantum Hamiltonian:* Our goal is to choose functions $\beta_\tau(\tau)$ and $\beta(\tau)$ so that the transfer matrix operator T has the form $T = e^{-\tau H} \approx I - \tau H$ for some quantum Hamiltonian H (independent of τ) acting on a 2-dimensional vector space. This is the τ-*continuum Hamiltonian* for the model. Here the idea is that statistical mechanics properties of the lattice action system (e.g. magnetization per site, average magnetization, two-point correlations, correlation length) will map to properties of the operator H related to its eigenvalues and eigenvectors.

For example, we could choose $\beta_\tau(\tau) = -\frac{1}{2}\log\tau$ and $\beta(\tau) = \tau$. Then we have

$$T = \begin{bmatrix} e^{-\tau h} & \tau \\ \tau & e^{\tau h}\end{bmatrix} \approx I_2 - \tau\begin{bmatrix} h & -1 \\ -1 & -h\end{bmatrix}.$$

From this it follows that

$$H = \begin{bmatrix} h & -1 \\ -1 & -h\end{bmatrix} = -\sigma_1 + h\sigma_3,$$

where σ_1, σ_2, and σ_3 are the Pauli matrices.

More generally, we can ask what constraints β and β_τ must satisfy to be able to do this. Analyzing each of the four matrix entries shows that for small τ we need

$$e^{-\beta h} \approx 1 - \tau H_{0,0}, \quad e^{-2\beta_\tau} \approx -\tau H_{1,0} = -\tau H_{0,1}, \quad \text{and } e^{\beta h} \approx 1 - \tau H_{1,1}.$$

Since $H_{1,0}$ and $H_{0,1}$ are constant with respect to τ we need $-2\beta_\tau \approx \log(\lambda\tau)$ for small τ and some $\lambda \in \mathbb{R}_{>0}$. Similarly, if β is small when τ is small we have $e^{-\beta h} \approx 1 - \beta h \approx 1 - \tau H_{0,0}$ and $e^{\beta h} \approx 1 + \beta h \approx 1 - \tau H_{1,1}$. So to first order we have $\beta \approx \tau$ and we see $H_{0,0} = -H_{1,1} = \mu h$ for some $\mu \in \mathbb{R} \setminus \{0\}$.

Putting all of this together, we see that the general τ-continuum Hamiltonian for this model will have the form

$$H = \begin{bmatrix} \mu h & -\lambda \\ -\lambda & -\mu h \end{bmatrix} = -\lambda\sigma_1 + \mu h \sigma_3.$$

Now we could also attempt to find a second order approximation of T so that $T \approx I - \tau H + \frac{1}{2}\tau^2 H^2$ for an operator H not depending on τ. However, following the approach above will show us that there is no easy solution in this case. To get a higher order approximation (or the actual matrix logarithm solving $T = e^{-\tau H}$ for H), we may need to allow H to depend on τ.

In the second order case, we need

$$\begin{aligned} e^{-\beta h} &\approx 1 - \tau H_{0,0} + \frac{\tau^2}{2}(H_{0,0}^2 + H_{0,1}^2), \\ e^{-2\beta_\tau} &\approx -\tau H_{0,1} + \frac{\tau^2}{2}(H_{0,0}H_{0,1} + H_{0,1}H_{1,1}), \\ e^{\beta h} &\approx 1 - \tau H_{1,1} + \frac{\tau^2}{2}(H_{0,1}^2 + H_{0,0}^2) \end{aligned}$$

for small τ, where we have used the fact that H should be real and symmetric and so $H_{0,1} = H_{1,0}$. From the first and third equations, we see that $H_{0,0}$ and $H_{1,1}$ must depend on h. But the second equation has no dependence on h, so the dependence of $H_{0,1}$ on h must be inversely related to the $H_{0,0}$ and $H_{1,1}$ dependence. On the other hand adding the first and third equations yields

$$\cosh(\beta h) \approx 1 - \frac{\tau}{2}(H_{0,0} + H_{1,1}) + \frac{\tau^2}{4}(H_{0,0}^2 + 2H_{0,1}^2 + H_{1,1}^2),$$

which naturally suggests taking $\beta = \mu\tau$, $H_{0,0} = -H_{1,1} = \mu h$, and $H_{0,1} = H1,0 = 0$. But this is a contradiction.

2.3. 1+1 Ising Model. We may analyze the 1+1 Ising model similarly to the 1+0 model. We are going to relate a statistical mechanics system on a 2-dimensional lattice (with one temporal and one spatial dimension) to a quantum mechanical Hamiltonian on a 1-dimensional system of interacting spins. See [1] and [6].

As in the 1+0 Ising model, we will work with a lattice with periodic boundary conditions. Suppose $\mathcal{L}$ is a $N_x \times N_\tau$ lattice of points in $\mathbb{Z}^2$ with $\vec{\tau}$ a unit vector in the time direction and $\vec{x}$ a unit vector in the spatial direction. Then the action for the model is

$$\mathcal{S} = -\sum_\ell \beta_\tau s_\ell s_{\ell+\vec{\tau}} + \beta s_\ell s_{\ell+\vec{x}},$$

where the sum is over all lattice points $\ell \in \mathcal{L}$.

We want to write $\mathcal{S} = \sum_{j=1}^{N_\tau} L(j, j+1)$ for some $L(j, j+1)$ that describes the interaction between the spatial rows j and $j+1$. To do this, we first rewrite

$$\mathcal{S} = \sum_\ell \frac{\beta_\tau}{2}(s_\ell - s_{\ell+\vec{\tau}})^2 - \frac{1}{2}\beta(s_\ell s_{\ell+\vec{x}} + s_{\ell+\vec{\tau}} s_{\ell+\vec{\tau}+\vec{x}}),$$

using the periodic boundary conditions. Note that the new $\mathcal{S}$ differs from the previous one by a normalization constant so that the first term is 0 when all of the spins are aligned, rather than being $-N\beta_\tau$. We can now define

$$L(j, j+1) := \sum_{\ell=1}^{N_x} \frac{\beta_\tau}{2}(s_\ell - \tilde{s}_\ell)^2 - \frac{\beta}{2}(s_\ell s_{\ell+1} + \tilde{s}_\ell \tilde{s}_{\ell+1}),$$

where the sum is over the N_x indices in the spatial row, $\{s\}$ is the configuration of row j and $\{\tilde{s}\}$ is the configuration of row $j+1$.

Then the partition function is

$$Z = \sum_{\{s\}} e^{-L(1,2)} e^{-L(2,3)} \dots e^{-L(N_\tau,1)}.$$

As in the previous section, we express Z as the trace of the N_τ^{th} power of a transfer matrix $\hat{T}$ describing the transition between rows. Since there are 2^{N_x} configurations for each row, $\hat{T}$ will be a $2^{N_x} \times 2^{N_x}$ matrix. The elements of $\hat{T}$ can be organized by the number of spin flips between configurations, since these determine the value of the first term of $L(j, j+1)$. We want to find β, β_τ so that for τ near 0, $\hat{T} \approx 1 - \tau \hat{H}$.

$$\begin{aligned}
\hat{T}|_{0\,\text{flips}} &= e^{\beta \sum_{\ell=1}^{N_x} s_\ell s_{\ell+1}}, \\
\hat{T}|_{1\,\text{flip}} &= e^{-2\beta_\tau} e^{\frac{\beta}{2}\sum_{\ell=1}^{N_x}(s_\ell s_{\ell+1} + \tilde{s}_\ell \tilde{s}_{\ell+1})}, \\
\vdots \qquad & \qquad \vdots \\
\hat{T}|_{k\,\text{flips}} &= e^{-2k\beta_\tau} e^{\frac{\beta}{2}\sum_{\ell=1}^{N_x}(s_\ell s_{\ell+1} + \tilde{s}_\ell \tilde{s}_{\ell+1})}
\end{aligned}$$

The nicest solution to this is to choose $\beta = \lambda\tau$ and $\tau = e^{-2\beta_\tau}$ for some $\lambda \in \mathbb{R}_{>0}$. Then to first order approximation in τ,

$$\begin{aligned}
\hat{T}|_{0\,\text{flips}} &\approx 1 - \mu\lambda\tau, \\
\hat{T}|_{1\,\text{flip}} &\approx \tau(1 + \kappa\lambda\tau) \approx \tau, \\
\vdots \qquad & \qquad \vdots \\
\hat{T}|_{k\,\text{flips}} &\approx \tau^k(1 + \kappa\lambda\tau) \approx 0,
\end{aligned}$$

where $\mu = \sum_{\ell=1}^{N_x} s_\ell s_{\ell+1}$ and $\kappa = \frac{1}{2}\sum_{\ell=1}^{N_x}(s_\ell s_{\ell+1} + \tilde{s}_\ell \tilde{s}_{\ell+1})$ are independent of τ.

We can now express $\hat{T}$ as $I - \tau\hat{H}$ where the Hamiltonian $\hat{H}$ is given in terms of the Pauli operators at each site ℓ, $\hat{\sigma}_1(\ell)$ and $\hat{\sigma}_3(\ell)$:

$$\hat{H} = -\lambda \sum_{\ell=1}^{N_x} \hat{\sigma}_3(\ell)\hat{\sigma}_3(\ell+1) - \sum_{\ell=1}^{N_x} \hat{\sigma}_1(\ell). \tag{1}$$

2.3.1. *Phase transition of 1+1 Ising model.* There are a variety of ways to see that the 1+1 Ising model has a phase transition.

2.3.2. *Jordan-Wigner transform and solution in terms of Fermionic operators.* A Hamiltonian of the form of Eq. (1) can be rewritten in terms of Fermion operators $\{a_j, a_j^\dagger\}_{j=1}^n$ that satisfy the *canonical commutation relations* (CCRs)

$$\{a_j, a_k^\dagger\} = \delta_{k,j} I; \qquad \{a_j, a_k\} = 0,$$

where $\{A, B\} = AB + BA$ is the anticommutator of two operators [6],[8],[9].

We can summarize this transformation as follows:

(1) Use duality to swap the roles of $\hat{\sigma}_1$ and $\hat{\sigma}_3$ in Eq. (1).
(2) Use raising and lowering operators to rewrite H in terms of fermion operators. (Jordan-Wigner transform)
(3) Convert the resulting operators and quadratic Hamiltonian to momentum space.
(4) Diagonalize the momentum Fermionic Hamiltonian.
(5) Determine the eigenvalues of the resulting Hamiltonian.

3. O(N) Model

These models are extensively treated in [1],[4],[6]. As in the Ising model, we have a lattice $\mathcal{L} \subset \mathbb{Z}^d$ of M points, usually cubic. The spins at each site i of the lattice are now $\vec{x_i} \in \Omega_0 = S^{N-1} = \{\vec{x} \in \mathbb{R}^N \mid \|\vec{x}\| = 1\}$. The action is

$$\mathcal{S} = \sum_{i,j} -J_{i,j}\vec{x_i} \cdot \vec{x_j},$$

where the sum is over nearest-neighbor pairs in $\mathcal{L}$. Frequently we will take the interaction constants $J_{i,j}$ to depend only on which coordinate of the lattice the points differ in. As in the earlier examples, it may be helpful to renormalize the action so that when all the $\vec{x_i}$ are equal we get $\mathcal{S} = 0$. This results in a renormalized action

$$\mathcal{S} = \sum_{i,j} \frac{1}{2} J_{i,j}(\vec{x_i} - \vec{x_j})^2.$$

As before, we are interested in thinking of our d-dimensional lattice as having 1 time dimension and $d-1$ spatial dimensions and finding a τ-continuum quantum mechanical Hamiltonian H that corresponds to the action as the lattice spacing τ goes to 0. Since the spin configuration space is now continuous, the partition function will now be an integral rather than a summation:

$$Z = \int_{(S^N)^M} d\vec{x_1} \dots d\vec{x_M} e^{-\beta \sum_{i,j} \frac{1}{2} J_{i,j}(\vec{x_i} - \vec{x_j})^2}.$$

3.1. O(2) model on a 1-dimensional lattice. First we consider the $O(2)$ model on a 1-dimensional lattice of N points with periodic boundary conditions and a magnetic field of strength h. Since our configuration space is S^1, we can parameterize the spins $\vec{x_i} = (\cos(\theta_i), \sin(\theta_i))$ for $\theta \in [0, 2\pi)$. Under this parameterization, the action becomes

$$\begin{aligned}
\mathcal{S} &= -\beta_\tau \sum_{j=1}^{N} \cos(\theta_j)\cos(\theta_{j+1}) + \sin(\theta_j)\sin(\theta_{j+1}) - \beta h \sum_{j=1}^{N} \cos(\theta_j) \\
&= -\beta_\tau \sum_{j=1}^{N} \cos(\theta_j - \theta_{j+1}) - \frac{\beta h}{2} \sum_{j=1}^{N} \left(\cos(\theta_j) + \cos(\theta_{j+1})\right).
\end{aligned}$$

The corresponding partition function is then

$$Z = \int_{[0,2\pi]^N} \left(\prod_{j=1}^{N} d\theta_j\right) e^{\beta_\tau \sum_{j=1}^{N} \cos(\theta_j - \theta_{j+1})} e^{\frac{\beta h}{2} \sum_{j=1}^{N} \cos(\theta_j) + \cos(\theta_{j+1})}.$$

We now wish find an analogue of the transfer matrix from the analysis of the Ising model. To do this, we need the concept of integral operators. Given any L^2 function $f(x, y)$ on a domain

$E \times E$, we can define the operator $L_f : L^2(E) \to L^2(E)$ by

$$(L_f g)(x) = \int_E f(x,y)g(y)\, dy$$

for any $g \in L^2(E)$.

Define $f(\theta,\phi) : [0,2\pi] \times [0,2\pi]$ by

$$f(\theta,\phi) = e^{\beta_\tau \cos(\theta-\phi)} e^{\frac{\beta h}{2}(\cos(\theta)+\cos(\phi))}.$$

Then the integral operator $\hat{T} = L_f$ plays the same role as the transfer matrix above. Since f is in $L^2([0,2\pi]^2)$, $\hat{T}$ is trace-class and we have

$$Z = \int_{[0,2\pi]^N} \left(\prod_{j=1}^N d\theta_j \right) f(\theta_1,\theta_2) \dots f(\theta_N,\theta_1) = \operatorname{Tr} \hat{T}^N.$$

Again, our new goal is to write $\hat{T} = I - \tau \hat{H}$ when τ is small for some Hamiltonian on the one site Hilbert space $L^2([0,2\pi])$. Recall that the set $B = \{\psi_m := \frac{1}{\sqrt{2\pi}} e^{im\theta} \mid m \in \mathbb{Z}\}$ forms an orthonormal Hilbert basis for $L^2([0,2\pi])$. We will approximate the action of $\hat{T}$ on elements ψ_m of this basis in order to find our approximate Hamiltonian $\hat{H}$.

$$\begin{aligned}(\hat{T}\psi_m)(\theta) &= \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} e^{\beta_\tau \cos(\theta-\phi)} e^{\frac{\beta h}{2}(\cos(\theta)+\cos(\phi))} e^{im\phi}\, d\phi \\ &= \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}\beta h \cos(\theta)} \int_0^{2\pi} e^{\beta_\tau \cos(\theta-\phi)} e^{\frac{1}{2}\beta h \cos(\phi)} e^{im\phi}\, d\phi. \qquad (2)\end{aligned}$$

We can use a Fourier transform to rewrite the $\cos(\theta-\phi)$ portion of the exponential in Eq. (2).

$$e^{-\beta_\tau + \beta_\tau \cos(\theta-\phi)} = \sum_{\ell \in \mathbb{Z}} e^{i\ell(\theta-\phi)} I_\ell(\beta_\tau), \qquad (3)$$

where $I_\ell(\beta)$ is the Bessel function of imaginary argument. In particular, we will let $\beta_\tau = \tau^{-1}$ so that β_τ is large when τ goes to 0. Then we can approximate $I_\ell(\beta_\tau)$ by the Gaussian $e^{-\ell^2/2\beta_\tau}$ and Eq. (3) becomes

$$\begin{aligned} e^{\beta_\tau \cos(\theta-\phi)} &\approx e^{\beta_\tau} \sum_{\ell\in\mathbb{Z}} e^{i\ell(\theta-\phi)} e^{-\ell^2/2\beta_\tau} \qquad (4) \\ &= e^{1/\tau} \sum_{\ell\in\mathbb{Z}} e^{i\ell(\theta-\phi)} e^{-\tau\ell^2/2}. \qquad (5)\end{aligned}$$

After substituting Eq. (5) into Eq. (2), we can use absolute convergence of the sum to rewrite the expression further.

$$\begin{aligned}(\hat{T}\psi_m)(\theta) &\approx \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}\beta h \cos(\theta)} \int_0^{2\pi} e^{1/\tau} \left(\sum_{\ell\in\mathbb{Z}} e^{i\ell(\theta-\phi)} e^{-\tau\ell^2/2} \right) e^{\frac{1}{2}\beta h \cos(\phi)} e^{im\phi}\, d\phi \\ &= \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}\beta h \cos(\theta) + 1/\tau} \sum_{\ell\in\mathbb{Z}} e^{i\ell\theta - \tau\ell^2/2} \left(\int_0^{2\pi} e^{\frac{1}{2}\beta h\cos(\phi)} e^{i(m-\ell)\phi} \right) d\phi. \qquad (6)\end{aligned}$$

At this point, we want $\beta\beta_\tau$ to remain finite as τ goes to 0, so $\beta = \lambda\tau$ for some constant λ. Thus $\frac{1}{2}\beta h \cos(\phi)$ goes to 0 as τ goes to 0 and we can expand the corresponding exponential in

Eq. (6). We then use orthogonality to evaluate the resulting integral.

$$\begin{aligned}
(\hat{T}\psi_m)(\theta) &\approx \frac{1}{\sqrt{2\pi}} e^{\frac{\lambda\tau h}{2}\cos(\theta)+1/\tau} \sum_{\ell\in\mathbb{Z}} e^{i\ell\theta-\tau\ell^2/2} \left(\int_0^{2\pi} (1+\left(\frac{\lambda\tau h}{2}\cos(\phi)\right) e^{i(m-\ell)\phi} \right) d\phi, \\
&= \frac{1}{\sqrt{2\pi}} e^{\frac{\lambda\tau h}{2}\cos(\theta)+1/\tau} \sum_{\ell\in\mathbb{Z}} e^{i\ell\theta-\tau\ell^2/2} \left(2\pi\delta_{\ell,m} + \frac{\lambda\tau h}{2}\int_0^{2\pi} \left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right) e^{i(m-\ell)\phi} \right) d\phi, \\
&= \frac{1}{\sqrt{2\pi}} e^{\frac{\lambda\tau h}{2}\cos(\theta)+1/\tau} \sum_{\ell\in\mathbb{Z}} 2\pi e^{i\ell\theta-\tau\ell^2/2} \left(\delta_{\ell,m} + \frac{\lambda\tau h}{4}(\delta_{m+1,\ell}+\delta_{m-1,\ell}) \right), \\
&= \sqrt{2\pi} e^{\frac{\lambda\tau h}{2}\cos(\theta)+1/\tau} (e^{im\theta-\tau m^2/2} + \frac{\lambda\tau h}{4}(e^{i(m+1)\theta-\tau(m+1)^2/2} + e^{i(m-1)\theta-\tau(m-1)^2/2})).
\end{aligned}$$

At this point, we need to rescale the problem to drop the factor of $e^{\frac{1}{\tau}}$, as this term goes to infinity as τ goes to 0. After we do this, we will write the action of $\hat{T}$ on ψ_m in terms of the operators J_z and $J_\pm$ defined on our basis and extended linearly by

$$J_z\psi_m = m\cdot\psi_m, \quad J_\pm\psi_m = \psi_{m\pm1}.$$

To do this we use the fact that τ is small to expand the remaining exponentials that do not depend on θ and drop all terms involving a power of τ greater than one.

$$\begin{aligned}
(\hat{T}\psi_m)(\theta) &\approx \sqrt{2\pi}\left(1+\frac{1}{2}\lambda\tau h\cos(\theta)\right)\left(\left(1-\frac{1}{2}\tau m^2\right)e^{im\theta} + \frac{\lambda\tau h}{4}\left(e^{i(m+1)\theta}+e^{i(m-1)\theta}\right)\right), \\
&\approx \sqrt{2\pi}\left(\left(1-\frac{1}{2}\tau m^2+\frac{1}{2}\lambda\tau h\cos(\theta)\right)e^{im\theta} + \frac{\lambda\tau h}{4}\left(e^{i(m+1)\theta}+e^{i(m-1)\theta}\right)\right), \\
&= \sqrt{2\pi}\left(\left(1-\frac{1}{2}\tau m^2\right)e^{im\theta} + \frac{\lambda\tau h}{2}\left(e^{i(m+1)\theta}+e^{i(m-1)\theta}\right)\right), \\
&= \left(2\pi\left(I-\tau\left(\frac{J_z^2}{2}-\frac{\lambda h}{2}(J_+ + J_-)\right)\right)\psi_m\right)(\theta). \qquad (7)
\end{aligned}$$

From Eq. (7) we see that the desired Hamiltonian is

$$\hat{H} = 2\pi\left(\frac{1}{2}J_z^2 - \frac{\lambda h}{2}(J_+ + J_-)\right). \qquad (8)$$

3.2. O(2) model on a 2-dimensional lattice. We can perform a similar analysis on a 2-dimensional $M\times N$ lattice where there are M spatial rows and N temporal columns. For simplicity, we will drop the magnetic field term and consider only the interactions between neighboring spins. We will fix all interaction constants in the temporal direction (with unit vector $\vec{\tau}$ and temporal lattice spacing τ) to be some $\beta_\tau(\tau)$. The interaction constants in the spatial direction (with unit vector $\vec{x}$) will be $\beta(\tau)$. For brevity we will omit the explicit dependence on τ.

The action then becomes

$$S = -\beta_\tau \sum_\ell \cos(\theta_\ell - \theta_{\ell+\vec{\tau}}) - \beta\sum_\ell \cos(\theta_\ell - \theta_{\ell+\vec{x}}).$$

It will be helpful later to adjust the action by a constant factor of $MN\beta_\tau - \frac{MN}{2}\log(2\pi\beta_\tau)$. We also write θ_j^k for the angle in row k and column j of the lattice and rewrite the second term to be symmetric between adjacent rows. The resulting action is

$$S = \sum_{k=1}^{M} L(\vec{\theta^k}, \vec{\theta}^{k+1}),$$

where

$$L(\vec{\theta},\vec{\phi}) = -\frac{N}{2}\log(2\pi\beta_\tau) - \beta_\tau\sum_{j=1}^{N}(-1+\cos(\theta_j-\phi_j)) - \frac{\beta}{2}\sum_{j=1}^{N}\left(\cos(\theta_j-\theta_{j+1})+\cos(\phi_j-\phi_{j+1})\right)$$

Then

$$Z = \int d\vec{\theta}^1 \dots d\vec{\theta}^M e^{-L(\vec{\theta}^1,\vec{\theta}^2)} \dots e^{-L(\vec{\theta}^N,\vec{\theta}^1)} = \mathrm{Tr}\left(\hat{T}^M\right),$$

where $\hat{T}$ is the integral operator on the Hilbert space $L^2([0,2\pi]^N) \cong L^2([0,2\pi])^{\otimes N}$ defined by

$$(\hat{T}f)(\vec{\theta}) = \int_{[0,2\pi]^N} e^{-L(\vec{\theta},\vec{\phi})} f(\vec{\phi})\, d\vec{\phi}.$$

As in the previous section, we will analyze the action of $\hat{T}$ on elements of the orthornomal Hilbert basis

$$B = \left\{ \psi_{\vec{m}}(\vec{\theta}) := \frac{1}{(2\pi)^{N/2}} \prod_{j=1}^{N} e^{im_j\theta_j} \mid \vec{m} \in \mathbb{Z}^N \right\}$$

of $L^2([0,2\pi]^N)$. This will go very similarly and we will need the following approximations for small τ and $m \in \mathbb{Z}$. We will also take $\beta_\tau = 1/\tau$ and $\beta = \lambda\tau$ for some constant λ.

$$\int_0^{2\pi} e^{\frac{1}{\tau}(-1+\cos(\theta-\phi))} e^{im\phi}\, d\phi \approx \frac{\sqrt{\tau}}{\sqrt{2\pi}} \int_0^{2\pi} \sum_{\ell\in\mathbb{Z}} e^{i\ell(\theta-\phi)-\ell^2\tau/2+im\phi}\, d\phi = \sqrt{2\pi\tau}e^{-m^2\tau/2}e^{im\theta}. \quad (9)$$

$$e^{\frac{1}{2}\lambda\tau\sum_{j=1}^N \cos(\phi_j-\phi_{j+1})} \approx 1 + \frac{\lambda\tau}{4}\sum_{j=1}^{N}(e^{i\phi_j}e^{-i\phi_{j+1}} + e^{-i\phi_j}e^{i\phi_{j+1}}) \quad (10)$$

We will write our Hamiltonian in terms of the operators $J_z(j), J_+(j)$, and $J_-(j)$ defined on the basis $\{\psi_{\vec{m}}\}$ by

$$\begin{aligned}(J_z(j)\psi_{\vec{m}})(\vec{\theta}) &:= m_j\psi_{\vec{m}},\\ (J_\pm(j)\psi_{\vec{m}})(\vec{\theta}) &:= e^{\pm i\theta_j}\psi_{\vec{m}}.\end{aligned}$$

Now

$$\begin{aligned}(\hat{T}\psi_{\vec{m}})(\vec{\theta}) &= \frac{1}{(2\pi)^{N/2}}\int_{[0,2\pi]^N} e^{-L(\vec{\theta},\vec{\phi})} e^{\sum_{j=1}^N im_j\phi_j}\, d\vec{\phi}.\\ &= F(\vec{\theta})\int e^{\frac{1}{\tau}\sum_{j=1}^N(-1+\cos(\theta_j-\phi_j)+im_j\phi_j)+\frac{\lambda\tau}{2}\sum_{j=1}^N\cos(\phi_j-\phi_{j+1})}\, d\vec{\phi},\end{aligned}$$

where $F(\vec{\theta}) = \left(\frac{1}{\tau}\right)^{N/2} e^{\frac{\lambda\tau}{2}\sum_{j=1}^{N}\cos(\theta_j-\theta_{j+1})}$. We can now use Eq. (9) and Eq. (10). Whenever we take an expansion in powers of τ we will neglect any terms with order at least 2.

$$(\hat{T}\psi_{\vec{m}})(\vec{\theta}) \approx F(\vec{\theta}) \int e^{\frac{1}{\tau}\sum_{j=1}^{N}(-1+\cos(\theta_j-\phi_j)+im_j\phi_j)} \left(1+\frac{\lambda\tau}{4}\sum_{j=1}^{N}(e^{i\phi_j}e^{-i\phi_{j+1}}+e^{-i\phi_j}e^{i\phi_{j+1}})\right) d\vec{\phi},$$

$$\approx F(\vec{\theta}) \left(\prod_{j=1}^{N}\sqrt{2\pi\tau}e^{-m_j^2\tau/2}e^{im_j\theta_j} + \sum_{\substack{1\le k,\ell\le N\\ k\neq\ell}}\prod_{j=1}^{N}\sqrt{2\pi\tau}e^{i(m_j+\delta_{k,j}-\delta_{\ell,j})\theta_j}e^{-\tau(m_j+\delta_{k,j}-\delta_{\ell,j})^2/2}\right),$$

$$\approx F(\vec{\theta}) \left(\prod_{j=1}^{N}\sqrt{2\pi\tau}(1-m_j^2\tau/2)e^{im_j\theta_j} + \sum_{k\neq\ell}\prod_{j=1}^{N}\sqrt{2\pi\tau}(1-\tau(m_j+\delta_{k,j}-\delta_{\ell,j})^2/2)e^{i(m_j+\delta_{k,j}-\delta_{\ell,j})\theta_j}\right),$$

$$\approx F(\vec{\theta})(2\pi\sqrt{\tau})^N \left(\left(I-\frac{\tau}{2}\sum_{j=1}^{N}J_z^2(j)+\frac{\lambda\tau}{4}\sum_{j=1}^{N}(J_+(j)J_-(j+1)+J_-(j)J_+(j+1))\right)\right)\psi_{\vec{m}}(\vec{\theta}),$$

$$\approx (2\pi)^N\left(I-\frac{\tau}{2}\sum_{j=1}^{N}J_z^2(j)-\lambda(J_+(j)J_-(j+1)+J_-(j)J_+(j+1))\right)\psi_{\vec{m}}(\vec{\theta}).$$

It now follows that the desired Hamiltonian is

$$\hat{H} := (2\pi)^N\frac{1}{2}\sum_{j=1}^{N}J_z^2(j)-\lambda(J_+(j)J_-(j+1)+J_-(j)J_+(j+1)).$$

4. Spherical Model

Detailed analysis of the spherical model in various dimensions can be found in [1],[5],[10].

5. Lattice Gauge Theories

References

[1] E. Fradkin and L. Susskind. Order and disorder in gauge systems and magnets. *Phys. Rev. D*, 17:2637–2658, May 1978.

[2] S. Friedli and Y. Velenik. *Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction.* Cambridge University Press, 2017.

[3] B. C. Hall. *Quantum Theory for Mathematicians.* Springer-Verlag New York, 2013.

[4] C. J. Hamer, J. B. Kogut, and L. Susskind. Strong-coupling expansions and phase diagrams for the O(2), O(3), and O(4) Heisenberg spin systems in two dimensions. *Phys. Rev. D*, 19:3091–3105, May 1979.

[5] M. Henkel and C. Hoeger. Hamiltonian formulation of the spherical model in d=r+1 dimensions. *Zeitschrift für Physik B Condensed Matter*, 55(1):67–73, Mar 1984.

[6] J. B. Kogut. An introduction to lattice gauge theory and spin systems. *Rev. Mod. Phys.*, 51:659–713, Oct 1979.

[7] M. Mueller, D. A. Johnston, and W. Janke. Exact solutions to plaquette ising models with free and periodic boundaries. *Nuclear Physics B*, 914:388 – 404, 2017.

[8] M. A. Nielsen. The Fermionic canonical commutation relations and the Jordan-Wigner transform. `http://michaelnielsen.org/blog/archive/notes/fermions_and_jordan_wigner.pdf`, July 2005.

[9] T. D. Schultz, D. C. Mattis, and E. H. Lieb. Two-dimensional ising model as a soluble problem of many fermions. *Rev. Mod. Phys.*, 36:856–871, Jul 1964.

[10] C. J. Thompson. Spherical model as an instance of eigenvalue degeneracy. *Journal of Mathematical Physics*, 9(7):1059–1062, 1968.