NOTES FROM NSF-MSGI INTERNSHIP

HUNTER LEHMANN

1. Introduction

We will explore the connection between statistical mechanics models and quantum Hamiltonians, primarily via examples.

More stuff here later. Basic objects/definitions.

2. Ising Models

The Ising models are a family of statistical mechanics models with nearest-neighbor interaction. Given any lattice $\mathcal{L} \subset \mathbb{Z}^d$ with N total points, define a *spin* at each site of the lattice via a variable $s \in \Omega_0 = \{\pm 1\}$. A *configuration* is a collection $\{s\} = \{s_\ell\}_{\ell \in \mathcal{L}}$. For convenience, we will usually work with cubical lattices and impose periodic boundary conditions. The *action* associated to this model is the function

$$\mathcal{S}(\{s\}, \tau) = -\beta_\tau(\tau) \sum_{i \sim j} s_i s_j - \beta(\tau) h \sum_i s_i,$$

where τ is the lattice spacing in the "time" direction, h is the strength of an external magnetic field, and the first sum is over nearest neighbor sites of the lattice. The coefficients β_τ and β may depend on the time direction lattice spacing and are otherwise chosen to reflect some physical situation (e.g. they may be taken to be proportional to the "inverse temperature" $\frac{1}{kT}$ where k is Boltzmann's constant).

We will be interested in the *partition function* Z defined as

$$Z = \sum_{\{s\} \in \Omega_0^N} e^{-\mathcal{S}(\{s\}, \tau)}.$$

2.1. 1+0 Ising. In the 1+0 Ising model, we take a 1-dimensional statistical mechanics system and relate it to a 0-dimensional (point) quantum Hamiltonian. The definitions above simplify to the following. The lattice is now a set of N points $x_0, \ldots, x_{N-1}$ on a circle so that $s_0 = s_N$. The action is given by

$$\mathcal{S}(\{s\}, \tau) = -\beta_\tau(\tau) \sum_{i=0}^{N-1} s_i s_{i+1} - \beta(\tau) h \sum_{i=0}^{N-1} s_i.$$

It will be helpful to rewrite this expression to be symmetric and expressed in terms of sums and differences of the s_i. Up to a constant factor of $-\beta_\tau \sum_{i=0}^{N-1} s_i$, we find

$$\mathcal{S}(\{s\}, \tau) = \frac{1}{2}\beta_\tau(\tau) \sum_{i=0}^{N-1} (s_i - s_{i+1})^2 - \frac{1}{2}\beta(\tau) h \sum_{i=0}^{N-1} (s_i + s_{i+1}).$$

Note that we needed periodicity in order to rewrite the second term in this way. Now the partition function becomes

$$Z = \sum_{\{s\}\in\Omega_0^N} e^{\frac{1}{2}\beta_\tau(\tau)\sum_{i=0}^{N-1}(s_i-s_{i+1})^2-\frac{1}{2}\beta(\tau)h\sum_{i=0}^{N-1}(s_i+s_{i+1})}$$
$$= \sum_{s_0\in\{\pm1\}}\cdots\sum_{s_{N-1}\in\{\pm1\}} e^{\frac{1}{2}\beta_\tau(s_0-s_1)^2}e^{-\frac{1}{2}\beta h(s_0+s_1)}\cdot\ldots\cdot e^{\frac{1}{2}\beta_\tau(s_{N-1}-s_0)^2}e^{-\frac{1}{2}\beta h(s_{N-1}+s_0)}.$$

However, each factor in the product depends only on what the values of $s_i - s_{i+1}$ and s_i+s_{i+1} are. We can construct the *transfer matrix*, T, to record this data, indexed by the possible spins at each site. So we have

$$T_{-1,-1} = e^{-\beta h}, \quad T_{-1,1} = T_{1,-1} = e^{-2\beta_\tau}, \quad \text{and } T_{1,1} = e^{\beta h}.$$

Thus

$$T = \begin{bmatrix} e^{-\beta h} & e^{-2\beta_\tau} \\ e^{-2\beta_\tau} & e^{\beta h} \end{bmatrix}.$$

Now we can write $Z = \operatorname{Tr} T^N$.

Our goal is to choose functions $\beta_\tau(\tau)$ and $\beta(\tau)$ so that the transfer matrix operator T has the form $T = e^{-\tau H} \approx I - \tau H$ for some quantum Hamiltonian H (independent of τ) acting on a 2-dimensional vector space. This is the *τ-continuum Hamiltonian* for the model. Here the idea is that statistical mechanics properties of the lattice action system (e.g. magnetization per site, average magnetization, two-point correlations, correlation length) will map to properties of the operator H related to its eigenvalues and eigenvectors.

For example, we could choose $\beta_\tau(\tau) = -\frac{1}{2}\log\tau$ and $\beta(\tau) = \tau$. Then we have

$$T = \begin{bmatrix} e^{-\tau h} & \tau \\ \tau & e^{\tau h} \end{bmatrix} \approx I_2 - \tau \begin{bmatrix} h & -1 \\ -1 & -h \end{bmatrix}.$$

From this it follows that

$$H = \begin{bmatrix} h & -1 \\ -1 & -h \end{bmatrix} = -\sigma_1 + h\sigma_3,$$

where σ_1, σ_2, and σ_3 are the Pauli matrices.

More generally, we can ask what constraints β and β_τ must satisfy to be able to do this. Analyzing each of the four matrix entries shows that for small τ we need

$$e^{-\beta h} \approx 1 - \tau H_{0,0}, \quad e^{-2\beta_\tau} \approx -\tau H_{1,0} = -\tau H_{0,1}, \quad \text{and } e^{\beta h} \approx 1 - \tau H_{1,1}.$$

Since $H_{1,0}$ and $H_{0,1}$ are constant with respect to τ we need $-2\beta_\tau \approx \log(-\lambda\tau)$ for small τ and some $\lambda \in \mathbb{R}_{>0}$. Similarly, if β is small when τ is small we have $e^{-\beta h} \approx 1 - \beta h \approx 1 - \tau H_{0,0}$ and $e^{\beta h} \approx 1 + \beta h \approx 1 - \tau H_{1,1}$. So to first order we have $\beta \approx \tau$ and we see $H_{0,0} = -H_{1,1} = \mu h$ for some $\mu \in \mathbb{R} \setminus \{0\}$.

Putting all of this together, we see that the general τ-continuum Hamiltonian for this model will have the form

$$H = \begin{bmatrix} \mu h & -\lambda \\ -\lambda & -\mu h \end{bmatrix} = -\lambda\sigma_1 + \mu h\sigma_3.$$

Now we could also attempt to find a second order approximation of T so that $T \approx I - \tau H + \frac{1}{2}\tau^2 H^2$ for an operator H not depending on τ. However, following the approach above will show us that in fact this is not possible. To get a higher order approximation (or the actual matrix logarithm solving $T = e^{-\tau H}$ for H), we need to allow H to depend on τ.

2.2. Infinite Lattice Ising Models. The Ising model can be generalized to an *infinite volume model* by letting the lattice $\mathcal{L}$ grow to $\mathbb{Z}^d$ appropriately. We'll briefly sketch the ideas here - a detailed exposition can be found in [1] in chapters 3 and 6.

References

[1] S. Friedli and Y. Velenik. *Statistical Mechanics of Lattice Systems: A Concrete Mathematical Introduction*. Cambridge University Press, 2017.