1)

$ax + by - d = 0$

$a^2 + b^2 = 1$, so vector (a, b) is a unit normal vector to the line.

$$d_{\text{perpendicular}} = \frac{|ax_i + by_i - d|}{\underbrace{\sqrt{a^2+b^2}}_{=1}} = |ax_i + by_i - d|$$

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2)

$$E = \sum_{i=1}^{n} (ax_i + by_i - d)^2$$

$$\frac{\partial E}{\partial d} = \sum_{i=1}^{n} 2(ax_i + by_i - d)(-1)$$

$$= -2 \sum_{i=1}^{n} (ax_i + by_i - d) = 0$$ // divide by (-2)

$$= \sum_{i=1}^{n} (ax_i + by_i - d) = 0$$

$$= \sum_{i=1}^{n} (ax_i + by_i) - \sum_{i=1}^{n} d = 0$$

$$= \sum_{i=1}^{n} (ax_i + by_i) - n \cdot d = 0$$

$$= d = \frac{1}{n} \sum_{i=1}^{n} (ax_i + by_i)$$

$$\Rightarrow d = a \cdot \vec{x} + b\,\vec{y}, \quad \text{where} \quad \left(\vec{x} = \frac{1}{n} \sum_{i=1}^{n} x_i, \quad \vec{y} = \frac{1}{n} \sum_{i=1}^{n} y_i \right)$$

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3)

$$E = \sum_{i=1}^{n} (ax_i + by_i - \underline{\underline{d}})^2$$

$$d = a \cdot \vec{x} + b \cdot \vec{y} \Rightarrow E = \sum_{i=1}^{n} (ax_i + by_i - a\vec{x} - b\vec{y})^2$$

$$\Rightarrow E = \sum_{i=1}^{n} [a(x_i - \vec{x}) + b(y_i - \vec{y})]^2$$

$$= \left\| \underbrace{\begin{bmatrix} x_1 - \vec{x} & y_1 - \vec{y} \\ \vdots & \\ x_n - \vec{x} & y_n - \vec{y} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{v}} \right\|^2$$

$$E = \sum_{i=1}^{n} \left[[a\ b] \begin{bmatrix} x_i - \vec{x} \\ y_i - \vec{y} \end{bmatrix} \right]^2$$

$$E = \sum_{i=1}^{n} (v^T U_i)^2$$

$$= v^T U^T U v \quad = (a\,b) U^T U (a\,b)^T$$

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4)

$a^2+b^2=1$, $(a\ b)$ is unit vector.

$$\| U (a\ b)^T \|^2 = (a\,b)\, U^T U\, (a\ b)^T$$

It can be solved as an eigenvalue problem

$$U^T U v = \lambda v$$

The solution for $(a\ b)^T$ is the eigenvector corresponding to the smallest eigenvalue of $U^T U$

we can compute d using the formula from stage two.

$$\Rightarrow d = a \cdot \vec{x} + b \cdot \vec{y}$$