Exercise 1)

$\boxed{\text{part a.}}$

$$E = \sum_{i=1}^{n} \| x_i' - Mx_i - t \|^2$$

$$= \sum_{i=1}^{n} (x_i' - Mx_i - t)^T (x_i' - Mx_i - t)$$

$$\frac{\partial E}{\partial M} = 2\sum_{i=1}^{n} (x_i' - Mx_i - t) x_i^T$$

$$\frac{\partial E}{\partial t} = 2\sum_{i=1}^{n} (x_i' - Mx_i - t)$$

$\boxed{\text{part b}}$

$$\frac{\partial E}{\partial M} = 2\sum_{i=1}^{n} (x_i' - Mx_i - t) x_i^T = 0$$

$$\Rightarrow \sum_{i=1}^{n} (x_i' - Mx_i - t) x_i^T = 0$$

$$\frac{\partial E}{\partial t} = 2\sum_{i=1}^{n} (x_i' - Mx_i - t) = 0$$

$$\Rightarrow \sum_{i=1}^{n} (x_i' - Mx_i - t) = 0$$

Let h be a vector that includes the unknown parameter M and t.

$$h = \begin{bmatrix} a \\ b \\ c \\ d \\ t_x \\ t_y \end{bmatrix}, \quad Sh = u, \text{ where } S \text{ is } 6\times6 \text{ matrix}$$

Coordinates of the original points x_i

u is 6×1 matrix

↳ transformed points x_i'

Part C

$x_1 = (0,0) \rightarrow x'_1 = (1,2)$

$x_2 = (1,0) \rightarrow x'_2 = (3,2)$

$x_3 = (0,1) \rightarrow x'_3 = (1,4)$

$x'_i = Mx_i + t$, where $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $t = \begin{bmatrix} t_x \\ t_y \end{bmatrix}$

x_1 : $a \cdot 0 + b \cdot 0 + t_x = 1 \Rightarrow t_x = 1$

$c \cdot 0 + d \cdot 0 + t_y = 2 \Rightarrow t_y = 2$

x_2 : $a \cdot 1 + b \cdot 0 + t_x = 3 \Rightarrow a + 1 = 3 \Rightarrow a = 2$

$c \cdot 1 + d \cdot 0 + t_y = 2 \Rightarrow c + 2 = 2 \Rightarrow c = 0$

x_3 : $a \cdot 0 + b \cdot 1 + t_x = 1 \Rightarrow b + 1 = 1 \Rightarrow b = 0$

$c \cdot 0 + d \cdot 1 + t_y = 4 \Rightarrow d + 2 = 4 \Rightarrow d = 2$

$$M = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad t = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\therefore \; x'_i = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} x_i + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\Rightarrow$ This one scales the original coordinates by a factor of 2 & translate them by (1,2)

Exercise 2)

Part a

$$x_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}, \quad v = x_2 - x_1 = \begin{bmatrix} x_2 - x_1 \\ y_2 - y_1 \end{bmatrix}$$

$$x'_1 = \begin{bmatrix} x'_1 \\ y'_1 \end{bmatrix}, \quad x'_2 = \begin{bmatrix} x'_2 \\ y'_2 \end{bmatrix}, \quad v' = x'_2 - x'_1 = \begin{bmatrix} x'_2 - x'_1 \\ y'_2 - y'_2 \end{bmatrix}$$

dot product

$$\cos(\theta) = \frac{v' \cdot v}{|v'||v|}$$

$$\sin(\theta) = \frac{v' \times v}{|v'||v|}$$

$$\theta = \operatorname{atan2}(\sin(\theta), \cos(\theta))$$

Part b

$$s = \frac{|v'|}{|v|} = \frac{\sqrt{(x'_2 - x'_1)^2 + (y'_2 - y_1)^2}}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}}$$

Part c

$t = x'_1 - sRx_1$, where x'_1 is transformed point

x_1 is the original point

s is the scale factor

R is the rotation matrix

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$\boxed{\text{Part d}}$

$$\left(\frac{1}{2}, 0\right) \to (0,0) \quad , \quad \left(0, \frac{1}{2}\right) \to (-1, -1)$$

$$x_1 = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} , \quad x_2 = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$$

$$x_1' = \begin{bmatrix} 0 \\ 0 \end{bmatrix} , \quad x_2' = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$V = x_2 - x_1 = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$V' = x_2' - x_1' = \begin{bmatrix} -1 \\ -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$S = \frac{V' \cdot V}{|V'||V|} = \frac{\cancel{(-1)\left(\frac{1}{2}\right)} + \cancel{(-1)\left(\frac{1}{2}\right)}}{\cancel{\sqrt{2}} \cdot \frac{1}{\cancel{\sqrt{2}}}} = 0$$

$$\sin(\theta) = \frac{V' \times V}{|V'||V|} = \frac{0 + (-1)}{\sqrt{2} \cdot \frac{1}{\sqrt{2}}} = -1$$

$$\theta = \frac{3\pi}{2} \text{ radians} = 270^\circ$$

$$t = x_1' - sRx_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - 2 \cdot \begin{bmatrix} \cos\left(\frac{3\pi}{2}\right) & -\sin\left(\frac{3\pi}{2}\right) \\ \sin\left(\frac{3\pi}{2}\right) & \cos\left(\frac{3\pi}{2}\right) \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \end{bmatrix} - 2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore t = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$