Exercise Round 2

1) $$E[x] = \int x\, p(x)\, dx \qquad \int := \int_{-\infty}^{+\infty}$$

$$Var[x] = \int (x - E[x])^2\, p(x)\, dx$$

$$\int_{-\infty}^{+\infty} p(x)\, dx = 1$$

a) $$Var[x] = E[x^2] - E^2[x]$$

$$Var[x] = \int (x - E[x])^2\, p(x)\, dx$$

Expand: $$= \int \left(x^2 - 2x E[x] + E^2[x]\right) p(x)\, dx$$

$$= \int x^2 p(x)\, dx - \int 2x\, E[x]\, p(x)\, dx + \int E^2[x]\, p(x)\, dx$$

$$E[h(x)] = \int h(x)\, p(x)\, dx$$

$E[x]$ and $E^2[x]$ are constant

$$= E[x^2] - 2E[x]\underbrace{\int x\, p(x)\, dx} + E^2[x]\underbrace{\int p(x)\, dx}$$

$$= E[x^2] - 2E^2[x] + E^2[x]$$

$$= E[x^2] - (E[x])^2$$

2) $$\underbrace{P(x)} = N(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right)$$

$$E[x] = \int x \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma}} \exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right) dx$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int x \underbrace{\exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right)} dx \qquad (1)$$

$$\frac{d}{dx}\left(\exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right)\right) = \frac{-2}{2\sigma^2}(x-\mu)\exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right)$$

$$\int (x - \mu + \mu) \exp\left(-1 \ldots (x-\mu)^2\right) dx \qquad (2)$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int (\underbrace{x - \mu} + \mu) \exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right) dx \quad \leftarrow \text{②}$$

Expand $\rightarrow$

$$= \frac{-\sigma^2}{\sqrt{2\pi}\sigma} \int \underbrace{\frac{1}{-\sigma^2}}(x-\mu) \underbrace{\exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right)} dx \quad ③$$

$$+ \frac{1}{\sqrt{2\pi}\sigma} \int \mu \exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right) dx \quad \underbrace{\int p(x)dx = 1}_{④}$$

$$= \frac{-\sigma^2}{\sqrt{2\pi}\sigma} \int \frac{d}{dx}\left(\exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right)\right) dx \quad ⑤$$

$$+ \mu$$

$$= \frac{-\sigma^2}{\sqrt{2\pi}\sigma} \left(\exp\left(\frac{-1}{2\sigma^2}(x-\mu)^2\right)\right) \Bigg|_{-\infty}^{+\infty} + \mu$$

$$= 0 + \mu = \mu$$

2) we have: $z = Lx$, $E[x] = m$, $Cov[x] = P$

show: $E[z] = Lm$, $Cov[z] = LPL^T$

$$E[z] = E[Lx] = \int Lx\, p(x)\, dx = L \underbrace{\int x\, p(x)\, dx}$$

$$= L\, E[x] = Lm$$

$$Cov[z] = Cov[Lx] = E\left[(Lx - E[Lx])(Lx - E[Lx])^T\right]$$

$$= E\left[(Lx - Lm)(Lx - Lm)^T\right]$$

$$= E\left[L(x-m)(x-m)^T L^T\right]$$

$$= L\, \underbrace{E\left[(x-m)(x-m)^T\right]}\, L^T$$

$$= L\, \underbrace{Cov[x]}_{P}\, L^T = LPL^T$$

3) $y_n = g(x_n) + r_n$, $var\{r_n\} = \sigma_n^2$

a)

we define $y_n = \frac{y_n}{\sigma_n}$, $g(x_n) = \frac{g(x_n)}{\sigma_n}$, $\hat{r}_n = \frac{r_n}{\sigma_n}$

$$\Rightarrow \quad \hat{y}_n = \hat{g}(x_n) + \hat{r}_n \qquad var\{\hat{r}_n\} = ?$$

$$var[\hat{r}_n] = var\left[\frac{r_n}{\sigma_n}\right] = \frac{1}{\sigma_n} var[r_n] \frac{1}{\sigma_n} = \frac{\sigma_n^2}{\sigma_n^2} = 1$$

we saw that $\begin{cases} Z = L x, & Cov[x] = P \\ Cor[z] = L P L^T \end{cases}$

b) $y_n = g(x_n) + r_n$, $var[r_n] = \sigma_n^2$ WLS problem

$$J_1 = \sum_n \frac{1}{\sigma_n^2} (y_n - g(x_n))^2$$

$\hat{y}_n = \hat{g}(x_n) + \hat{r}_n$, $var[\hat{r}_n] = 1$ LS problem

$$J = \sum (\hat{y} \; \hat{g}(x_n))^2 = \sum (y_n \; g(x_n))^2$$

$$J_2 = \sum_n \left(y_n - g(x_n)\right) = \sum_n \left(\frac{y_n}{\sigma_n} - \frac{g(x_n)}{\sigma_n}\right)$$

$$= \sum_n \left(\frac{1}{\sigma_n}\right)^2 \left(y_n - g(x_n)\right)^2$$

$$\boxed{J_1 = J_2}$$