Solutions of Basics of Sensor Fusion Exercise Round 2

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September 2024

Exercise 1

(a)

We get by straightforward calcuation:

$$
\begin{aligned}
\operatorname{var}\{x\} &= \int (x - \mathrm{E}\{x\})^2 p(x) dx \\
&= \int (x^2 - 2x\mathrm{E}\{x\} + \mathrm{E}\{x\}^2) p(x) dx \\
&= \underbrace{\int x^2 p(x) dx}_{\mathrm{E}\{x^2\}} - 2\mathrm{E}\{x\} \underbrace{\int x p(x) dx}_{\mathrm{E}\{x\}} + \mathrm{E}\{x\}^2 \underbrace{\int p(x) dx}_{1} \\
&= \mathrm{E}\{x^2\} - 2\mathrm{E}\{x\}\mathrm{E}\{x\} + \mathrm{E}\{x\}^2 \\
&= \mathrm{E}\{x^2\} - \mathrm{E}\{x\}^2
\end{aligned}
\tag{1}
$$

(b)

Gaussian distribution is

$$
p(x) = (2\pi\sigma^2)^{-1/2} \exp\left(-\frac{1}{2\sigma^2}(x - \mu)^2\right) \tag{2}
$$

For the expected value we get

$$
\begin{aligned}
\int_{-\infty}^{\infty} xp(x)dx &= (2\pi\sigma^2)^{-1/2} \int x \, \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx \\
&= (2\pi\sigma^2)^{-1/2} \int (x-\mu+\mu) \, \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx \\
&= (2\pi\sigma^2)^{-1/2} \int (x-\mu) \, \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx \\
&+ (2\pi\sigma^2)^{-1/2} \int \mu \, \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx \\
&= -(2\pi\sigma^2)^{-1/2}\sigma^2 \int [-(x-\mu)/\sigma^2] \, \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx \\
&+ \mu \underbrace{\int (2\pi\sigma^2)^{-1/2} \, \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right) dx}_{=1} \\
&= -(2\pi\sigma^2)^{-1/2}\sigma^2 \underbrace{\exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)\Bigg|_{-\infty}^{\infty}}_{0} \\
&+ \mu \\
&= \mu.
\end{aligned} \tag{3}
$$

Another way: you could also use change of variable technique as: $z = \frac{x-\mu}{\sigma}$. Then we have $x = \sigma\, z + \mu$ and $dx = \sigma\, dz$. By substitution, we will get the same result.

Exercise 2

Given $\mathrm{E}\{\mathbf{x}\} = \mathbf{m}$ (there is a typo in exercise paper) and $\mathrm{Cov}\{\mathbf{x}\} = \mathbf{P}$. Then

$$
\begin{aligned}
\mathrm{E}\{\mathbf{z}\} &= \int \mathbf{L}\,\mathbf{x}\, p(\mathbf{x})\, d\mathbf{x} \\
&= \mathbf{L} \int \mathbf{x}\, p(\mathbf{x})\, d\mathbf{x} \\
&= \mathbf{L}\, \mathrm{E}\{\mathbf{x}\} = \mathbf{L}\,\mathbf{m},
\end{aligned} \tag{4}
$$

and

$$\begin{aligned}
\mathrm{Cov}\{\mathbf{z}\} &= \mathrm{E}\{(\mathbf{z} - \mathrm{E}\{\mathbf{z}\})\,(\mathbf{z} - \mathrm{E}\{\mathbf{z}\})^\top\} \\
&= \mathrm{E}\{(\mathbf{z} - \mathbf{L}\,\mathbf{m})\,(\mathbf{z} - \mathbf{L}\,\mathbf{m})^\top\} \\
&= \mathrm{E}\{(\mathbf{L}\,\mathbf{x} - \mathbf{L}\,\mathbf{m})\,(\mathbf{L}\,\mathbf{x} - \mathbf{L}\,\mathbf{m})^\top\} \\
&= \mathrm{E}\{\mathbf{L}\,(\mathbf{x} - \mathbf{m})\,(\mathbf{x} - \mathbf{m})^\top \mathbf{L}^\top\} \\
&= \mathbf{L}\,\mathrm{E}\{(\mathbf{x} - \mathbf{m})\,(\mathbf{x} - \mathbf{m})^\top\}\mathbf{L}^\top \\
&= \mathbf{L}\,\mathbf{P}\,\mathbf{L}^\top
\end{aligned} \tag{5}$$

Exercise 3

(a)

Original model

$$y_n = g(x_n) + r_n, \quad \mathrm{E}\{r_n\} = 0, \mathrm{var}\{r_n\} = \sigma_n^2 \tag{6}$$

We have

$$\hat{y}_n = \hat{g}(x_n) + \hat{r}_n, \quad \mathrm{E}\{\hat{r}_n\} = 0, \mathrm{var}\{\hat{r}_n\} = 1 \tag{7}$$

For the variance we get

$$\begin{aligned}
\mathrm{var}\{\hat{r}_n\} &= \mathrm{var}\{r_n/\sigma_n\} = (1/\sigma_n)\mathrm{var}\{r_n\}(1/\sigma_n) \\
&= (1/\sigma_n)\sigma_n^2(1/\sigma_n) = 1
\end{aligned} \tag{8}$$

(b)

Weighted least squares problem is (weights are $w_n = 1/\sigma_n^2$):

$$\sum_n \frac{1}{\sigma_n^2}(y_n - g(x_n))^2 \tag{9}$$

For the modified problem we get

$$\begin{aligned}
\sum_n (\hat{y}_n - \hat{g}(x_n))^2 &= \sum_n (y_n/\sigma_n - g(x_n)/\sigma_n)^2 \\
&= \sum_n \frac{1}{\sigma_n^2}(y_n - g(x_n))^2
\end{aligned} \tag{10}$$