Solutions of Basics of Sensor Fusion Exercise Round 7

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Exercise 1

(a)

Our differential equation is

$$\dot{x}(t) = \lambda\, x(t)\,(1 - x(t)) \tag{1}$$

given that $x(0) = x_0$. We can rewrite the ODE as

$$f(\dot{x}, x) = 0 \tag{2}$$

where

$$f(\dot{x}, x) = \dot{x} - \lambda\, x\,(1 - x). \tag{3}$$

Function $g : \mathbb{R}^n \to \mathbb{R}$ is linear if it satisfies (we have $c \in \mathbb{R}$ and $\mathbf{y}, \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{R}^n$)

$$\begin{aligned} g(\mathbf{y}_1 + \mathbf{y}_2) &= g(\mathbf{y}_1) + g(\mathbf{y}_2), \\ g(c\,\mathbf{y}) &= c\, g(\mathbf{y}). \end{aligned} \tag{4}$$

If we show that at least one of the equations in (4) is not satisfied then we can conclude that the function is nonlinear.

Thus we have $\mathbf{y} = [\dot{x}\ x]^\top$. We get

$$\begin{aligned} f(\dot{x}_1 + \dot{x}_2, x_1 + x_2) &= \dot{x}_1 + \dot{x}_2 - \lambda\,(x_1 + x_2)\,(1 - x_1 - x_2) \\ &= \dot{x}_1 + \dot{x}_2 - \lambda\,[(x_1 + x_2) - (x_1 + x_2)^2] \\ &= \dot{x}_1 + \dot{x}_2 - \lambda\,[x_1 + x_2 - x_1^2 - x_2^2 - 2x_1x_2] \end{aligned} \tag{5}$$

But then

$$\begin{aligned} f(\dot{x}_1, x_1) + f(\dot{x}_2, x_2) &= \dot{x}_1 - \lambda\, x_1\,(1 - x_1) + \dot{x}_2 - \lambda\, x_2\,(1 - x_2) \\ &= \dot{x}_1 + \dot{x}_2 - \lambda\,[x_1\,(1 - x_1) + x_2\,(1 - x_2)] \\ &= \dot{x}_1 + \dot{x}_2 - \lambda\,[x_1 + x_2 - x_1^2 - x_2^2] \\ &\neq f(\dot{x}_1 + \dot{x}_2, x_1 + x_2) \end{aligned} \tag{6}$$

which shows that f and hence the ODE is not linear.

(b)

Our differential equation is

$$\frac{dx}{dt} = \lambda\, x\,(1-x) \tag{7}$$

So we proceed

$$\begin{aligned}
\frac{dx}{x\,(1-x)} &= \lambda\, dt, \qquad \int \cdot \\
\int_{x_0}^{x} \frac{dx}{x\,(1-x)} &= \int_0^t \lambda\, dt, \\
\int_{x_0}^{x} \frac{dx}{x\,(1-x)} &= \lambda\, t, \\
\int_{x_0}^{x} \frac{dx}{x} + \frac{dx}{(1-x)} &= \lambda\, t, \\
[\ln x - \ln(1-x)]|_{x_0}^{x} &= \lambda\, t, \\
\ln \frac{x}{1-x} - \ln \frac{x_0}{1-x_0} &= \lambda\, t, \\
\ln \frac{x}{1-x} = \ln \frac{x_0}{1-x_0} & + \lambda\, t, \\
\frac{x}{1-x} = \exp(\ln \frac{x_0}{1-x_0}) & \exp(\lambda\, t), \\
\frac{x}{1-x} = \frac{x_0}{1-x_0} & \exp(\lambda\, t),
\end{aligned} \tag{8}$$

After simplification we get

$$x = \frac{x_0 \exp(\lambda t)}{x_0 \exp(\lambda t) + 1 - x_0} \tag{9}$$

Exercise 3

Canonical form is

$$\begin{gathered}
\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} u_1 \cos(x_3) \\ u_1 \sin(x_3) \\ u_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \\
\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}) + \mathbf{B}\,\mathbf{w}
\end{gathered} \tag{10}$$

where

$$\begin{gathered}
\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} p_x \\ p_y \\ \varphi \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} v \\ \omega_{gyro} \end{bmatrix}, \quad \mathbf{f}(\mathbf{x}, \mathbf{u}) = \begin{bmatrix} u_1 \cos(x_3) \\ u_1 \sin(x_3) \\ u_2 \end{bmatrix} \\
\mathbf{B} = \mathbf{I}, \quad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}
\end{gathered} \tag{11}$$