Solutions of Basics of Sensor Fusion Exercise Round 5

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Exercise 1

Here we go:

Require: Initial parameter guess $\hat{\mathbf{x}}^{(0)}$, data $\mathbf{y}$, function $\mathbf{g}(\mathbf{x})$, Jacobian $\mathbf{G}_{\mathbf{x}}(\mathbf{x})$, and the grid size N_g

Ensure: Parameter estimate $\hat{\mathbf{x}}_{\mathrm{WLS}}$

Set $i \leftarrow 0$

repeat

  Calculate the update direction

$$\Delta \mathbf{x}^{(i+1)} = \mathbf{G}_{\mathbf{x}}^{\top}(\hat{\mathbf{x}}^{(i)})\, \mathbf{R}^{-1}(\mathbf{y} - \mathbf{g}(\hat{\mathbf{x}}^{(i)}))$$

  Define $J_{\mathrm{WLS}}(\mathbf{x}) = (\mathbf{y} - \mathbf{g}(\mathbf{x}))^{\top} \mathbf{R}^{-1} (\mathbf{y} - \mathbf{g}(\mathbf{x}))$

  Set $\gamma^* \leftarrow 0$ and $J^* \leftarrow J_{\mathrm{WLS}}\left(\hat{\mathbf{x}}^{(i)}\right)$

  for $j \in \{1, 2, \ldots, N_g\}$ **do**

    Set $\gamma \leftarrow j / N_g$

    if $J_{\mathrm{WLS}}\left(\hat{\mathbf{x}}^{(i)} + \gamma \Delta \hat{\mathbf{x}}^{(i+1)}\right) < J^*$ **then**

      Set $\gamma^* \leftarrow \gamma$

      $J^* \leftarrow J_{\mathrm{WLS}}\left(\hat{\mathbf{x}}^{(i)} + \gamma^* \Delta \hat{\mathbf{x}}^{(i+1)}\right)$

    end if

  end for

  Calculate

$$\hat{\mathbf{x}}^{(i+1)} = \hat{\mathbf{x}}^{(i)} + \gamma^* \Delta \mathbf{x}^{(i+1)}$$

  Set $i \leftarrow i + 1$

until Converged

Exercise 2

(a)

We have $g(x)$ 1d and thus the Jacobian is just the derivative $G_x(x) = g'(x)$. We also have $R = 1$. So the iteration is

$$\begin{aligned} x^{(i+1)} &= x^{(i)} + \gamma \frac{1}{[g'(x^{(i)})]^2} g'(x^{(i)})(y - g(x^{(i)})) \\ &= x^{(i)} + \gamma \frac{1}{g'(x^{(i)})}(y - g(x^{(i)})) \end{aligned} \tag{1}$$

(b)

Based on the question, $J(x)$ has a unique global minimum x^* which means that:

If $x = x^*$ then $\frac{\partial J(x^*)}{\partial x} = -2g(x^*)(y - g(x^*)) = 0$
And because x^* is unique then for all $x \neq x^*$ then $\frac{\partial J(x)}{\partial x} = -2g(x)(y - g(x)) \neq 0$

Now first we assume that $x^{(0)} \neq x^*$ and we know that $g'(x^{(0)}) \neq 0$. Let us try to find γ such that

$$x^* = x^{(0)} + \gamma \frac{1}{g'(x^{(0)})}(y - g(x^{(0)})) \tag{2}$$

This gives

$$\gamma = \frac{g'(x^{(0)})(x^* - x^{(0)})}{(y - g(x^{(0)}))} \tag{3}$$

This works when $y - g(x^{(0)}) \neq 0$ and this is true because we assumed that $x^{(0)} \neq x^*$ and we know that for all $x \neq x^*$ then $\frac{\partial J(x)}{\partial x} = -2g(x)(y - g(x)) \neq 0$.

But if $x^{(0)} = x^*$ then

$$x^* = x^* + \gamma \frac{1}{g'(x^{(0)})}(y - g(x^{(0)})) \tag{4}$$

So we should set $\gamma = 0$

(c)

If the γ calculated in Equation 3 is in the grid search vector then the method converges in one step.

No, it is not applicable to multidimensional model.