Solutions of Basics of Sensor Fusion Exercise Round 6

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Exercise 1

We have ODE

$$m\,a(t) + k\,p(t) + \eta\,v(t) = 0 \tag{1}$$

It is an ODE because $v = dp/dt = \dot{p}$, $a = dv/dt = \dot{v} = \ddot{p}$. So we have

$$m\,\ddot{p} + k\,p + \eta\,\dot{p} = 0 \tag{2}$$

Let us use the Ansatz

$$p(t) = C\,\exp(\lambda t) \tag{3}$$

Then we have

$$\begin{aligned} \dot{p}(t) &= C\,\lambda\,\exp(\lambda t), \\ \ddot{p}(t) &= C\,\lambda^2\,\exp(\lambda t), \end{aligned} \tag{4}$$

Substituting to ODE we have

$$\begin{aligned} m\,\ddot{p} + k\,p + \eta\,\dot{p} &= 0, \\ m\,C\,\lambda^2\,\exp(\lambda t) + k\,C\,\exp(\lambda t) + \eta\,C\,\lambda\,\exp(\lambda t) &= 0, \\ m\,\lambda^2 + k + \eta\,\lambda &= 0, \\ \lambda^2 + \frac{\eta}{m}\,\lambda + \frac{k}{m} &= 0, \end{aligned} \tag{5}$$

The solution to the last equation is

$$\lambda = \frac{1}{2}\left[-\frac{\eta}{m} \pm \sqrt{\frac{\eta^2}{m^2} - 4\frac{k}{m}}\right] \tag{6}$$

- If $\frac{\eta^2}{m^2} - 4\frac{k}{m} > 0$, then we have

$$\begin{aligned} a &= \frac{1}{2}\left[-\frac{\eta}{m} + \sqrt{\frac{\eta^2}{m^2} - 4\frac{k}{m}}\right] \\ b &= \frac{1}{2}\left[-\frac{\eta}{m} - \sqrt{\frac{\eta^2}{m^2} - 4\frac{k}{m}}\right] \end{aligned} \tag{7}$$

and the solution will be

$$p(t) = A\,\exp(a\,t) + B\,\exp(b\,t) \tag{8}$$

- If $\frac{\eta^2}{m^2} - 4\frac{k}{m} < 0$, then we have

$$\begin{aligned} a &= -\frac{\eta}{2m}, \\ b &= \frac{1}{2}\sqrt{4\frac{k}{m} - \frac{\eta^2}{m^2}} \end{aligned} \tag{9}$$

and the solution will be

$$p(t) = A\,\exp(a\,t)\,\cos(b\,t) + B\,\exp(a\,t)\,\sin(b\,t). \tag{10}$$

- When $\frac{\eta^2}{m^2} - 4\frac{k}{m} = 0$, we get one more solution, which you can figure out.

Side note: The full solution that you saw in equation (8) can be obtained using Laplace transform.

Laplace transform is defied as:

$$L(x(t)) = X(s) = \int_0^\infty x(t)e^{-st}\,dt \tag{11}$$

Using this definition we have the following property:

$$L(\frac{dx}{dt}) = s\,X(s) - x_0 \tag{12}$$

where x_0 is the initial value of the $x(t)$.

Now, in order to find the Laplace transform, first, we derive the matrix form of the ODE:

$$\begin{aligned} x_1 &= p, \\ x_2 &= v \end{aligned} \tag{13}$$

Then

$$\begin{aligned} \dot{x_1} &= \dot{p} = v = x_2 \\ \dot{x_2} &= \dot{v} = a = -\frac{k}{m}p - \frac{\eta}{m}\dot{p} = -\frac{k}{m}x_1 - \frac{\eta}{m}x_2 \end{aligned} \tag{14}$$

which can be written as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dot{p} \\ -\frac{k}{m}\,p - \frac{\eta}{m}\,\dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\eta}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{15}$$

So this is

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} \tag{16}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\eta}{m} \end{bmatrix} \tag{17}$$

Now we get the Laplace transform of equation (24) using equation (12)

$$sX(s) - \mathbf{x}_0 = A\,X(s), \qquad X(s) = \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} \tag{18}$$

Which can be rewritten as :

$$\begin{aligned} (sI - A)X(s) &= \mathbf{x}_0 \\ X(s) &= (sI - A)^{-1}\mathbf{x}_0 \\ &= \begin{bmatrix} s & -1 \\ \frac{k}{m} & s + \frac{\eta}{m} \end{bmatrix}^{-1} \begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix} \\ &= \frac{1}{s^2 + \frac{\eta}{m}s + \frac{k}{m}} \begin{bmatrix} s + \frac{\eta}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix} \begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix} \\ &= \frac{1}{(s-a)(s-b)} \begin{bmatrix} s + \frac{\eta}{m} & 1 \\ -\frac{k}{m} & s \end{bmatrix} \begin{bmatrix} x_{1,0} \\ x_{2,0} \end{bmatrix} \end{aligned} \tag{19}$$

So we have:

$$X_1(s) = \frac{A}{s-a} + \frac{B}{s-b} \tag{20}$$

A and B can be obtained by solving $A + B = x_{1,0}$ and $Ab + Ba = \frac{\eta}{m}x_{1,0} + x_{2,0}$

Exercise 2

$$\begin{aligned} x_1 &= p, \\ x_2 &= v \end{aligned} \tag{21}$$

Then

$$\begin{aligned} \dot{x_1} &= \dot{p} = v = x_2 \\ \dot{x_2} &= \dot{v} = a = -\frac{k}{m}p - \frac{\eta}{m}\dot{p} = -\frac{k}{m}x_1 - \frac{\eta}{m}x_2 \end{aligned} \tag{22}$$

which can be written as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dot{p} \\ -\frac{k}{m}\,p - \frac{\eta}{m}\,\dot{p} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\eta}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{23}$$

So this is

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} \tag{24}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\eta}{m} \end{bmatrix} \tag{25}$$

The solution is given given by

$$\mathbf{x}(t) = \exp(t\,\mathbf{A})\,\mathbf{x}(0). \tag{26}$$

where exp is the matrix exponential.

d) More generally

$$m\,\ddot{p} + k\,p + \eta\,\dot{p} = u(t) \tag{27}$$

$$\begin{aligned} x_1 &= p, \\ x_2 &= v \end{aligned} \tag{28}$$

Then

$$\begin{aligned} \dot{x_1} &= \dot{p} = v = x_2 \\ \dot{x_2} &= \dot{v} = a = -\frac{k}{m}p - \frac{\eta}{m}\dot{p} = -\frac{k}{m}x_1 - \frac{\eta}{m}x_2 + \frac{u}{m} \end{aligned} \tag{29}$$

which can be written as

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dot{p} \\ -\frac{k}{m}\,p - \frac{\eta}{m}\,\dot{p} + \frac{u}{m} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\eta}{m} + \frac{u}{m} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{30}$$

So this is

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} + \mathbf{B}\,u \tag{31}$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{\eta}{m} \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \tag{32}$$

The solution is given as

$$\mathbf{x}(t) = \exp(t\,\mathbf{A})\,\mathbf{x}(0) + \int_0^t \exp((t-s)\,\mathbf{A})\,\mathbf{B}\,u(s)\,ds \tag{33}$$