Exercise 7

1) $\dot{x} = \lambda x(1-x)$ $\qquad x(0) = x_0$

a)

Definition: $g(y)$ is linear if

x ① $g(y_1 + y_2) = g(y_1) + g(y_2)$

② $g(cy) = c\, g(y)$

Rewrite the ODE: $g(x) = \dot{x} - \lambda x(1-x) = 0$

① $g(x_1 + x_2) = \dot{x}_1 + \dot{x}_2 - \lambda(x_1 + x_2)(1 - x_1 - x_2)$

$= \dot{x}_1 + \dot{x}_2 - \lambda\left(x_1 + x_2 - (x_1 + x_2)^2\right)$

$= \dot{x}_1 + \dot{x}_2 - \lambda\left(x_1 + x_2 - x_1^2 - x_2^2 - \boxed{2x_1x_2}\right)$

$g(x_1) + g(x_2) = \dot{x}_1 - \lambda x_1(1-x_1) + \dot{x}_2 - \lambda x_2(1-x_2)$

$$= x_1 + x_2 - \lambda(x_1 + x_2 - x_1 - x_2)$$

x

$$g(x_1 + x_2) \neq g(x_1) + g(x_2)$$

② $$g(cx) = c\dot{x} - \lambda cx(1 - cx)$$

$$= c\dot{x} - \lambda cx + \lambda \boxed{c^2} x^2$$

$$c\,g(x) = c(\dot{x} - \lambda x(1-x))$$

$$= c\dot{x} - \lambda cx + \lambda \boxed{c} x^2$$

b) $$\frac{dx}{dt} = \lambda x(1-x)$$

$$\int_{x_0}^{x} \frac{dx}{x(1-x)} = \int_0^t \lambda\, dt \rightarrow \int_{x_0}^{x} \left(\frac{1}{x} + \frac{1}{1-x}\right) dx = \int_0^t \lambda\, dt$$

$$\left(\ln x - \ln(1-x)\right)\Big|_{x_0}^{x} = \lambda t \Big|_0^t$$

$$\ln x - \ln(1-x) - \ln(x_0) + \ln(1-x_0) = \lambda t$$

$$\ln \frac{x}{1-x} = \ln \frac{x_0}{1-x_0} + \lambda t$$

exp

$$\frac{x}{1-x} = \boxed{\frac{x_0}{1-x_0} e^{\lambda t}} := a$$

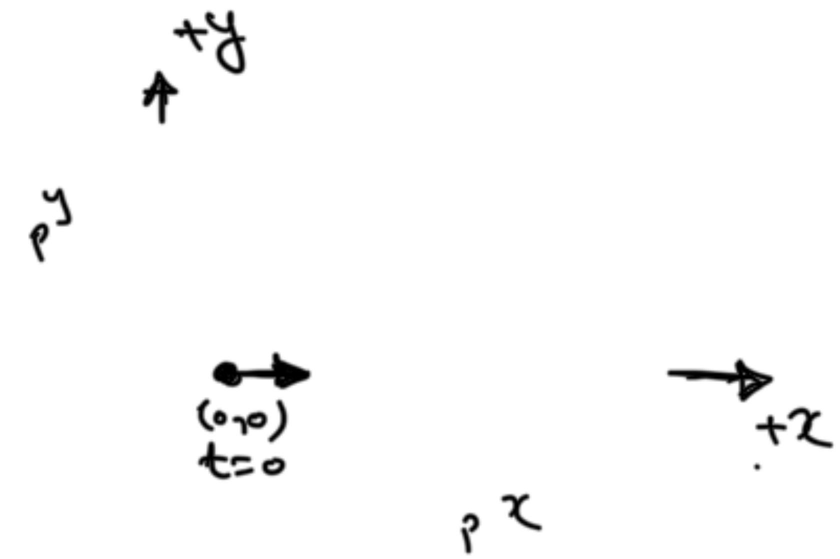
$$\frac{x}{1-x} = a \to x = a - ax$$

$$(1+a)x = a \to x = \frac{a}{1+a}$$

$$\boxed{x(t) = \frac{\frac{x_0}{1-x_0} e^{\lambda t}}{1 + \frac{x_0}{1-x_0} e^{\lambda t}} = \frac{x_0 e^{\lambda t}}{1 - x_0 + x_0 e^{\lambda t}}}$$

3)

$$\left.\begin{aligned} \dot{p}^x(t) &= v(t)\cos(\varphi(t)) + w_1(t) \\ \dot{p}^y(t) &= v(t)\sin(\varphi(t)) + w_2(t) \end{aligned}\right\} \quad \begin{aligned} p^x(0) &= 0 \\ p^y(0) &= 0 \\ \varphi(0) &= 0 \end{aligned}$$



$$\dot{\varphi}(t) = \omega_{gyro}(t) + \omega_3(t) \quad \}$$

a) $$\dot{x}(t) = f(x(t), u(t)) + B_w(x(t))\, w(t)$$

$$x(t) = \begin{bmatrix} p^x(t) \\ p^y(t) \\ \varphi(t) \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad u(t) = \begin{bmatrix} v(t) \\ \omega_{gyro}(t) \end{bmatrix} = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}$$

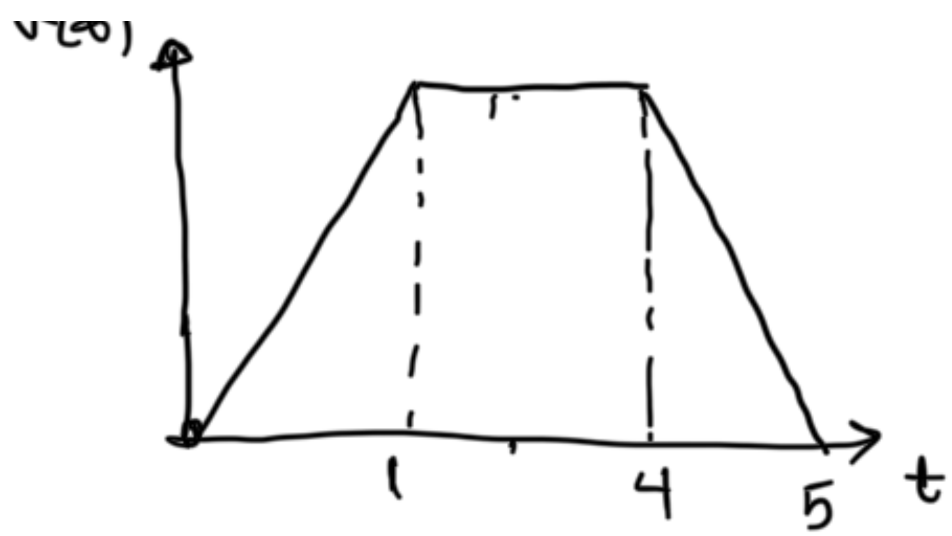
$$f(x(t), u(t)) = \begin{bmatrix} u_1 \cos(x_3) \\ u_1 \sin(x_3) \\ u_2 \end{bmatrix}, \qquad \underbrace{B_w(x(t))}\, w(t) = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

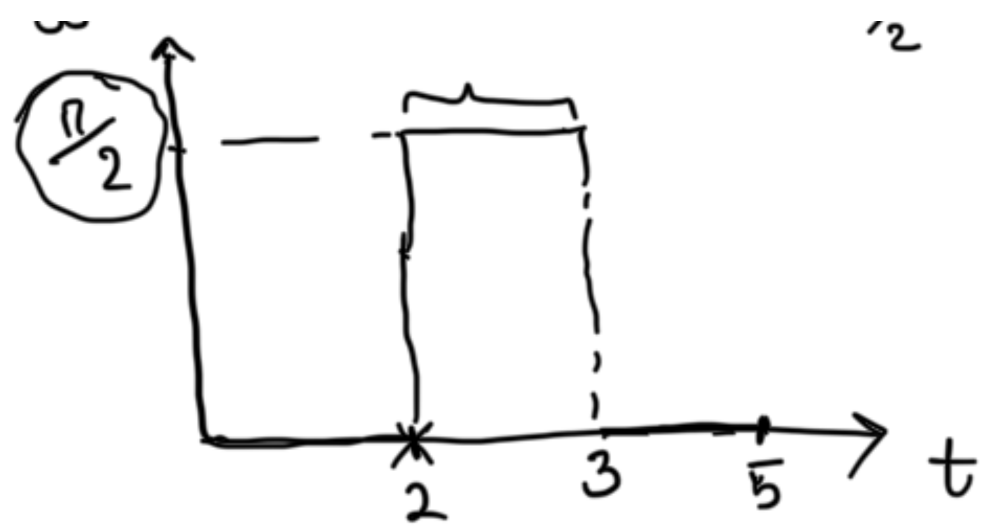
$$B_w(x(t)) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

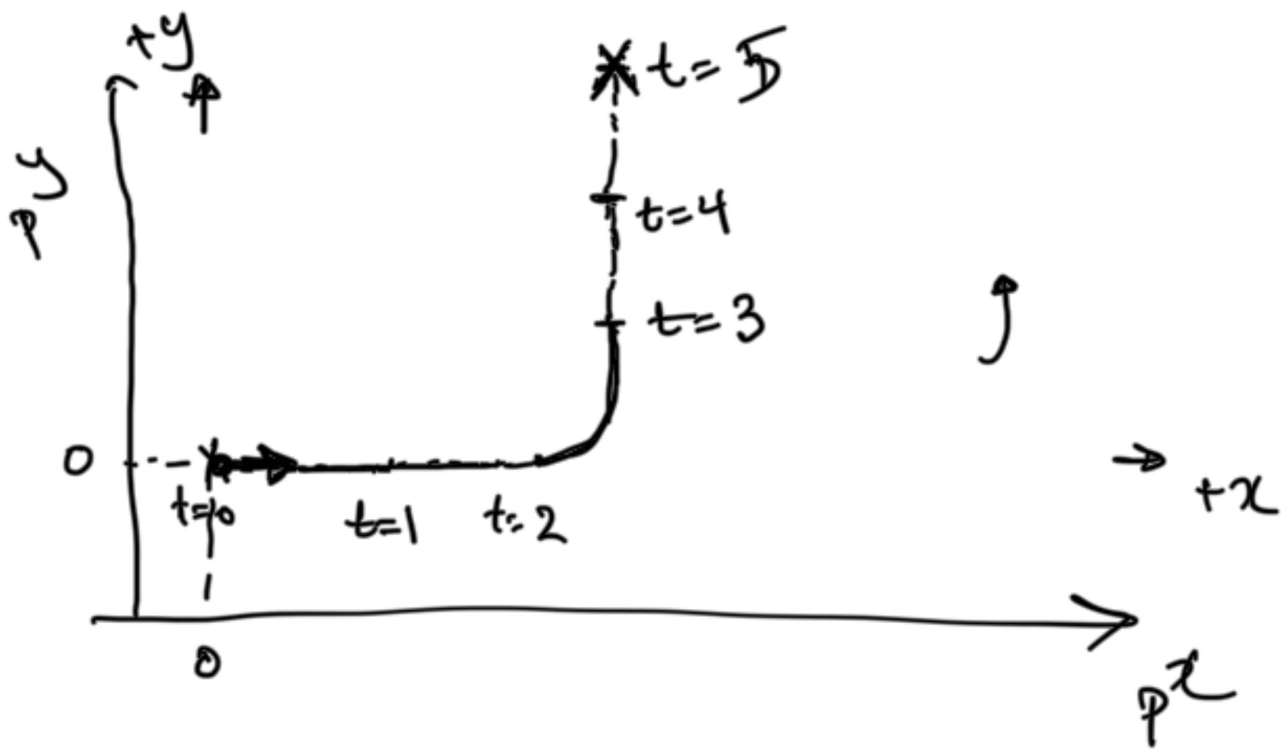
$$w(t) = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}$$

b) $v(t)$:

ω_{gyro} -







$$\omega_{gyro} = \text{angular velocity}$$

$$= \frac{\text{rotation}}{\text{second}}$$