Queuing System

1. Model Assument

   (1) Input Process:

   customer arrives alone

   The time interval between any two customers arriving at the service desk obey Exponential Distribution

   (2) Queuing process:

   There is no limit to the number of people in the queue, and there is only one queue

   When the service desk is idle, customers will enter the service

   (3) Service process:

   The service hours of the two service desks are independent of each other

   The time each customer receives service obeys Exponential Distribution

2. Symbol Description

T_{1i}	The time when i^{th}customer reaches the service disk
T_{2i}	The time when i^{th}customer leaves the service disk
t_{11i}	The time when i^{th}customer reaches the restaurant
t_{12i}	The time when i^{th}customer chooses their food
t_{21i}	The time when i^{th}Customer receives service
t_{wi}	The time when i^{th} customer waits
Δt_{1i}	The time interval between any two customers arriving at the service d
Δt_{2i}	The time interval between any two customers leaving at the service de
m	The number of serice desks
n	The number of customers
N_T The total number of customers	
n	The number of customers in the queuing system
P_n	The probability of n customers in the queuing system
$N(t)$	The number of customers at t time
λ	The Exponential Distribution of arriving time interval (start with t tin
μ	The Exponential Distribution of leaving time (start with t time)
L	The number of waiting customers

where $i = 1, 2, 3, ...$

3. Model Estimate and slove

(1) Input Process:

$T_{1i} = t_{11i} + T_{12i}$

$\Delta t_{1i} = T_{1i} - T_{1i-1} = t_{11i} + t_{12i} - (t_{11i-1} + t_{12i-1})$

$$E(\Delta t_{1i}) = \frac{\sum_{i=1}^{n} \Delta t_{1i}}{n} = \frac{\sum_{i=1}^{n} [t_{11i} + t_{12i} - (t_{11i-1} + t_{12i-1})]}{n} = \frac{\sum_{i=1}^{n} [t_{11i} - t_{11i-1}]}{n} + \frac{\sum_{i=1}^{n} [t_{12i} - t_{12i-1}]}{n}$$

SInce t_{11i} obey Exponential Distribution, $\dfrac{\sum_{i=1}^{n} [t_{11i} - t_{11i-1}]}{n} = 1/\lambda$

Since t_{12i} obey Normal Distribution, $\dfrac{\sum_{i=1}^{n} [t_{12i} - t_{12i-1}]}{n} = 0$

So, $E(\Delta t_{1i}) = \dfrac{\sum\limits_{i=1}^{n} \Delta t_{1i}}{n} = 1/\lambda.$

That is the time interval between any two customers arriving at the service desk obey Exponential Distribution with λ

(2) Service and Queuing process:

$T_{2i} = T_{1i} + t_{21i} + t_{wi}$

$\Delta t_{2i} = T_{1i} - T_{1i-1} = T_{1i} + t_{21i} + t_{wi} - (T_{1i-1} + t_{21i-1} + t_{wi-1})$

$E(\Delta t_{2i}) = \dfrac{\sum\limits_{i=1}^{n} T_{1i} + t_{21i} + t_{wi} - (T_{1i-1} + t_{21i-1} + t_{wi-1})}{n} = \dfrac{\sum\limits_{i=1}^{n} t_{21i}}{n}$

SInce t_{21i} obey Exponential Distribution, $\frac{\sum_{i=1}^{n} t_{21i}}{n} = \frac{1}{\mu}$

So, $E(\Delta t_{2i}) = \dfrac{\sum\limits_{i=1}^{n} \Delta t_{2i}}{n} = \dfrac{1}{\mu}.$

That is the time interval between any two customers leaving at the service desk obey Exponential Distribution with μ

(3) Whole process:

All adjacent states can be obtained by one-step transition. When the system reaches equilibrium, record as P_n, where $n = 0, 1, 2, ...$ Since "arriving" and "leaving" are alternated, we can think that these two events have equal probability. When the system runs for a period of time, for state n, each state remains stable, so the equilibrium equation can be established:

$$P_{n+1} = \frac{\lambda_n}{\mu_{n+1}} P_n + \frac{1}{\mu + 1}(\mu_n P_n - \lambda_{n-1} P_{n-1}) = \frac{\prod\limits_{i=0}^{n} \lambda_i}{\prod\limits_{i=i}^{n} \mu_i} P_0$$

Let

$$C_n = \frac{\prod_{i=0}^{n} \lambda_i}{\prod_{i=i}^{n} \mu_i}$$

Since

$$\sum_{i=0}^{\infty} P_n = 1$$

$$P_0 = \frac{1}{1 + \sum\limits_{n=1}^{\infty} C_n}$$

When $m > 1$,if $n \geq m$ $\mu_n = m\mu$, where $m = 1, 2, ...$; else $\mu_n = n\mu$, where $n = 0, 1, ..., m-1$

$$C_n = \frac{(\lambda/\mu)^n}{n!}, n = 1, 2, ..., m$$

else

$$C_n = \frac{(\lambda/\mu)^m}{m!} (\frac{\lambda}{m\mu})^n, n = m+1, ...$$

(Each step needs to choose the "Entrance")

$$P_0 = [\sum_{n=0}^{m-1} \frac{(\lambda/\mu)^n}{n!} + \frac{(\lambda/\mu)^m}{m!(1 - C_n)}]^{-1}$$

When $n \geq m$, customers should wait,

$$E(L) = \sum_{n=m+1}^{\infty} (n-m)P_n = \frac{P_0(\lambda/\mu)^m \lambda/(m\mu)}{m!(1 - (\lambda/(m\mu))^2}$$

Other special situation:

If $\sum_{n=1}^{\infty} C_n$ is not convergence ($\frac{\lambda}{m\mu} \geq 1$), the queue system cannot be in stable condition. And we need to give a number N_t for the total customers in this system.

That is $\frac{\lambda}{m\mu} = 1$,

$$P_n = \frac{1}{n+1}$$

else

$$P_n = \frac{(1 - \frac{\lambda}{m\mu})(\frac{\lambda}{m\mu})^n}{1 - \frac{\lambda}{m\mu} N_T + 1}$$

So, when $\frac{\lambda}{m\mu} = 1$

$$E(L) = \sum_{n=0}^{N_T} (n-1)P_n = \frac{N_T(N_T-1)}{2(N_T+1)}$$

else

$$E(L) = \frac{(\frac{\lambda}{m\mu})^2}{1-\frac{\lambda}{m\mu}} - \frac{(N_T+\frac{\lambda}{m\mu})(\frac{\lambda}{m\mu})^{N_T+1}}{1-(\frac{\lambda}{m\mu})^{N_T+1}}$$

By $Little's\ Law: L = \lambda\omega$,

$$E(t_w) = \frac{E(L)}{\lambda}$$

4. Solution for the question

   $\lambda = 120$,$\mu = 60$, $m = 2$, and total time is 2, so, $N_T = 240$ and $\frac{\lambda}{m\mu} = 1$

   That is

$$P_n = \frac{1}{N_T+1} = \frac{1}{241}$$

$$E(L) = \frac{N_T(N_T-1)}{2(N_T+1)} = \frac{120 * 239}{241}$$

$$E(t_w) = \frac{E(L)}{\lambda} = \frac{239}{241} \approx 0.9917$$

Appendix

```
x = zeros(1,240);
for i = 1:10000
    x(i) = normrnd(8,2,1);
    y(i) = -1/2*log(rand(1));
end
y = cumsum(y);
z = x+y;
z = sort(z);
z = diff(z);
pd = makedist("Exponential");
qqplot(z,pd);
lambda = 1/mean(z);
k = zeros(1,240);
for i = 1:240
    k(i) = -1/lambda*log(rand(1));

end
d = k-z;
mean(d)
```