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Assignment 2

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1. (1) likelihood function: $L = \prod_{i=1}^{n} f(\varepsilon_i) = \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\varepsilon_i^2\right] = \frac{1}{(2\pi\sigma^2)^{\frac{n}{2}}} \exp\left[-\frac{1}{2\sigma^2}(y-X\beta)^T(y-X\beta)\right]$

(2) $l(\beta,\sigma^2) = -\frac{1}{2\sigma^2}(y-X\beta)^T(y-X\beta) - \frac{n}{2}\log(2\pi\sigma^2)$

(3) $\frac{\partial l(\beta,\sigma^2)}{\partial \beta} = \frac{1}{2\sigma^2} 2X^T(y-X\beta) = 0 \Rightarrow \beta = (X^TX)^{-1}X^Ty$

$\frac{\partial l(\beta,\sigma^2)}{\partial \sigma^2} = \frac{1}{2\sigma^4}(y-X\beta)^T(y-X\beta) - \frac{n}{2\sigma^2} = 0 \Longrightarrow \sigma^2 = \frac{(y-X\beta)^T(y-X\beta)}{n}$

2. Since the model is $y_t = \beta_0 + \beta_1 t + \varepsilon_t = X\beta + \varepsilon_t$ ($Var(\varepsilon_t) = \sigma^2$) where $X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & T \end{bmatrix}_{T\times 2}$ $\beta = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$

$X^TX = \begin{bmatrix} 1 & \cdots & 1 \\ 1\ 2 & \cdots & T \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & T \end{bmatrix} = \begin{bmatrix} T & \sum_{i=1}^{T} i \\ \sum_{i=1}^{T} i & \sum_{i=1}^{T} i^2 \end{bmatrix} = \begin{bmatrix} T & \frac{T(T+1)}{2} \\ \frac{T(T+1)}{2} & \frac{T(T+1)(2T+1)}{6} \end{bmatrix}$ $\det(X^TX) = \frac{T^2(T+1)(2T+1)}{6} - \frac{T^2(T+1)^2}{4} = \frac{T^2(T+1)(T-1)}{12}$

$(X^TX)^{-1} = \frac{12}{T^2(T+1)(T-1)}\begin{bmatrix} \frac{T(T+1)(2T+1)}{6} & -\frac{T(T+1)}{2} \\ -\frac{T(T+1)}{2} & T \end{bmatrix} = \begin{bmatrix} \frac{2(2T+1)}{T(T-1)} & -\frac{6}{T(T-1)} \\ -\frac{6}{T(T-1)} & \frac{12}{T(T+1)(T-1)} \end{bmatrix}$

$X^Ty = \begin{bmatrix} 1 & \cdots\cdots & 1 \\ 1\ 2 & \cdots & T \end{bmatrix}\begin{bmatrix} y_1 \\ \vdots \\ y_t \\ \vdots \\ y_T \end{bmatrix} = \begin{bmatrix} \sum_{t=1}^{T} y_t \\ \sum_{t=1}^{T} t y_t \end{bmatrix}$

$\hat\beta = \begin{bmatrix} \hat\beta_0 \\ \hat\beta_1 \end{bmatrix} = (X^TX)^{-1}X^Ty = \begin{bmatrix} \frac{2(2T+1)}{T(T-1)} & -\frac{6}{T(T-1)} \\ -\frac{6}{T(T-1)} & \frac{12}{T(T+1)(T-1)} \end{bmatrix}\begin{bmatrix} \sum_{t=1}^{T} y_t \\ \sum_{t=1}^{T} t y_t \end{bmatrix} = \begin{bmatrix} \frac{2(2T+1)}{T(T-1)}\sum_{t=1}^{T} y_t - \frac{6}{T(T-1)}\sum_{t=1}^{T} t y_t \\ -\frac{6}{T(T-1)}\sum_{t=1}^{T} y_t + \frac{12}{T(T+1)(T-1)}\sum_{t=1}^{T} t y_t \end{bmatrix}$

$Var(\hat\beta) = \sigma^2(X^TX)^{-1} = \begin{bmatrix} \frac{2\sigma^2(2T+1)}{T(T-1)} & -\frac{6\sigma^2}{T(T-1)} \\ \frac{-6\sigma^2}{T(T-1)} & \frac{12\sigma^2}{T(T+1)(T-1)} \end{bmatrix}$

So $Var(\hat\beta_0) = \frac{2\sigma^2(2T+1)}{T(T-1)}$ $Var(\hat\beta_1) = \frac{12\sigma^2}{T(T^2-1)}$

3. $\hat y = \hat\beta_0 + \hat\beta_1 x_1 = \tilde{X}\tilde{\beta}$. where $\tilde X = (1 \quad x_1)$, $\tilde\beta = \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$

Actual: $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 = X\beta$, where $X = (1 \quad x_1 \quad x_2) = (\tilde X \quad x_2)$, $\beta = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \tilde\beta \\ \beta_2 \end{pmatrix}$

$E(\beta) = E[(\tilde X^T\tilde X)^{-1}\tilde X^T y] = (\tilde X^T\tilde X)^{-1}\tilde X^T E(y) = (\tilde X^T\tilde X)^{-1}\tilde X^T X\beta = (\tilde X^T\tilde X)^{-1}\tilde X^T[\tilde X \quad x_2]\begin{bmatrix} \tilde\beta \\ \beta_2 \end{bmatrix} = \tilde\beta + (\tilde X^T\tilde X)^{-1}\tilde X^T x_2\beta_2$

(2) $E(\hat\beta_1) = \beta_1 + (X^TX)^{-1}X^Tx_2\beta_2$

4. keep taking the difference on y_t and x_t, until the residuals are uncorrelated

$y'_t = y_t - \phi y_{t-1} = \beta_0 + \beta_1 x_t + \varepsilon_t - \phi(\beta_0 + \beta_1 x_{t-1} + \varepsilon_{t-1}) = (1-\phi)\beta_0 + \beta_1(x_t - x_{t-1}) + \varepsilon_t - \phi\varepsilon_{t-1}$

$y''_t = y'_t - \phi y'_{t-1} = (1-\phi)\beta_0 + \beta_1(x_t - x_{t-1}) + \varepsilon_t - \phi\varepsilon_{t-1} - \phi[(1-\phi)\beta_0 + \beta_1(x_{t-1} - x_{t-2}) + \varepsilon_{t-1} - \phi\varepsilon_{t-2}]$

$= (1-2\phi+\phi^2)\beta_0 + \beta_1(x_t + x_{t-2} - 2x_{t-1}) + \underbrace{\varepsilon_t - 2\phi\varepsilon_{t-1} + \phi^2\varepsilon_{t-2}}_{\text{error term}}$

Since $\phi = \frac{\sum_{t=2}^{T} e_t e_{t-1}}{\sum_{t=1}^{T} e_t^2}$, $\varepsilon_t = \rho_1\varepsilon_{t-1} + \rho_2\varepsilon_{t-2} + a_t$

So $\rho_1 = 2\phi = \frac{2\sum_{t=2}^{T} e_t e_{t-1}}{\sum e_t^2}$ $\rho_2 = -\phi^2 = -\left(\frac{2\sum_{t=2}^{T} e_t e_{t-1}}{\sum_{t=1}^{T} e_t^2}\right)^2$

$$\text{So} \quad \rho_1 = 2\phi = \frac{2\sum_{t=2}^{T} e_t e_{t-1}}{\sum_{t=1}^{T} e_t^2} \qquad \rho_2 = -\phi^2 = -\left(\frac{2\sum_{t=2}^{T} e_t e_{t-1}}{\sum_{t=1}^{T} e_t^2}\right)^2$$

5.

(a) $X_t = a + bt + S_t + Y_t$ $\quad S_t$. seasonal 12.

Since S_t is a seasonal component with period 12.

$$S_t = S_{t-12} = S_{t-24} = \cdots$$

$$\nabla\nabla_{12} X_t = (1-B)(1-B^{12}) X_t = (1-B)(1-B^{12})\left[a + bt + S_t + Y_t\right] = (1-B)\left[a - a + bt - b(t-12) + S_t - S_{t-12} + Y_t - Y_{t-12}\right]$$

$$= (1-B)\left[12b + Y_t - Y_{t-12}\right] = Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13}$$

$$E(\nabla\nabla_{12} X_t) = E\left[Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13}\right] = 0 - 0 + 0 + 0 = 0$$

$$\gamma_{\nabla\nabla_{12} X_t}(t, t+k) = \mathrm{Cov}(Y_t - Y_{t-1} - Y_{t-12} + Y_{t-13},\ Y_{t+k} - Y_{t+k-1} - Y_{t+k-12} + Y_{t+k-13})$$

$$= \gamma_k - \gamma_{k-1} - \gamma_{k-12} + \gamma_{k-13} - \gamma_{k+1} + \gamma_k + \gamma_{k-11} - \gamma_{k-12} - \gamma_{k+12} + \gamma_{k+11} + \gamma_k - \gamma_{k-1} + \gamma_{k+13} - \gamma_{k+12} - \gamma_{k+1} + \gamma_k$$

$$= 4\gamma_k - 2\gamma_{k-1} - 2\gamma_{k+1} - 2\gamma_{k-12} - 2\gamma_{k+12} + \gamma_{k+11} + \gamma_{k-11} + \gamma_{k-13} + \gamma_{k+13}$$

$$= \begin{cases} 4\sigma^2 + f_1(k). & k = 0 \\ -2\sigma^2 + f_2(k) & k = \pm 1 \\ -2\sigma^2 + f_3(k) & k = \pm 12 \\ \sigma^2 + f_4(k) & k = \pm 11 \\ \sigma^2 + f_5(k) & k = \pm 13 \\ \text{independent of } t. & \text{ow} \end{cases}$$

So $\nabla\nabla_{12} X_t = (1-B)(1-B^{12}) X_t$ is stationary ts.

(b) $X_t = (a + bt) S_t + Y_t$

$$\nabla_{12}^2 X_t = (1-B^{12})^2 X_t = \left[(a+bt)S_t + Y_t - (a + b(t-12))S_{t-12} + Y_{t-12}\right] - \left[(a + b(t-12))S_{t-12} + Y_{t-12} - (a + b(t-24))S_{t-24} + Y_{t-24}\right]$$

$$= 12b S_t + Y_t - Y_{t-12} - (12b S_t + Y_{t-12} + Y_{t-24}) = Y_t + Y_{t-24} - 2Y_{t-12}$$

$$E\left[\nabla_{12}^2 (X_t)\right] = E\left[Y_t + Y_{t-24} - 2Y_{t-12}\right] = 0$$

$$\gamma_{\nabla_{12}^2}(X_t, X_{t+k}) = \mathrm{Cov}(X_t, X_{t+k}) = \mathrm{Cov}(Y_t + Y_{t-24} - 2Y_{t-12},\ Y_{t+k} + Y_{t+k-24} - 2Y_{t+k-12})$$

$$= \gamma_k + \gamma_{k-24} - 2\gamma_{k-12} + \gamma_{k+24} + \gamma_k - 2\gamma_{k+12} - 2\gamma_{k+12} - 2\gamma_{k-12} + 4\gamma_k$$

$$= 6\gamma_k - 4\gamma_{k-12} + 4\gamma_{k+12} + \gamma_{k+24} + \gamma_{k-24}$$

$$= \begin{cases} 6\sigma^2 + f_1(x). & k = 0 \\ 4\sigma^2 + f_2(x). & k = \pm 12 \\ \sigma^2 + f_3(x) & k = \pm 24. \\ \text{independent with } t & \text{ow}. \end{cases}$$

So $\nabla_{12}^2 (X_t, X_{t+24})$ is a stationary ts.