Assignment 1

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1. span $= N$ MA, μ, σ^2

(a) moving average is N.

Solution:
suppose data come from ts. variable X

so. $Var(X) = \sigma^2$. $E(X) = \mu$

Since $M_t = \frac{1}{N}\sum_{i=t-N}^{t} X_i$.

So $Var(M_t) = Var\left(\frac{1}{N}\sum_{i=t-N}^{t} X_i\right) = \frac{1}{N^2} Var\left(\sum_{i=t-N}^{t} X_i\right)$

$\xrightarrow{X_i \text{ are uncorrelated}} = \frac{1}{N^2}\cdot N\sigma^2 = \frac{\sigma^2}{N}$

(b) $Cov(M_t, M_{t+k}) = Cov\left(\sum_{i=t-N}^{t} M_i, \sum_{j=t-N+k}^{t+1} M_j\right)$

$= Cov(M_{t-N} + \cdots + M_t \quad M_{t-N+k} + \cdots + M_{t+k})$

Since M_i. are uncorrelated.

If $t > t-N+k \Rightarrow N-k > 0$

$= Cov(M_{t-N+k} + \cdots + M_t, M_{t-N+k} + \cdots + M_t)$

$= Var(M_{t-N+k} + \cdots + M_t) = Var\left(\sum_{j=1}^{N-k} M_j\right)$

$\overset{\text{uncorrelated}}{\sim} \sum_{j=1}^{N-k} Var(M_j) = \sigma^2 \sum_{j=1}^{N-k}\left(\frac{1}{N}\right)^2$

(3) when $k < N$

$\rho_k = \frac{c_k}{c_0} = \frac{\sigma^2 \sum_{j=1}^{N-k}\left(\frac{1}{N}\right)^2}{\sigma^2/N} = \frac{\sigma^2 (N-k)\cdot\left(\frac{1}{N}\right)^2}{\sigma^2/N} = 1 - \frac{|k|}{N}$

when $k \geq N$. Since $\{M_t\}$ is uncorrelated

so there is no intersection between $[1, N-k]$ and $[k, N+k]$. so $\rho_k = 0$ when $k \geq N$

2. $M_T^w = \sum_{t=T-N+1}^{T} a_{T+1} y_i$

(a) Because we want to smooth the time series so that analysing it to draw conclusion

(b)

$Var(M_T^*) = Var\left(\sum_{t=T-N+1}^{T} a_{T+1-i} y_t\right) = \sigma^2 \sum_{j=1}^{N} a_j^2$

(c) $Cov(M_T^w, M_{T+k}^w) = Cov\left(\sum_{t=T-N+1}^{T} a_{T+1-t} y_t, \sum_{t=T-N+1+k}^{T+k} a_{T+1-t+k} y_{t+k}\right)$

If $N > |k|$, so $T-N+1+k < T$

$= Cov\left(\sum_{t=T-N+1}^{T-N+k} a_{T+1-t} y_t, \sum_{t=T-N+1+k}^{T} a_{T+1-t+k} y_{t+k}\right)$

$+ Cov\left(\sum_{t=T-N+1}^{T-N+k} a_{T+1-t} y_t, \sum_{t=T+1}^{T+k} a_{T+1-t+k} y_{t+k}\right)$

$+ Cov\left(\sum_{t=T-N+1+k}^{T} a_{T+1-k} y_t, \sum_{t=T-N+1+k}^{T} a_{T+1-t+k} y_{t+k}\right)$ ✓

$+ Cov\left(\sum_{t=T-N+1+k}^{T} a_{T+1-k} y_t, \sum_{t=T+1}^{T} a_{T+1-t+k} y_{t+k}\right)$

$= Cov\left(\sum_{t=T-N+1+k}^{T} a_{T+1-k} y_t, \sum_{t=T-N+1+k}^{T} a_{T+1-t+k} y_{t+k}\right)$

$\Longleftrightarrow \sum_{j=1}^{N-k} Cov(a_j y_t, a_{j+k} y_{t+k})$

$= \sum_{j=1}^{n+k} a_j a_{j+k} Cov(y_t, y_{t+k}) = \sigma^2 \sum_{j=1}^{n+k} a_j a_{j+k}$

(d) when $0 < k < N$

$\rho_k = \frac{Cov(M_T^w, M_{T+k}^w)}{Var(M_T^w)} = \frac{\sigma^2 \sum_{j=1}^{N-k} a_j a_{j+k}}{\sigma^2 \sum_{j=1}^{N} a_j^2}$

when $k \geq N$

For $Cov(M_T^w, M_{T+k}^w)$, since $k \geq N$, and M_T^w is the weighted average ts with time t.

So $T-N+1+k > T$., that means there is no intersection part in plot ⊛. Since $\{y_t\}$ is uncorrelated and a_i is a constant. so the $Cov(M_T^w, M_{T+k}^w) = 0 \Rightarrow \rho_k = 0$ for $k \geq N$

3.

a)

<1> Strictly stationary:

$f(y_t, y_{t+1}, y_{t+2}, \cdots, y_{t+n}) \overset{\text{independent}}{=} f(y_t) f(y_{t+1}) \cdots f(y_{t+n})$

$\xrightarrow{\text{iid. } y_t} f(y_{t+n})^n = f^n(y_{t+k+n}) = f(y_{t+k+n}) f(y_{t+k+n-1}) \cdots f(y_{t+k})$.

$\overset{\text{independent}}{=} f(y_{t+k}, \cdots, y_{t+k+n})$

So $\{y_t\}$ is strictly stationary.

<2> Weakly stationary

$E(y_t) = E(X) = \mu$ (constant).

$Cov(y_t, y_{t+k}) = Cov(x_i, x_j) = \begin{cases} \sigma^2 & k=0 \\ 0 & k \neq 0 \end{cases}$ constant

So $\{y_t\}$ is weakly stationary

(2)

autocovariance function $= C_k = \begin{cases} 0 & k \neq 0 \\ \frac{\sigma^2}{N} & k = 0 \end{cases}$

4.

a) ① If Y_t, Y_{t-k} are all even

$Cov(Y_t, Y_{t-k}) = Cov(X_t + 5, X_{t+k} + 5) = Cov(X_t, X_{t+k})$

② If Y_t, Y_{t-k} are all odd

$Cov(Y_t, Y_{t-k}) = Cov(X_t, X_{t+k})$

③ If Y_t, Y_{t-k} are one even one odd

$Cov(Y_t, Y_{t-k}) = Cov(Y_{t+k}, Y_t) = Cov(X_t, X_{t+k})$.

Since $\{X_t\}$ is stationary, so $E(X_t)$ and $Cov(X_t, X_{t+k})$ are all constant, so $Cov(Y_t, Y_{t+k})$ is constant and it is free of lag k

b) Suppose $E(X) = \mu$.

when Y_t is odd. $E(Y_t) = E(X_t) = \mu$

when Y_t is even $E(Y_t) = E(X_t) + 5 = \mu + 5$

> It is not free of t.
> so $\{y_t\}$ is not stationary

5.

a) Since $\{Z_t\} \overset{iid}{\sim} N(0,1)$.

$E(X_t) = \begin{cases} E(Z_t) = 0 & \text{when } t \text{ is odd} \\ E\left(\frac{Z_{t-1}^2 - 1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}\left[E(Z_{t-1}^2) - 0^2 - 1\right] = \frac{1}{\sqrt{2}}\left[1 - 0^2 - 1\right] = 0 & \text{when } t \text{ is even} \end{cases}$

$Var(X_t) = \begin{cases} Var(Z_t) = 1 & \text{when } t \text{ is odd.} \\ Var\left(\frac{Z_{t-1}^2 - 1}{\sqrt{2}}\right) = \frac{1}{2} Var(Z_{t-1}^2) = 1 & \text{when } t \text{ is even} \end{cases}$

so $\{X_t\} \sim WN(0,1)$.

b). NO, Because the distribution of $\{X_t\}$ are varies by the odd and even conditions