a) $\frac{d\phi}{dt} = 2t \quad \phi(0)=0 \qquad \int d\phi = \int 2t\,dt$

$$\underset{\overset{\uparrow}{\text{since } \phi(0)=0 \rightarrow c=0}}{\phi = t^2 + c}$$

$$\phi = t^2$$

$$\boxed{\phi(1)=1}$$

b) 2 steps: $K=2 \quad \Delta t = 0.5$ for $t=0$ to $t=1$

$f(t, \phi(t)) = \frac{d\phi}{dt} = 2t \qquad \hat{\phi}_i = \hat{\phi}_{i-1} + \Delta t \cdot f(t_{i-1}, \phi_{i-1})$

$$\hat{\phi}_{0.5} = \underset{\overset{\uparrow}{0}}{\hat{\phi}_0} + 0.5 \cdot f(t_0, \underset{\overset{\uparrow}{0}}{\phi_0})$$

$\phi = t^2$

$0 = t^2$

$0 = t$

$$\hat{\phi}_{0.5} = 0 + 0.5 \cdot (2(0)) = 0$$

$$\hat{\phi}_1 = \underset{\overset{\uparrow}{0}}{\hat{\phi}_{0.5}} + 0.5 \cdot f(\underset{\overset{\uparrow}{0.5}}{t_{0.5}}, \underset{\overset{\uparrow}{0}}{\phi_{0.5}})$$

$$\boxed{\hat{\phi}_1 = 0.5}$$

*use fraction

3 steps: $K=3 \quad \Delta t = 1/3$ for $t=0$ to $t=1$

$$\hat{\phi}_{1/3} = \hat{\phi}_0 + \frac{1}{3} \cdot f(t_0, \phi_0) = 0$$

$$\hat{\phi}_{2/3} = \underset{\overset{\uparrow}{0}}{\hat{\phi}_{1/3}} + \frac{1}{3} \cdot f(\underset{\overset{\uparrow}{1/3}}{t_{1/3}}, \underset{\overset{\uparrow}{0}}{\phi_{1/3}}) = 0 + \frac{1}{3} \cdot 2 \cdot \frac{1}{3} = \frac{2}{9}$$

$$\hat{\emptyset}_1 = \hat{\emptyset}_{2/3} + \frac{1}{3} \cdot f(t_{2/3}, \emptyset_{2/3}) = \frac{2}{9} + \frac{1}{3} \cdot \frac{2}{3} \cdot 2$$

$$\boxed{2/3 = \hat{\emptyset}_1}$$

c) for n steps of equal length $\Rightarrow \Delta t = \frac{1}{n}$

$$\hat{\emptyset}_1 = \hat{\emptyset}_0 + \frac{1}{n} \cdot 2t_0$$

$$\hat{\emptyset}_2 = \hat{\emptyset}_1 + \frac{1}{n} \cdot 2t_1 \quad \hat{\emptyset}_2 = \hat{\emptyset}_0 + \frac{1}{n} \cdot 2t_0 + \frac{1}{n} \cdot 2t_1$$

$$\hat{\emptyset}_3 = \hat{\emptyset}_2 + \frac{1}{n} \cdot 2t_2 \quad \hat{\emptyset}_3 = \hat{\emptyset}_0 + \frac{2}{n}(t_0 + t_1) + \frac{1}{n} \cdot 2t_2$$

$$\vdots \quad \hat{\emptyset}_4 = \hat{\emptyset}_0 + \frac{2}{n}(t_0 + t_1 + t_2) + \frac{1}{n} \cdot 2t_3$$

$$\hat{\emptyset}_n = \hat{\emptyset}_{n-1} + \frac{1}{n} \cdot 2t_{n-1} \quad \Downarrow$$

$$\hat{\emptyset}_n = \hat{\emptyset}_0 + \frac{2}{n}\underbrace{(t_0 + t_1 + \ldots + t_{n-1})}_{(0) + \left(\frac{1}{n}\right) + \left(\frac{2}{n}\right) + \left(\frac{3}{n}\right) + \ldots + \left(\frac{n-1}{n}\right)}$$

$$0$$

$$1 + 2 + \ldots + (n-1) = \frac{(n-1) \cdot n}{2}$$

$$\hat{\emptyset}_n = \frac{(n-1) \cdot \cancel{n}}{\cancel{2}n} \cdot \frac{\cancel{2}}{\cancel{n}} = \frac{n-1}{n} \rightarrow \lim_{n \to \infty} \frac{n-1}{n} = 1 \checkmark$$

absolute error: $\left|1 - \frac{n-1}{n}\right| = \left|\frac{n-n+1}{n}\right| = \frac{1}{n} = \Delta t \checkmark$

Yes, the Forward Euler approximation converges to $\emptyset(1)$ as $n \to \infty$ because the timestep size is very small when $n \to \infty$, so the approximation is as accurate as possible, because it is 'skipping over' essentially no data of the function. Thus, the approximation is very close to the true value.

d) The derivations done in parts (a)-(c) agree with the plots in part (d) because, for larger values of n, the plot of the Forward Euler approximation gets closer to 0 as the time approaches 1. The plot of the absolute error also confirms this because the error gets very small very quickly for larger values of n. For example, the approximation for n=4 does not come close to the true value at time=1, but the approximation for n=256 essentially reaches 1. Similarly, the absolute error for n=4 decreases much slower than that of n=256. The error for n=4 gets close to 0 around t=0.75, but for n=256 it is at around t=0.55. Also, the plot of error is linear with respect to time, which is expected, because error scales with Δt, as shown in part (c). This is because Forward Euler is first-order.