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Contents

1 Assignment 1

- know the difference between $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and ${}_nP_k = \frac{n!}{(n-k)!}$ and when to use them: e.g does order matter, is there replacement if drawing etc.
- Probability laws, like:
  - for event space $\Omega = \{\omega_1, \ldots, \omega_n\}$,
  $$\mathbb{P}\left(\bigcup_{i=1}^{n} \omega_i\right) = 1$$
  consequence for $\Omega \supseteq A, A^c = \{x \in \Omega \mid x \notin A\}$:
  $$\mathbb{P}(A^c) = 1 - \mathbb{P}(A)$$
  - independence:
  $$A_1 \perp\!\!\!\perp A_2 \perp\!\!\!\perp \ldots \perp\!\!\!\perp A_n \iff \left\{\mathbb{P}(\bigcap_{i=1}^{n} A_i) = \prod_{i=1}^{n} \mathbb{P}(A_i)\right\}$$
  - conditional probability
  $$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \implies \left\{A \perp\!\!\!\perp B \implies \left[\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A)\mathbb{P}(B)}{\mathbb{P}(B)} = \mathbb{P}(A)\right]\right\}$$
- Set theory: e.g inclusion and exclusion principle (†) and everything leading up to it
  $$(\dagger) \quad \mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \Big[\mathbb{P}(A \cap B) + \mathbb{P}(B \cap C) + \mathbb{P}(A \cap C)\Big] + \mathbb{P}(A \cap B \cap C)$$
  if you can figure this out from a venn diagram you're good imo
- understand the probability of cards, dice, and balls in bins leading to:
- (stars and bars theorem)

  Given:

  $n > 0$

  $X_1 + X_2 + \ldots + X_k = n$ for $X_i \in \{1, 2, \ldots, n-(k-1)\}$

  the number of distinct ordered tuples $(X_1, \ldots, X_k)$ is $\binom{n-1}{k-1}$

  if we allow $X_i = 0$, then the number of distinct ordered tuples $(X_1, \ldots, X_k)$ is $\binom{n+k-1}{k-1}$
- Bayes' Theorem
  $$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(B \mid A)\mathbb{P}(A)^{\dagger}}{\mathbb{P}(B)}$$
  proof:
  $$\begin{aligned}\mathbb{P}(A \mid B) &= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \quad \text{(conditional probability)} \\ &= \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(B)} \quad \text{(commutative intersection)} \\ &= \frac{\mathbb{P}(B \mid A)\mathbb{P}(A)}{\mathbb{P}(B)} \quad \text{(conditional probability)}\end{aligned}$$

$\blacksquare$

- use of conditional probability:

  Let $A = A_1 \cup A_2 \cup A_3$ where $A_i \cap A_j = \emptyset \ \forall (i,j) \in \{1,2,3\}^2 \mid i \neq j$

  (A is made up of 3 parts A_1, A_2, A_3 that don't intersect)

  then because $A = \bigcup_{i=1}^{3} A_i$,

$$\mathbb{P}(A) = \sum_{i=1}^{3} \mathbb{P}(A \mid A_i)\mathbb{P}(A_i)$$

  proof:

$$\begin{aligned}
\sum_{i=1}^{3} \mathbb{P}(A \mid A_i)\mathbb{P}(A_i) &= \sum_{i=1}^{3} \mathbb{P}(A \cap A_i) \quad \text{(conditional } \mathbb{P}\text{)} \\
&= \sum_{i=1}^{3} \mathbb{P}(A_i) \quad \left\{ A_i \subseteq A \implies A_i \cap A = A_i \right\} \\
&= \mathbb{P}\left(\bigcup_{i=1}^{3} A_i \right) \quad \text{(because)} A_i \perp\!\!\!\perp A_j \ \forall \ i \neq j \\
&= \mathbb{P}(A)
\end{aligned}$$

  if you understand these proofs you get the gist of probability ■

- Assignment 1 Question 13(c) is a good exercise in probability. I have not looked at the posted solution though.

2 Assignment 2

- distributions – the ones that are on the midterm sheet. know when and how to use them, how to use their means and variances.

  be able to recognize which distribution bets fits the the question.

  ex:

  5 dice rolls $X_i \in \{1, 2, 3, 4, 5, 6\}$

  find probability that the number of $X_i = 3$ is 2 (2 dice rolled were 3)

  chance that each $X_i = 3 \sim \text{Bernoulli}(p = \frac{1}{6})$

  $\implies$ chance that all n independent $X_i = 3 \sim \text{Binomial}(n, p = \frac{1}{6})$.

  $\implies \mathbb{P}(\text{two 3's rolled})) = \binom{5}{2} \left(\frac{1}{6}\right)^2 \left(1 - \frac{1}{6}\right)^3 = \dots$

- basic integration from calc
- RV X with CDF F of pdf f:

$$F(\alpha) = \int_{-\infty}^{\alpha} f(x)\, dx$$

  $\longrightarrow f$ has support $\mathcal{S}_f$, so the integral can be thought of as

$$F(\alpha) = \mathbb{P}\left(X \leq \alpha\right) = \int_{\mathcal{S}^\star} f(x)\, dx$$

  where $\mathcal{S}^\star = (-\infty, \alpha) \cap \mathcal{S}_f$.

  a.k.a: only integrate over $(-\infty, \alpha)$ and where $f(x) \neq 0$.

- RV X with CDF F of pmf f:

$$F(\alpha) = \mathbb{P}(X \leq \alpha) = \sum_{i=-\infty}^{\alpha} \mathbb{P}(X = i)$$

- Properties: $F(\infty) = 1, F(-\infty) = 0$ (as limits, iykyk)
- $\mathbb{P}(X > a) = 1 - \mathbb{P}(X \leq A) = 1 - F(a)$
- $\mathbb{P}(a \leq X \leq B) = F(b) - F(a)$
- expectation and variance:

$$\mathbb{E}\left(\sum_{i=1}^{n} a_i X_i\right) = \left(\sum_{i=1}^{n} a_i \mathbb{E}\left(X_i\right)\right) \;\; \forall\; a_i \in \mathbb{R}, n \in \mathbb{N} \text{ and any random variables } X_i, i \in \{1, \dots, n\}$$

$$\mathbb{E}(X^n) = \int_{-\infty}^{\infty} x^n \cdot f(x)\, dx \quad \text{or} \quad \mathbb{E}(X^n) = \sum_{i=-\infty}^{\infty} i^n \cdot \mathbb{P}(X = i)$$

$$\text{Var}(X) = \mathbb{E}\left[\left(X - \mathbb{E}\left(X\right)\right)^2\right] = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

$$\text{Var}(X - a) = \text{Var}(X)$$

$$\text{Var}(a + 2X) = 2^2\, \text{Var}(X)$$

- conditional:

  given joint density $f_{X,Y}(x,y)$:

  $$f_x(x) = \int_{\mathcal{S}_y} f_{X,Y}(d,y)\,dy$$

  where $\mathcal{S}_y = \{y \in \mathbb{R} \mid f_{X,Y}(x,y) \neq 0\}$

- if you're given f_x, f_y can you know what the original $f_{X,Y}$ was? not necessarily, but can you think of a case when you could with a bit more information?

  copulas can help with this if you have an afternoon to lose learning about them.

- Markov Inequality

  $$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a} \quad \forall\ a > 0$$

- Chebyshev Inequality

  $$\mathbb{P}\left(|X - \mathbb{E}\left(X\right)| \geq k\sigma_x\right) \leq \frac{1}{k^2}$$

3 Assignment 3

- how to normalize a Normal probability and use Z-scores

$$X \sim \mathcal{N}(\mu, \sigma^2) \implies \frac{X - \mu}{\sigma} = Z \sim \mathcal{N}(0, 1)$$

$\Phi_Z(z)$ is the CDF of the *standard normal distribution*:

$$\begin{aligned}
\Phi_Z(z) &= \mathbb{P}(Z \leq z) \\
\Phi_Z(-z) &= \mathbb{P}(Z \leq -z) = 1 - \mathbb{P}(Z > z) \\
\mathbb{P}(Z > z) &= 1 - \mathbb{P}(Z \leq z) = 1 - \Phi_Z(z) \\
\mathbb{P}(a \leq Z \leq b) &= \Phi_Z(b) - \Phi_Z(a) \\
\mathbb{P}\left(X \leq a\right) &= \mathbb{P}\left(\frac{X - \mathbb{E}(X)}{\sqrt{\mathrm{Var}(X)}} \leq \frac{a - \mathbb{E}(X)}{\sqrt{\mathrm{Var}(X)}}\right) \\
&= \mathbb{P}\left(Z \leq \frac{a - \mathbb{E}(X)}{\sqrt{\mathrm{Var}(X)}}\right) \\
&= \Phi_Z\left(\frac{a - \mathbb{E}(X)}{\sqrt{\mathrm{Var}(X)}}\right)
\end{aligned}$$

- using CLT to estimate (after knowing what CLT is):

  $\longrightarrow$ estimate X with CLT:

  find $\mathbb{E}(X), \mathrm{Var}(X)$ based on information given: e.g X might be a sum of other variables, like

$$\chi_k^2 = \sum_{i=1}^{k} Z_i^2, \quad Z_i \sim \mathcal{N}(0, 1)$$

  estimate your probability with

$$\mathbb{P}(X \leq a) \simeq \Phi_Z\left(\frac{a - \mathbb{E}(X)}{\sqrt{\mathrm{Var}(X)}}\right)$$

- sample mean

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \xrightarrow[n \to \infty]{\dagger} \quad \mu_x$$

  †: law of large numbers

$$\mathbb{E}(\overline{X}) = \mu_x$$

$$\mathrm{Var}(\overline{X}) = \frac{\mathrm{Var}(X_i)}{n} \quad \text{when } iid$$

- sample variance:

$$s_x^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \overline{X})^2$$

$$\frac{(n-1)s_x^2}{\sigma_x^2} \sim \chi_{n-1}^2$$