Probaron que: en $[-L, L]$, $\omega_n = \frac{\pi n}{L} = \frac{2\pi n}{T}$

$$\langle \cos \omega_n t \mid \sin \omega_m t \rangle = 0 \qquad T/2 = L$$

$$\langle \sin \omega_n t \mid \sin \omega_m t \rangle = 0 \quad \text{si} \quad m \neq n$$

$$\langle \cos \omega_n t \mid \cos \omega_m t \rangle = 0 \quad \text{si} \quad m \neq n.$$

$\therefore$ $\sin \omega_n t$, $\cos \omega_n t$ es una familia ortogonal.

Teorema: (Dirichlet)

Sea f, continua por tramos y con un numero finito de max/min, entonces

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \omega_n t + b_n \sin \omega_n t \right)$$

excepto en sus discontinuidades donde la serie es el promedio $(f(0^+) + f(0^-))/2$.

$$\langle f(t) \mid \cos \omega_m t \rangle = \left\langle \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \hat{c}\,\omega_n t + b_n \hat{s}\,\omega_n t) \,\middle|\, \hat{c}\,\omega_m t \right\rangle$$

$$\frac{a_0}{2} \underbrace{\langle 1 \mid \hat{c}\,\omega_m t \rangle}_{0}^{\int_{-L}^{L} 1\,\hat{c}\,\omega_m t\,dt} + \sum_{n=1}^{\infty} \left(a_n \overbrace{\langle \hat{c}\,\omega_n t \mid \hat{c}\,\omega_m t \rangle}^{0} + b_n \overbrace{\langle \hat{s}\,\omega_n t \mid \hat{c}\,\omega_m t \rangle}^{0} \right)$$

si $n \neq m$

$$= a_m \langle \hat{c}\,\omega_m t \mid \hat{c}\,\omega_m t \rangle$$

$$\langle \hat{c}\,\omega_m t \mid \hat{c}\,\omega_m t \rangle = \int_{-L}^{L} \hat{c}^2\,\omega_m t\,dt = \frac{1}{2}\int_{-L}^{L} \left(1 + \hat{c}(2\omega_m t)\right) dt$$

$$\omega_m L = \frac{\pi m \cancel{L}}{\cancel{L}}$$

$$\frac{2L}{2} + \Big(\underbrace{\frac{1}{4\omega_m}\hat{s}(\omega_m L) + \frac{1}{4\omega_m}\hat{s}(\omega_m L)}_{0}\Big)$$

$$\int_{-L}^{L} f\cdot \hat{c}\,\omega_m t\,dt = a_m \cdot L \Rightarrow \boxed{a_m = \frac{1}{L}\int_{-L}^{L} f\cdot \hat{c}\,\omega_m t\,dt}$$

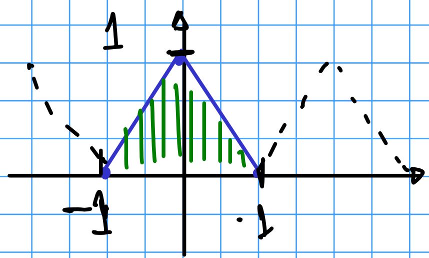
$$\frac{a_0}{2}\cdot 2L = \int_{-L}^{L} f\,dt \Rightarrow \boxed{a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt}$$

$$\boxed{b_n = \frac{1}{L}\int_{-L}^{L} f\cdot \hat{s}\,\omega_n t\,dt}$$

Ejemplo: Sea

$$f(t) = \begin{cases} 1+t & -1 < t < 0 \\ 1-t & 0 < t < 1 \end{cases}$$

Encuentre su representación como serie de Fourier real.



$L = 1$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt = \int_{-1}^{1} f(t)\,dt = 1.$$

$$a_m = \frac{1}{L}\int_{-L}^{L} f(t)\cos(\omega_m t)\,dt = \int_{-1}^{0}(1+t)\cos(\omega_m t)\,dt + \int_{0}^{1}(1-t)\cos\omega_m t\,dt$$

*	u	dv
+	$1+t$	$\cos(\omega_m t)$
-	1	$\frac{1}{\omega_m}\sin(\omega_m t)$
+	0	$\frac{-1}{\omega_m^2}\cos(\omega_m t)$

$$\left(\frac{1+t}{\omega_m}\sin(\omega_m t) + \frac{1}{\omega_m^2}\cos(\omega_m t)\right)\Bigg|_{-1}^{0}$$

$$\omega_m = \frac{\pi m}{L} = \pi m$$

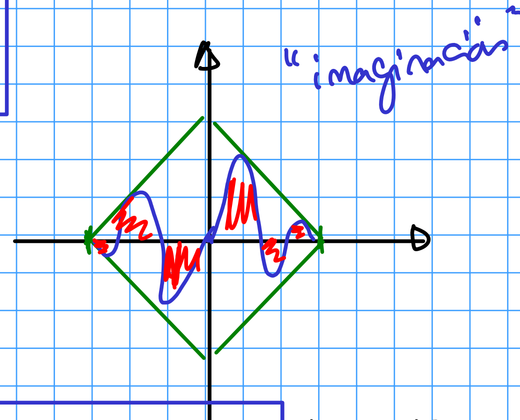
$$\left(\frac{1}{\pi m}\cdot 0 + \frac{1}{\pi^2 m^2}\cdot 1\right) - \left(\frac{0}{\omega_m}\cancel{\sin(-\pi)} + \frac{1}{\pi^2 m^2}\cos(-\pi m)\right)$$

0

$$\int_{-1}^{0}(1+t)\cos\omega_m t\,dt = \frac{1}{\pi^2 m^2}\left(1 - \cos(\pi m)\right)$$

$$\int_0^1 (1-t)\,\hat{c}\,w_m t\,dt = \frac{1}{\pi^2 m^2}\Big(1 - \cos(\pi m)\Big)$$

$$\boxed{a_m = \frac{2}{\pi^2 m^2}\Big(1 - \cos(\pi m)\Big)}$$



$$b_m = \frac{1}{L}\int_{-L}^{L} f(t)\sin w_m t\,dt = 0.$$

$w_m = \pi m$

$$\boxed{f(t) = \frac{1}{2} + \sum_{m=1}^{\infty} \frac{2}{\pi^2 m^2}\Big(1 - \cos(\pi m)\Big)\cos(w_m t)}$$

$$= \frac{1}{2} + \frac{2}{\pi^2}\big(1 - \hat{c}(\pi)\big)\hat{c}t + \frac{2}{\pi^2\cdot 4}\big(\cancelto{0}{1 - \hat{c}(2\pi)}\big)\hat{c}(2\pi t) +$$

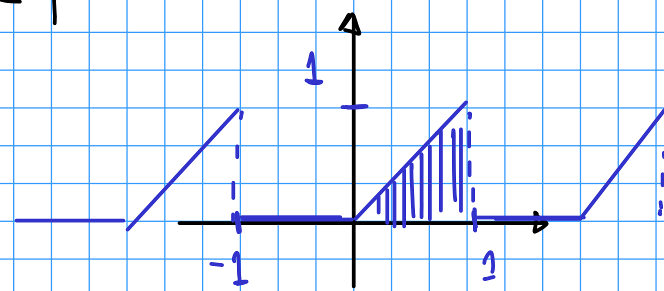
$$= \frac{1}{2} + \frac{4}{\pi^2}\hat{c}(\pi t) + \frac{4}{\pi^2\cdot 9}\hat{c}(3\pi t) + \frac{4}{\pi^2\cdot 25}\hat{c}(5\pi t) + \dots$$

$$\hat{c}(m\pi) = (-1)^m \Rightarrow 1 - \cos(m\pi) = \begin{cases} 0 & \text{si } m \text{ par} \\ 2 & \text{si } m \text{ impar} \end{cases}$$

$$f(t) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{4}{\pi^2(2k-1)^2}\cdot\cos\big(\pi(2k-1)t\big)$$

__Ejemplo:__ Sea

$$f(t) = \begin{cases} 0 & -1 < t < 0 \\ t & 0 < t < 1 \end{cases} \qquad f(t+2) = f(t)$$



calcule su serie de Fourier.

sol: $L = 1, \quad \omega_m = \frac{\pi m}{L} = \pi m \qquad m \in \mathbb{Z}^+$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f\,dt = \frac{1}{2}$$

	u	du
+	t	$\hat{c}(\pi m t)$
-	1	$\frac{\hat{s}(\pi m t)}{\pi m}$
+	0	$-\frac{\hat{c}(\pi m t)}{\pi^2 m^2}$

$$a_m = \frac{1}{L}\int_{-L}^{L} f(t)\,\hat{c}(\omega_m t)\,dt = \int_0^1 t\,\hat{c}(\pi m t)\,dt$$

$$= \left(\frac{t\,\hat{s}(\pi m t)}{\pi m} + \frac{\hat{c}(\pi m t)}{\pi^2 m^2}\right)\Bigg|_0^1 = \left(0 + \frac{(-1)^m}{\pi^2 m^2}\right) - \left(0 + \frac{1}{\pi^2 m^2}\right)$$

$$\boxed{a_m = \frac{(-1)^m - 1}{\pi^2 m^2}}$$

$$b_m = \frac{1}{L}\int_{-L}^{L} f(t)\,\hat{s}(\omega_m t)\,dt = \int_0^1 t\,\hat{s}(\pi m t)\,dt$$

$$\begin{array}{ccc} & u & dv \\ + & t & \hat{s}(\pi m t) \\ - & 1 & -\dfrac{\hat{c}(\pi m t)}{\pi m} \\ + & 0 & -\dfrac{\hat{s}(\pi m t)}{\pi^2 m^2} \end{array}$$

$$= \left(\frac{-t\,\hat{c}(m\pi t)}{m\pi} + \frac{\hat{s}(m\pi t)}{\pi^2 m^2} \right)\Bigg|_0^1$$

$$= \left(\frac{(-1)^{m+1}}{m\pi} + \cancelto{0}{\frac{\hat{s}(m\pi)}{\pi^2 m^2}} \right) - \left(0 + 0 \right) = \frac{(-1)^{m+1}}{m\pi}.$$

$$\boxed{b_m = \frac{(-1)^{m+1}}{m\pi}}$$

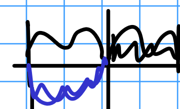
$$f(t) = \frac{1}{4} + \sum_{m=1}^{\infty} \left(\frac{(-1)^m - 1}{\pi^2 m^2}\,\hat{c}(\pi m t) + \frac{(-1)^{m+1}}{m\pi}\,\hat{s}(\pi m t) \right)$$

$$= \frac{1}{4} + \sum_{m=1}^{\infty} \left(\frac{-2}{\pi^2 (2m-1)}\,\hat{c}(\pi(2m-1)t) + \frac{(-1)^{m+1}}{m\pi}\,\hat{s}(\pi m t) \right)$$

$$\hat{s}x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\hat{c}x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

Paridad e Imparidad.



$f(-x) = f(x)$ Par

$f(-x) = -f(x)$ Impar.

(par)(par) = par

(par)(impar) = impar

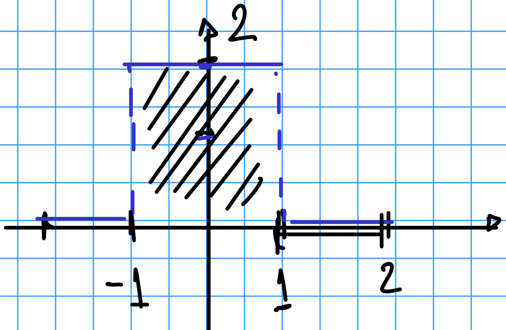
(impares)(impares) = par

$$\int_{-L}^{L} g(t)\,dt = \begin{cases} 2\displaystyle\int_0^L g(t)\,dt & g \text{ par} \\ 0 & g \text{ impar} \end{cases}$$

$$f(t+T)=f(t) \Rightarrow \int_{-L}^{L} f(t)\,dt = \int_{\alpha}^{\alpha+T} f(t)\,dt$$

Ejemplo: Sea

$$f(t)=\begin{cases} 0 & \text{si } -2<t<-1 \\ 2 & \text{si } -1<t<1 \\ 0 & \text{si } 1<t<2 \end{cases} \qquad f(t+4)=f(t)$$



$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt = \frac{1}{2}\cdot 4 = 2$$

$$b_m = \frac{1}{L}\int_{-L}^{L} \underbrace{f}_{\text{par}}\ \underbrace{\hat{s}\,\omega_m t}_{\text{imp}}\,dt = 0. \quad \text{por simetría}$$

$$\omega_m = \frac{\pi m}{L} =$$

$$a_m = \frac{1}{L}\int_{-L}^{L} \underbrace{f}_{\text{par}}\ \underbrace{\hat{c}(\omega_m t)}_{\text{par}}\,dt = \frac{2}{L}\int_{0}^{L} f\,\hat{c}(\omega_m t)\,dt = \frac{2}{2}\int_{0}^{1} 2\cdot\hat{c}\left(\frac{\pi m}{2}t\right)dt$$

$$2\cdot \left.\frac{\hat{s}\left(\frac{\pi m t}{2}\right)}{\frac{\pi m}{2}}\right|_{0}^{1} = \frac{4}{\pi m}\left(\hat{s}\left(\frac{\pi m}{2}\right) - 0\right) = \frac{4\,\hat{s}\left(\frac{\pi m}{2}\right)}{\pi m}$$

$$f(t) = 1 + \sum_{m=1}^{\infty} \frac{4\hat{s}\left(\frac{\pi m}{2}\right)}{\pi m} \cdot \hat{c}\left(\frac{\pi m t}{2}\right)$$

$$= 1 + \frac{4}{\pi}\hat{c}\left(\frac{\pi t}{2}\right) - \frac{4}{\pi \cdot 3}\hat{c}\left(\frac{3\pi t}{2}\right) + \frac{4}{5\pi}\hat{c}\left(\frac{5\pi t}{2}\right) + \dots$$

$$= \boxed{1 + \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)}\hat{c}\left(\frac{(2m-1)\pi t}{2}\right)}$$