Example: Given the intrinsic equations of circular helix as

$$k(s) = \frac{\alpha}{c^2} \qquad \tau(s) = \frac{\beta}{c^2} \qquad \text{where } c = \sqrt{\alpha^2+\beta^2}$$

Please derive the parametric equations of circular helix.

Solution: By Frenet-Serret Formulae, we have

$$\frac{dt}{ds} = \frac{\alpha}{c^2} n \qquad \frac{dn}{ds} = -\frac{\alpha}{c^2} t + \frac{\beta}{c^2} b \qquad \frac{db}{ds} = -\frac{\beta}{c^2} n$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\frac{d^2t}{ds^2} = \frac{\alpha}{c^2}\frac{dn}{ds} \qquad \frac{d^2n}{ds^2} = -\frac{\alpha}{c^2}\frac{dt}{ds} + \frac{\beta}{c^2}\boxed{\frac{db}{ds}}$$

$\because \frac{dt}{ds} = \frac{\alpha}{c^2} n$

$\therefore n = \frac{c^2}{\alpha}\frac{dt}{ds}$

$$\Downarrow$$

$$\frac{d^3t}{ds^3} = \frac{\alpha}{c^2}\frac{d^2n}{ds^2}$$

$$\frac{d^2n}{ds^2} = -\frac{\alpha}{c^2}\frac{dt}{ds} - \frac{\beta^2}{c^4} n = -\frac{\alpha}{c^2}\frac{dt}{ds} - \frac{\beta^2}{c^4}\cdot\frac{c^2}{\alpha}\frac{dt}{ds}$$

$$= -\frac{\alpha^2+\beta^2}{\alpha c^2}\frac{dt}{ds}$$

$$= -\frac{1}{\alpha}\frac{dt}{ds}$$

$$\frac{d^3t}{ds^3} = \frac{\alpha}{c^2}\left(-\frac{1}{\alpha}\right)\frac{dt}{ds}$$

$$\Rightarrow \frac{d^3t}{ds^3} + \frac{dt}{ds}\frac{1}{c^2} = 0.$$

$$\because t = \frac{dr}{ds} \Rightarrow \frac{d^4r}{ds^4} + \frac{1}{c^2}\frac{d^2r}{ds^2} = 0$$

The general solution to this fourth order differential equation is

$$r(s) = C_1 + C_2 s + C_3 \cos\frac{s}{c} + C_4 \sin\frac{s}{c}$$

where the coefficients C_1, C_2, C_3, C_4 can be determined by the initial conditions.

Thus, the parametric equation $r(s)$ can be obtained.