Reducing Cross Correlation Interference with Complementary Pairs

David Egolf

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Our Requirements and Goals

We want to fire N code pairs $\{P_1, P_2, ..., P_N\}$, in two transmit events. Each code pair P_i has two sequences: $\{P_{i,1}, P_{i,2}\}$. The length of each of these sequences, for all code pairs, is L.

Pairs Must be Complementary

Each code pair is required to be "complementary" - it must satisfy:

$$CCF(P_{i,1}, P_{i,1})[n] + CCF(P_{i,2}, P_{i,2})[n] = k_i\delta[n]$$

where $\delta[n]$ is the discrete delta function and CCF is the cross correlation function. For our purposes, the cross correlation function of two sequences f and g of length L is:

$$CCF(f, g)[n] = \sum_{m=-\infty}^{\infty} f[m]g[m+n]$$

where the sequences are zero for all values of m unless specified otherwise.

We Desire Good Cross-correlation Properties

With each pair P_i, we associate a transmit region T_i and a receive region R_i. When decoding RF scatter data at receive region R_i to produce D_i, we do it this way:

$$\begin{aligned} D_i &= D_{i,1} + D_{i,2} \\ D_{i,1} &= CCF(P_{i,1}, RF_{i,1}) \\ D_{i,2} &= CCF(P_{i,2}, RF_{i,2}) \end{aligned}$$

where $RF_{i,1}$ is the raw RF data received at receive region R_i after the first set of codes $\{P_{1,1}, P_{2,1}, ..., P_{N,1}\}$ have been transmitted, and $RF_{i,2}$ is the raw RF data received at receive region R_i after the second set of codes $\{P_{1,2}, P_{2,2}, ..., P_{N,2}\}$ have been transmitted.

We assume for the moment that the data received at R_i on the j_{th} transmit is the sum of the transmitted data, with delays:

$$RF_{i,j} = W_{i,1}(P_{1,j}) + W_{i,2}(P_{2,j}) + ... + W_{i,N}(P_{N,j})$$

where $W_{i,j}$ is a "wait" function, which delays a function by some delay $\Delta_{i,j}$ (where i is the associated receive region, and j is the associated code).

$$W_{i,j}(f[n]) = f[n - \Delta_{i,j}]$$

This function adds in the delay in arrival at region R_i of codes from code pair P_j.

So, the decoded data is:

$$\begin{aligned} D_i &= CCF(P_{i,1}, RF_{i,1}) + CCF(P_{i,2}, RF_{i,2}) \\ &= CCF\left(P_{i,1}, \sum_{m=1}^{N} W_{i,m}(P_{m,1})\right) + CCF\left(P_{i,2}, \sum_{m=1}^{N} W_{i,m}(P_{m,2})\right) \end{aligned}$$

To break this down further, we can realize that the cross correlation function distributes over a sum in the second argument:

$$\begin{aligned} CCF\left(f, \sum_{i=1}^{N} g_i\right) &= \sum_{m=1}^{L-n} f[m](g_1[m+n] + g_2[m+n] + ... + g_N[m+n]) \\ &= \sum_{m=1}^{L-n} f[m]g_1[m+n] + f[m]g_2[m+n] + ... + f[m]g_N[m+n]) \\ &= \sum_{m=1}^{L-n} f[m]g_1[m+n] + ... + \sum_{m=1}^{L-n} f[m]g_N[m+n] \\ &= CCF(f, g_1) + CCF(f, g_2) + ... + CCF(f, g_N) \\ \implies CCF\left(f, \sum_{i=1}^{N} g_i\right) &= \sum_{i=1}^{N} CCF(f, g_i) \end{aligned}$$

Therefore, the decoded data is:

$$\begin{aligned} D_i &= CCF\left(P_{i,1}, \sum_{m=1}^{N} W_{i,m}(P_{m,1})\right) + CCF\left(P_{i,2}, \sum_{m=1}^{N} W_{i,m}(P_{m,2})\right) \\ &= \sum_{m=1}^{N} CCF(P_{i,1}, W_{i,m}(P_{m,1})) + \sum_{m=1}^{N} CCF(P_{i,2}, W_{i,m}(P_{m,2})) \\ &= \sum_{m=1}^{N} CCF(P_{i,1}, W_{i,m}(P_{m,1})) + CCF(P_{i,2}, W_{i,m}(P_{m,2})) \end{aligned}$$

Now we want use the fact that each pair is complementary:

$$D_i = CCF(P_{i,1}, W_{i,i}(P_{i,1})) + CCF(P_{i,2}, W_{i,i}(P_{i,2})) + \sum_{m=\{1,\ldots,N\}, m\neq i} CCF(P_{i,1}, W_{i,m}(P_{m,1})) + CCF(P_{i,2}, W_{i,m}(P_{m,2}))$$

We need another property of the CCF function:

$$\begin{aligned} CCF(f[k], g[k-a])[n] &= \sum_{m=-\infty}^{\infty} f[m]g[(m-a)+n] \\ &= \sum_{m=-\infty}^{\infty} f[m]g[m+(n-a)] \\ &= CCF(f[k], g[k])[n-a] \end{aligned}$$

Applying this property and complementarity to the first two pairs yields:

$$\begin{aligned} CCF(P_{i,1}, W_{i,i}(P_{i,1})) + CCF(P_{i,2}, W_{i,i}(P_{i,2})) &= CCF(P_{i,1}[n], P_{i,1}[n-\Delta_{i,i}]) + CCF(P_{i,2}[n], P_{i,2}[n-\Delta_{i,i}]) \\ &= CCF(P_{i,1}, P_{i,1})[n-\Delta_{i,i}] + CCF(P_{i,2}, P_{i,2})[n-\Delta_{i,i}] \\ &= \big(CCF(P_{i,1}, P_{i,1}) + CCF(P_{i,2}, P_{i,2})\big)[n-\Delta_{i,i}] \\ &= \delta[n-\Delta_{i,i}] \end{aligned}$$

The decoded data is therefore:

$$D_i = k\delta[n-\Delta_{i,i}] + \sum_{m=\{1,\ldots,N\}, m\neq i} CCF(P_{i,1}, W_{i,m}(P_{m,1})) + CCF(P_{i,2}, W_{i,m}(P_{m,2}))$$

We want D_i to be as much like $\Delta_{i,i}$ as possible. Therefore:

$$\text{our rough goal is to make this small:} \quad \sum_{m=\{1,\ldots,N\}, m\neq i} CCF(P_{i,1}, W_{i,m}(P_{m,1})) + CCF(P_{i,2}, W_{i,m}(P_{m,2}))$$

The sum is a sequence, and so there are different things that we could minimize in this sequence. For the moment, let's minimize the maximum value over the sequence terms in the sum.

If we denote the m_{th} "side lobe term" as $SLT_{i,m}$:

$$SLT_{i,m}[n] = \left(CCF(P_{i,1}, W_{i,m}(P_{m,1})) + CCF(P_{i,2}, W_{i,m}(P_{m,2})\right)[n]$$

And define the "noise" N_i for the i_{th} receive region as follows:

$$N_i = \max_{m,n} \left|SLT_{i,m}[n]\right|$$

Then our goal is to minimize the worse case noise N:

$$N = \frac{\max_i N_i}{\min_i |k_i|}$$

If a set of pairs P has a small noise $N(P)$, then it is said to have good cross correlation properties.

Intuitively, we are making each term in the cross correlation interference sum small compared to the weakest signal that we can recover. We achieve this by making the largest term in the cross correlation interference sum as small as we can, while making the signal we receive from each complementary pair as large as we can.

Relevant Info From Papers

In this section, we cite relevant results regarding how good we can make the cross correlation interference relative the signal strength - how small we can make N.

Gran and Jensen: *Spatial encoding with code division technique for ultrasound imaging*