Understanding Convolution

David Egolf

September 12, 2016

Definition

The convolution of two sequences $x[n]$ and $h[n]$:

$$(x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Intuitively, we are placing a shifted copy of the sequence h centered at $x = k$, and multiplying this by the k_{th} element in the sequence $x[n]$. We do this for all elements $x[k]$ in the input sequence and add the results.

Note that convolution is commutative, distributive, and associative.

Output of LTI System

Assume T is a linear time invariant system. Then:

$$\begin{aligned} T\{\delta[n]\} &= h[n] \text{ (impulse response)} \\ \implies T\{x[n]\} &= \sum_{k=-\infty}^{k=\infty} x[k]h[n-k] \end{aligned}$$

Model Ultrasound System as LTI System

Consider a single ultrasound transducer, and assume that we use it to transmit a signal, which is then reflected and received by the transducer. Let us define an ultrasound system U that maps from transducer input excitation to the final signal decoded by the transducer:

$$U = T_x \circ M_x \circ R_x$$

where T_x is the transmission operator, M_x is the reflection operator (acts like a "mirror"), and R_x is the receiving operator.

If we ignore the transmission delay, and assume that the reflected signal is identical to the transmitted signal up to a change in amplitude, then:

$$M_x\{x[n]\} = A \cdot x[n]$$

where $A \in \mathbb{R}$.

In our simulations we assume that the both T_x and R_x are LTI systems, with the same impulse response. Call this common transducer impulse response h.

Using these definitions, we can calculate the output of the ultrasound system:

$$\begin{aligned} U\{x[n]\} &= T_x \circ M_x \circ R_x\{x[n]\} \\ &= h[n] * (A \cdot h[n] * x[n]) \\ &= A \cdot \ (h[n] * h[n]) * x[n] \end{aligned}$$

where we have used the fact that convolution is commutative.

Motivation

So, in order to understand the action of the ultrasound system, it would be useful to understand the properties of $h[n] * h[n]$, since this is the impulse response of the entire system (up to a scalar multiple).

Problem Statement

Investigate the properties of the self convolution $(h * h)[n]$ of a sequence $h : \mathbb{Z} \to \mathbb{R}$, in the context of an ultrasound system.

Solution

Causal

I assume there is no noise in the ultrasound system to be modeled. I assume that in a noise free ultrasound system, the system will not begin to transmit data prior to excitation, and the system will not begin to receive data prior to a transmitted signal hitting the receiver. Therefore:

$$h[n] = 0 \text{ for } n < 0 \implies (h * h)[n] = 0 \text{ for } n < 0$$

This implies that the LTI system $h * h$ is causal.

Stable

I assume that if we stop exciting the transducer, then after a finite amount of time the ultrasound receiver will stop receiving anything. That is:

$$(h * h)[n] = 0 \text{ for } n \geq N$$

This implies that we are working with a finite impulse response system, and therefore the system is stable.

Not Memory-less

The output $y[n]$ depends on all values of the input $h[n]$, not just the current value of n. Therefore, the system is not memory-less.

Equation for Output

The output of the system $(h * h)[n]$ is explicitly:

$$(h * h)[n] = \sum_{k=-\infty}^{\infty} h[k]h[n-k]$$

Since the system is causal, we only need to sum over the terms where $k \geq 0$ and $n - k \geq 0 \implies k \leq n$:

$$(h * h)[n] = \sum_{k=0}^{n} h[k]h[n-k]$$

To get some intuition, we write out this sum explicitly in the case when $h[0] = 1, h[1] = 2, h[2] = 3, h[3] = 4, h[4] = 5$ and $h[j] = 0$ for all other $j \in \mathbb{Z}$:

$$(h * h)[n] = h[0]h[n] + h[1]h[n-1] + h[2]h[n-2] + h[3]h[n-3] + h[4]h[n-4]$$

Nonzero Output Region As Function of Length

Let L be an integer called the "length" of the impulse response. We provide elements $h[0], h[1], ..., h[L-1]$ to MATLAB when specifying the impulse response. We require $L \geq 1$ and $h[n] = 0$ for all $n \geq L$.

We assume that our impulse response starts at zero and ends at zero, so set $h[0] = h[L-1] = 0$.

Using this information, we can rewrite the form of the output $(h * h)[n]$. We are interesting in determining exactly at which times the output can be nonzero. The output $(h * h)[n]$ will be zero at n if:

$$h[k]h[n-k] = 0 \text{ for } k = 0, 1, .., n$$

Since we assume $h[0] = 0$ and $(h * h)[n] = 0$ for all $n \geq L-1$, we can reduce the number of terms under consideration. Specifically, if $n \geq L-1$, then the output is zero, and if $n \leq 0$ then the output is zero. So, it remains to consider the cases in which $1 \leq n \leq L-2$. These are the only cases in which would possibly get nonzero output.

We can further restrict these cases by realizing that if $L \leq 2$, then the entire sequence is zero and so the output will be zero. So, we only need to consider the cases $1 \leq n \leq L-2$ where $L \geq 3$. We want to know for which of these n values we have a chance for nonzero output, as a function of L (clearly the maximum n for nonzero output will increase with L).

Our strategy is to start small and search for a pattern:

If $n = 1$:

$$h[k]h[n-k] = h[k]h[1-k]$$

Since $1 - k \leq 0$ for $k = 1, .., L-2$, the output is always zero in this case. As a result, we only need to consider the possible nonzero cases as consisting of $2 \leq n \leq L-2$.

If $n = 2$:

$$h[k]h[n-k] = h[k]h[2-k]$$

The possible nonzero $h[i]$ range is $i = 1, 2.., L-2$. Checking when we are in this range:

$$\begin{aligned}
1 \leq 2 - k &\leq L - 2 \\
&\implies k \leq 1 \text{ and} \\
&\implies 4 \leq L + k \implies k \geq 4 - L \\
&\text{Together:} \\
4 - L \leq k &\leq 1
\end{aligned}$$

In order to satisfy this inequality, we need:

$$4 - L \leq 1 \implies L \geq 3$$

We also need:

$$4 - L \leq k_{max}, 1 \geq k_{min}$$

Where k_{max} is the largest value k can take on, and k_{min} is the smallest value k can take on, while preserving the fact that $h[k]$ might be nonzero. From our work before, $k_{max} = L-2$ and $k_{min} = 1$.

So, we need:

$$\begin{gathered}
4 - L \leq L - 2, 1 \geq 1 \\
\implies 2L \geq 6 \implies L \geq 3
\end{gathered}$$

So, $(h * h)[2]$ has a chance to be nonzero when $L \geq 3$.

Let's generalize this argument, setting $n = a$:

$$h[k]h[n-k] = h[k]h[a-k]$$

The possible nonzero $h[i]$ range is $i = 1, 2.., L-2$. Checking when we are in this range:

$$\begin{aligned} 1 \leq a - k \leq L - 2 & \\ & \implies k \leq a - 1 \text{ and} \\ & \implies a + 2 \leq L + k \implies k \geq a + 2 - L \\ \text{Together:} & \\ a + 2 - L \leq k \leq a - 1 & \implies L \geq 3 \end{aligned}$$

We also need:

$$a + 2 - L \leq k_{max}, a - 1 \geq k_{min}$$

Where k_{max} is the largest value k can take on, and k_{min} is the smallest value k can take on, while preserving the chance that $h[k]$ might be nonzero. From our work before, $k_{max} = L - 2$ and $k_{min} = 1$.

So, we need:

$$\begin{aligned} a + 2 - L \leq L - 2, \ a - 1 \geq 1 \\ \implies 2L \geq a + 4 \implies L \geq \frac{a+4}{2} \end{aligned}$$

Substituting $n = a$, we find $(h * h)[n]$ has a chance to be nonzero when $L \geq \frac{n+4}{2}$. That means when we are computing the sum to get the response, we only need to add terms from $n = 1$ to $2L - 4$.