Understanding Convolution

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Definition

The convolution of two sequences $x[n]$ and $h[n]$:

$$(x * h)[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Intuitively, we are placing a shifted copy of the sequence h centered at $x = k$, and multiplying this by the k_{th} element in the sequence $x[n]$. We do this for all elements $x[k]$ in the input sequence and add the results.

Note that convolution is commutative, distributive, and associative.

Output of LTI System

Assume T is a linear time invariant system. Then:

$$\begin{aligned} T\{\delta[n]\} &= h[n] \text{ (impulse response)} \\ \implies T\{x[n]\} &= \sum_{k=-\infty}^{k=\infty} x[k]h[n-k] \end{aligned}$$

Model Ultrasound System as LTI System

Consider a single ultrasound transducer, and assume that we use it to transmit a signal, which is then reflected and received by the transducer. Let us define an ultrasound system U that maps from transducer input excitation to the final signal decoded by the transducer:

$$U = T_x \circ M_x \circ R_x$$

where T_x is the transmission operator, M_x is the reflection operator (acts like a "mirror"), and R_x is the receiving operator.

If we ignore the transmission delay, and assume that the reflected signal is identical to the transmitted signal up to a change in amplitude, then:

$$M_x\{x[n]\} = A \cdot x[n]$$

where $A \in \mathbb{R}$.

In our simulations we assume that the both T_x and R_x are LTI systems, with the same impulse response. Call this common transducer impulse response h.

Using these definitions, we can calculate the output of the ultrasound system:

$$\begin{aligned} U\{x[n]\} &= T_x \circ M_x \circ R_x\{x[n]\} \\ &= h[n] * (A \cdot h[n] * x[n]) \\ &= A \cdot \ (h[n] * h[n]) * x[n] \end{aligned}$$

where we have used the fact that convolution is commutative.

Motivation

So, in order to understand the action of the ultrasound system, it would be useful to understand the properties of $h[n] * h[n]$, since this is the impulse response of the entire system (up to a scalar multiple).

Problem Statement

Investigate the properties of the self convolution $(h * h)[n]$ of a sequence $h : \mathbb{Z} \to \mathbb{R}$, in the context of an ultrasound system.

Solution

Causal

I assume there is no noise in the ultrasound system to be modeled. I assume that in a noise free ultrasound system, the system will not begin to transmit data prior to excitation, and the system will not begin to receive data prior to a transmitted signal hitting the receiver. Therefore:

$$h[n] = 0 \text{ for } n < 0 \implies (h * h)[n] = 0 \text{ for } n < 0$$

This implies that the LTI system $h * h$ is causal.

Stable

I assume that if we stop exciting the transducer, then after a finite amount of time the ultrasound receiver will stop receiving anything. That is:

$$(h * h)[n] = 0 \text{ for } n \geq N$$

This implies that we are working with a finite impulse response system, and therefore the system is stable.

Not Memory-less

The output $y[n]$ depends on all values of the input $h[n]$, not just the current value of n. Therefore, the system is not memory-less.

Equation for Output

The output of the system $(h * h)[n]$ is explicitly:

$$(h * h)[n] = \sum_{k=-\infty}^{\infty} h[k]h[n-k]$$

Since the system is causal, we only need to sum over the terms where $k \geq 0$ and $n - k \geq 0 \implies k \leq n$:

$$(h * h)[n] = \sum_{k=0}^{n} h[k]h[n-k]$$

To get some intuition, we write out this sum explicitly in the case when $h[0] = 1, h[1] = 2, h[2] = 3, h[3] = 4, h[4] = 5$ and $h[j] = 0$ for all other $j \in \mathbb{Z}$:

$$(h * h)[n] = h[0]h[n] + h[1]h[n-1] + h[2]h[n-2] + h[3]h[n-3] + h[4]h[n-4]$$

Nonzero Output Region As Function of Length

Let L be an integer called the "length" of the impulse response. We provide elements $h[0], h[1], ..., h[L-1]$ to MATLAB when specifying the impulse response. We require $L \geq 1$ and $h[n] = 0$ for all $n \geq L$.

We assume that our impulse response starts at zero and ends at zero, so set $h[0] = h[L-1] = 0$.

Using this information, we can rewrite the form of the output $(h * h)[n]$. We are interesting in determining exactly at which times the output can be nonzero. The output $(h * h)[n]$ will be zero at n if:

$$h[k]h[n-k] = 0 \text{ for } k = 0, 1, .., n$$

Since we assume $h[0] = 0$ and $(h * h)[n] = 0$ for all $n \geq L-1$, we can reduce the number of terms under consideration. Specifically, if $n \geq L-1$, then the output is zero, and if $n \leq 0$ then the output is zero. So, it remains to consider the cases in which $1 \leq n \leq L-2$. These are the only cases in which would possibly get nonzero output.

We can further restrict these cases by realizing that if $L \leq 2$, then the entire sequence is zero and so the output will be zero. So, we only need to consider the cases $1 \leq n \leq L-2$ where $L \geq 3$. We want to know for which of these n values we have a chance for nonzero output, as a function of L (clearly the maximum n for nonzero output will increase with L).

Our strategy is to start small and search for a pattern:

If $n = 1$:

$$h[k]h[n-k] = h[k]h[1-k]$$

Since $1 - k \leq 0$ for $k = 1, .., L-2$, the output is always zero in this case. As a result, we only need to consider the possible nonzero cases as consisting of $2 \leq n \leq L-2$.

If $n = 2$:

$$h[k]h[n-k] = h[k]h[2-k]$$

The possible nonzero $h[i]$ range is $i = 1, 2.., L-2$. Checking when we are in this range:

$$\begin{aligned} 1 \leq 2-k &\leq L-2 \\ &\implies k \leq 1 \text{ and} \\ &\implies 4 \leq L+k \implies k \geq 4-L \\ &\text{Together:} \\ 4-L \leq k &\leq 1 \end{aligned}$$

In order to satisfy this inequality, we need:

$$4 - L \leq 1 \implies L \geq 3$$

We also need:

$$4 - L \leq k_{max}, 1 \geq k_{min}$$

Where k_{max} is the largest value k can take on, and k_{min} is the smallest value k can take on, while preserving the fact that $h[k]$ might be nonzero. From our work before, $k_{max} = L-2$ and $k_{min} = 1$.

So, we need:

$$\begin{aligned} 4 - L \leq L-2, 1 \geq 1 \\ \implies 2L \geq 6 \implies L \geq 3 \end{aligned}$$

So, $(h * h)[2]$ has a chance to be nonzero when $L \geq 3$.

Let's generalize this argument, setting $n = a$:

$$h[k]h[n-k] = h[k]h[a-k]$$

The possible nonzero $h[i]$ range is $i = 1, 2.., L-2$. Checking when we are in this range:

$$\begin{aligned} 1 \leq a-k \leq L-2 & \\ & \implies k \leq a-1 \text{ and} \\ & \implies a+2 \leq L+k \implies k \geq a+2-L \\ \text{Together:} & \\ a+2-L \leq k \leq a-1 & \implies L \geq 3 \end{aligned}$$

We also need:

$$a+2-L \leq k_{max}, a-1 \geq k_{min}$$

Where k_{max} is the largest value k can take on, and k_{min} is the smallest value k can take on, while preserving the chance that $h[k]$ might be nonzero. From our work before, $k_{max} = L-2$ and $k_{min} = 1$.

So, we need:

$$\begin{aligned} a+2-L \leq L-2,\ a-1 \geq 1 & \implies a \geq 2 \\ \implies 2L \geq a+4 & \implies L \geq \frac{a+4}{2} \end{aligned}$$

Substituting $n = a$, we find $(h*h)[n]$ has a chance to be nonzero when $L \geq \frac{n+4}{2}$ and when $n \geq 2$.

Largest and Smallest Nonzero $(h*h)[n]$

Using this information, we can find the first possibly nonzero term. Trying $n = 2$, the condition for the output to be nonzero is:

$$L \geq \frac{2+4}{2} = 3 \implies L \geq 3$$

So, $n = 2$ is the smallest value of n for which $(h*h)[n]$ is possibly not zero.

Next, let's find the largest $n = n_{max}$ for which $(h*h)[n]$ is possibly nonzero. We know that this $n \geq 2$. Also, for $(h*h)[n_{max}]$ to be possibly nonzero, we need:

$$\begin{aligned} & 2L \geq n_{max}+4 \\ \implies & n_{max} \leq 2L-4 \end{aligned}$$

Choosing the largest element in this set, we get $n_{max} = 2L-4$.

So, $(h*h)[n]$ is possibly nonzero for $2 \leq n \leq 2L-4$.

Properties of $h*h$ So Far

Adding this new information about the nonzero range for n:

$$(h*h)[n] = \sum_{k=0}^{n} h[k]h[n-k] \text{ (possibly nonzero for } 2 \leq n \leq 2L-4)$$

This system is LTI, causal, stable, and not memory-less.

Lack of Symmetry in Self Convolution

It turns out that the convolution of a sequence with itself is NOT symmetric (doesn't form a palindrome when written out), even though the autocorrelation of a sequence with itself is! For example, if $h = [0, 1, 2, 0]$ (starting at $n = 0$), then:

$$(h*h)[0] = \sum_{k=0}^{0} h[k]h[0-k] = h[0]h[0] = 0$$

$$(h*h)[1] = \sum_{k=0}^{1} h[k]h[1-k] = h[0]h[1] + h[1]h[0] = 0$$

$$(h*h)[2] = \sum_{k=0}^{2} h[k]h[2-k] = h[0]h[2] + h[1]h[1] + h[2]h[0] = 0 + 1 + 0 = 1$$

$$(h*h)[3] = \sum_{k=0}^{3} h[k]h[3-k] = h[0]h[3] + h[1]h[2] + h[2]h[1] + h[3]h[0] = 0 + 2 + 2 + 0 = 4$$

$$(h*h)[4] = \sum_{k=0}^{4} h[k]h[4-k] = h[0]h[4] + h[1]h[3] + h[2]h[2] + h[3]h[1] + h[4]h[0] = 0 + 0 + 4 + 0 = 4$$

And since $n_{max} = 2L - 4 = 4$, $(h*h)[n] = 0$ for all larger n.

Sufficient Condition for Symmetry in Self Convolution

The autocorrelation of a sequence is known to be symmetric. The autocorrelation for a real sequence is:

$$ACF(h[n]) = \sum_{k=-\infty}^{\infty} h[k]h[k-n]$$

For comparison, here is the the definition of a sequence convolved with itself:

$$\begin{aligned}(h*h)[n] &= \sum_{k=-\infty}^{\infty} h[k]h[n-k] \\ &= \sum_{k=-\infty}^{\infty} h[k]h[-(k-n)]\end{aligned}$$

If we assume that $h[n]$ satisfies $h[n] = h[-n]$ (it is symmetric in time), then $h[-(k-n)] = h[n-k]$ and we find $ACF(h[n]) = (h*h[n])$ in this case. So, a time symmetric sequence $h[n] = h[-n]$ has a symmetric self convolution.

Incorporating Sinusoidal Shape

The impulse function $h[n]$ we use in simulation is of the form:

$$h[n] = a[n]\sin(w \cdot n)$$

where $a[n]$ is a sequence of real numbers and $w \in \mathbb{R}$. I assume that we use a natural number m of cycles (in order to ensures that $h[0] = h[L-1] = 0$ and also to produce an output that integrates to zero - which helps reduce side lobe energy). This tells us the value of w:

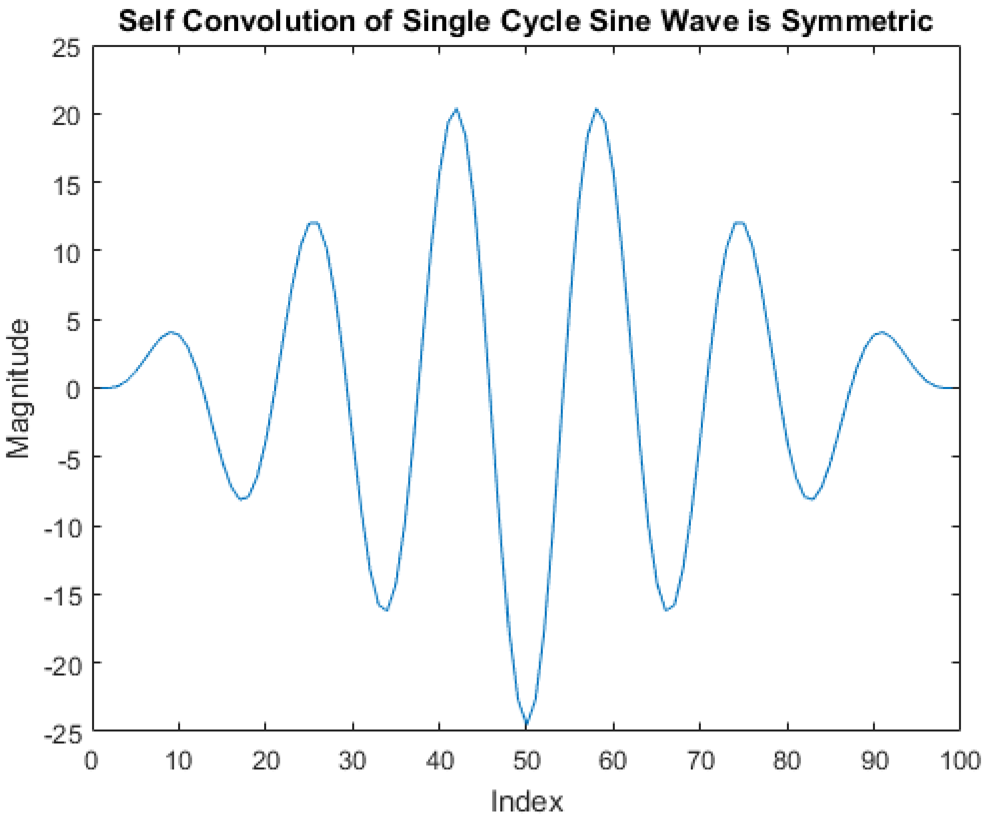
$$\begin{aligned} w \cdot (L-1) &= 2\pi m \\ \implies w &= \frac{2\pi m}{L-1}\end{aligned}$$

So, the impulse $h[n]$ is of the form, where $m \in \mathbb{N}$ is the number of cycles used:

$$h[n] = \begin{cases} a[n]\sin(\frac{2\pi m}{L-1} \cdot n) & 0 \le n \le L-1 \\ 0 & \text{for all other } n \end{cases}$$

Consequences of Sinusoidal Shape

For simplicity, we begin by assuming $a[n] = 1$ for all n. MATLAB evidencce seems to suggest that with the additional assumption of sinusoidal shape, the impulse response of the ultrasound system $h * h$ is now symmetric:



The most immediate symmetry present in an impulse response of this form is as follows:

$$h[a] = -h[L-1-a] \; \forall a \in \mathbb{Z}$$
$$\implies h[a-u] = -h[L-1-(a-u)] = -h[L-1-a+u)]$$

(ASSUMED FOR NOW)

If a sequence has this symmetry, it its self convolution symmetric? That is, we want to show:

$$(h*h)[2+a] = \sum_{k=0}^{2+a} h[k]h[2+a-k] = (h*h)[2L-4-a] = \sum_{k=0}^{2L-4-a} h[k]h[2L-4-a-k]$$
$$\iff \sum_{k=0}^{2+a} h[k]h[2+a-k] = \sum_{k=0}^{2L-4-a} h[k]h[2L-4-a-k]$$

Define u so that $k = L-1-u$. Then $h[k] = h[L-1-u] = -h[u]$, and $2L-4-a-k = -a+L+u-3$. Since $u = L-k-1$, if $k = 0$ then $u = L-1$. If $k = 2L-4-a$ then $u = a-L+3$:

$$\iff \sum_{k=0}^{2+a} h[k]h[2+a-k] = \sum_{u=L-1}^{a-L+3} h[L-1-u]h[L-3-a+u] \quad = \sum_{u=L-1}^{a-L+3} -h[u]h[L-3-a+u]$$

Rewriting the second part of the second sum:

$$h[L-3-a+u] = h[L-1-a+(u-2)] = -h[a-(u-2)] = -h[2+a-u]$$

Plugging this back in, renaming u to k, and continuing the same chain of implications:

$$\iff \sum_{k=0}^{2+a} h[k]h[2+a-k] = \sum_{k=L-1}^{a-L+3} h[k]h[2+a-k]$$

Call $A = 2 + a$:

$$\iff \sum_{k=0}^{A} h[k]h[A-k] = \sum_{k=L-1}^{A-(L-1)} h[k]h[A-k]$$

This seems to work in MATLAB. Maybe look to apply the fact that h goes to zero outside a certain range now? Could also look at cases on $A - (L - 1)$