Inverse Problems in Imaging, GV08, 2015

Answer 3 questions

Marks for each part of each question are indicated in square brackets

Calculators are permitted

1. In this question, consider the matrix

$$\mathsf{A} = \begin{pmatrix} 1 & -1 \\ 1 & -2 \\ 2 & 1 \end{pmatrix} ,$$

and the linear problem

$$\boldsymbol{b} = \mathsf{A}\boldsymbol{x} \quad \equiv \quad \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \mathsf{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} . \tag{1}$$

a. Write down the $2 \times 2$ matrix $\mathsf{B} = \mathsf{A}^{\mathrm{T}}\mathsf{A}$ and determine its eigenvalues $\lambda_1$, $\lambda_2$ and eigenvectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Thence determine $\boldsymbol{u}_1 = \lambda_1^{-1/2}\mathsf{A}\boldsymbol{v}_1$ and $\boldsymbol{u}_2 = \lambda_2^{-1/2}\mathsf{A}\boldsymbol{v}_2$, and a third vector $\boldsymbol{u}_3$ orthogonal to $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. Verify that the system $\{\boldsymbol{u}_i, \lambda_i^{1/2}, \boldsymbol{v}_i\}$ is the singular value decomposition of $\mathsf{A}$, and give an explicit form for the pseudo-inverse $\mathsf{A}^\dagger$ of $\mathsf{A}$.

[10 marks]

b. Using the pseudo-inverse, or otherwise, find the pseudo-solution $\boldsymbol{x}^\dagger$ to problem (1) for the case

$$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 5 \end{pmatrix} ,$$

and verify that it is the minimiser of the residual

$$||\boldsymbol{b} - \mathsf{A}\boldsymbol{x}^\dagger||^2 .$$

[6 marks]

c. On a 2D graph draw the lines representing the equations

$$x_1 - x_2 = -3\,, \quad x_1 - 2x_2 = 5\,, \quad 2x_1 + x_2 = 5\,,$$

and indicate the solution $\boldsymbol{x}^\dagger$ found in your answer to part b).

[5 marks]

d. Show that the solution to problem (1) could be recast as a constrained optimisation problem :

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimise}} & \Phi(\boldsymbol{b}, \mathsf{A}\boldsymbol{x}) := ||\boldsymbol{b} - \mathsf{A}\boldsymbol{x}||^2 \\ \text{subject to} & \boldsymbol{b}^\dagger = \mathsf{A}\,\boldsymbol{x} \text{ lies on the plane spanned by } \{\boldsymbol{u}_1, \boldsymbol{u}_2\}\,. \end{array}$$

[4 marks]

e. Consider the minimisation of the $p$-norm given by

$$|\boldsymbol{b} - \mathsf{A}\,\boldsymbol{x}^\dagger|^p\,, \qquad 1 < p < \infty\,.$$

By means of a sketch or otherwise, show that the solution can be found by first determining a point $\boldsymbol{b}_p^\dagger$ that lies on the plane defined by the vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ and has minimum distance in the p-norm $|\boldsymbol{b} - \boldsymbol{b}_p^\dagger|^p$, and secondly applying the pseudo-inverse $\mathsf{A}^\dagger$ to $\boldsymbol{b}_p^\dagger$. Suggest an algorithm for determining the point $\boldsymbol{b}_p^\dagger$, and discuss in particular how it behaves as $p \to 1$ and as $p \to \infty$. How would it behave if $0 < p < 1$ ?

[8 marks]

[Total 33 marks]

2. Let $f(x,y)$ represent an image as a function over two dimensions in space, and let $g = Af + n$ represent a noise corrupted blurred version of $f$ where $A$ represents convolution with a stationary kernel $G_\sigma(x,y) = \exp\left[-\frac{x^2+y^2}{2\sigma^2}\right]$ and $n \sim N(0,\Gamma_n)$ is a random vector taken from a zero-mean Normal distribution with covariance $\Gamma_n$.

   a. Briefly explain the terms *convolution*, *stationary kernel*, *Normal distribution*, and *covariance* in the above statement. What is the role of $\sigma$ in the expression for the kernel of the convolution ?

   [6 marks]

   b. Define the singular value decomposition (SVD) of a linear operator, such as that defined at the beginning of this question, and show how it can be used to determine whether the properties of existence, uniqueness and stability are present. Give the definition of the Moore-Penrose inverse of $A$ in terms of its SVD.

   [6 marks]

   c. For the particular convolution operator defined at the beginning of this question, show that the harmonic functions $\exp[-\mathrm{i}(k_x x + k_y y)]$ are its singular functions for any values of $k_x$, $k_y$, and give an expression for the corresponding singular values, in terms of the Fourier Transform of $G_\sigma$.

   [6 marks]

   d. Use the general expression for an inverse that you gave in part b) to show that the inverse of the convolution operator $A$ given at the beginning of this question is equivalent to division of the Fourier Transform of $g$ by the Fourier Transform of $G_\sigma$ followed by Inverse Fourier Transform. State the *convolution theorem* and show that it can be used to obtain the same result.

   [6 marks]

   e. Show that the addition of the noise term $n$ leads to a solution where high frequency noise is amplified.

   [3 marks]

f. It is proposed to dampen the effects of noise amplification by optimising a penalised function

$$f_* = \arg\min_f \left[ \Phi(f) = \int_{\mathbb{R}^2} \frac{1}{2}(g - Af)^2 + \alpha|\nabla f|^2 \mathrm{d}x\mathrm{d}y \right] ,$$

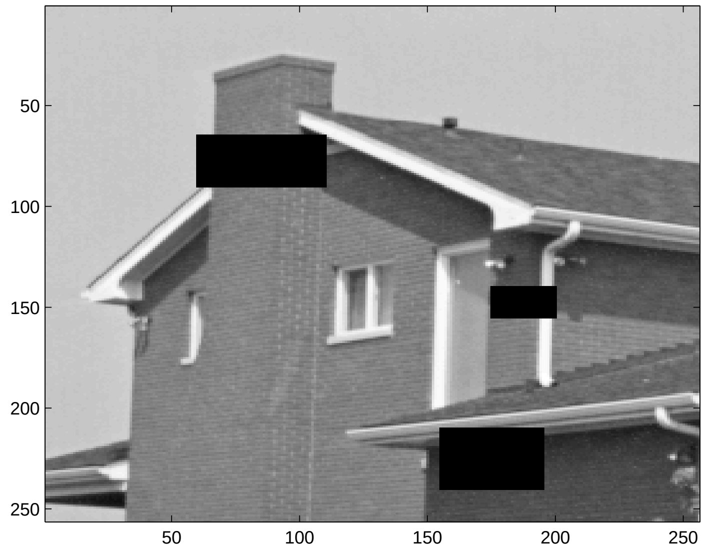
Show that this leads to a reconstruction scheme where the Fourier Transform of the noise term is scaled by the term $\frac{1}{\hat{G}_\sigma + \alpha(k_x^2 + k_y^2)}$. What is the effect on the recovered solution $f_*$ as $\alpha$ is varied from a small positive number to a large one ?

[6 marks]

[Total 33 marks]

3. This question is about inpainting, and the development of statistical priors.

You are working on a project to restore images which are degraded by missing pixels. An example is shown at the top of the page.




a. If the original picture has $n_{\text{pix}} = n_x \times n_y$ pixels and the total number of missing pixels over all the regions is $n_{\text{missing}}$, derive a format for the forward operator $\mathsf{A}$ that maps from a vector $\boldsymbol{f}$ of dimension $n_{\text{pix}}$ to a vector $\boldsymbol{g}$ of dimension $n_{\text{dat}} = n_{\text{pix}} - n_{\text{missing}}$.

[5 marks]

b. You propose to solve the restoration problem by optimising an expression

$$\Phi(\boldsymbol{f}) = \frac{1}{2}||\boldsymbol{g} - \mathsf{A}\boldsymbol{f}|| + \alpha\Psi(\boldsymbol{f})$$

where $\Psi(\boldsymbol{f})$ is a regularisation functional.

i. Why is regularisation essential for this problem ? What would happen if it were left out ?

[2 marks]

ii. Explain why the zero-order Tikhonov functional $\Psi(\boldsymbol{f}) = \frac{1}{2}\boldsymbol{f}^{\mathrm{T}}\boldsymbol{f}$ would be useless for this problem.

[2 marks]

iii. Describe the effect you would expect to see if the regularisation functional were the $1^{\text{st}}$-order Tikhonov functional $\Psi(\boldsymbol{f}) = \frac{1}{2}\boldsymbol{f}\mathsf{L}\boldsymbol{f}$ with $\mathsf{L}$ a discrete Laplacian.

[4 marks]

c. Describe the programming of a matrix-free Krylov method for minimising this expression, assuming that $n_x = n_y = 256$ and $n_{\text{missing}} = 4000$, and assuming a quadratic form for the regularisation functional.

[8 marks]

d. Suppose that a database of typical images were available. Describe how to develop a Gaussian statistical prior based on the mean and covariance of these images, and what form the regularisation functional would take.

[6 marks]

e. Discuss what types of non-Gaussian prior might be useful for this problem and explain how the implementation of the restoration method would change if the functional $\Psi(\boldsymbol{f})$ were i) convex, but not quadratic, ii) non-convex.

[6 marks]

[Total 33 marks]

4. This question is about the Radon Transform and 2D X-Ray tomography

   a. Give a definition of the forward projection operator of the Radon transform in 2D as an integral operator mapping a function $f(x, y)$ to a function $g(s, \theta)$. State also the back projection operator as the adjoint of the Radon transform.

   [5 marks]

   b. Show that the Radon Transform of a 2D Gaussian function $f(x, y) = \frac{1}{2\pi\sigma^2}\mathrm{e}^{-\frac{x^2+y^2}{2\sigma^2}}$ is a Gaussian function $g(s, \theta) = \frac{1}{\sqrt{2\pi\sigma^2}}\mathrm{e}^{-\frac{s^2}{2\sigma^2}}$ that does not depend on $\theta$. Apply the back projection operator to this function and thus show that the back projection operator is not itself the inverse of the forward projection.

   [6 marks]

   c. Show that the 1D Fourier Transform $\mathcal{F}_{s \to k}$ of the Radon Transform of a 2D function $f(x, y)$ is the same as the 2D Fourier Transform $\mathcal{F}_{x \to k_x, y \to k_y}$ of $f(x, y)$ sampled along the line $k_x = k \cos\theta, k_y = k \sin\theta$. Illustrate using the particular case of the Gaussian function of parts a) and b).

   [7 marks]

   d. Use your answer to part c) to show that an exact reconstruction formula can be found by either i) back projection followed by convolution with a 2D filter or ii) convolution with a 1D filter of the Radon transform, followed by back-projection. Give the form of the filter in both cases. Explain the relative merits of the different approaches, in regard to computational difficulty, stability with respect to noise and ease of implementation.

   [10 marks]

   e. What is the effect on tomographic reconstruction of using only limited angles? How could an acceptable solution be obtained by regularisation?

   [5 marks]

[Total 33 marks]

5. This question is about sparsity regularisation and the Iterative Shrinkage-Thresholding Algorithm (Iterative Soft-Thresholding Algorithm)

   a. What is meant by the ”$\ell_1$ penalty term” in inverse problems in imaging ? How is it related to the idea of sparsity-regularisation ? What choices of bases are suitable for defining sparsity in images ?

   [8 marks]

   b. The “Soft-Thresholding Operator” (also known as “Shrinkage Thresholding Operator”) is defined as

   $$S_\alpha(x) := \max(|x| - \alpha, 0)\mathrm{sign}(x)$$

   Sketch a graph of this function for positive and negative values of $x$ and a non-negative value of $\alpha$.

   [3 marks]

   c. Consider the minimisation of the functional

   $$\Phi(f) := \frac{1}{2}||g - Af||^2 + \lambda|f|_1$$

   where $A$ is a linear operator. By considering the derivative of $\Phi$ with respect to $f$ show that an iterative scheme for minimising $\Phi$ is given by the sequence

   $$f^{(k)} = \arg\min_f \left[\frac{1}{2\tau_k}||f - \left(f^{(k-1)} - \tau_k A^*(Af^{(k-1)} - g)\right)||^2 + \lambda|f|_1\right] .$$

   Thus show that the iterative solution given in part c) is also given by the sequence

   $$f^{(k)} = S_{\lambda\tau_k}\left(f^{(k-1)} - \tau_k A^*(Af^{(k-1)} - g)\right) .$$

   What is the role of the parameter $\tau_k$ and how should it be chosen ?

   [8 marks]

d. What is meant by a *proximal gradient method* ? Show how the proximal gradient method can be used to generalise the method of part c) to define the Iterative Shrinkage-Thresholding Algorithm (ISTA) for minimising problems of the type

$$\Phi(f) = \phi(f) + \psi(f)$$

where $\phi$ is smooth convex and continuously differentiable and $\psi(f)$ is continuous and convex but not everywhere smooth (i.e. it may have discontinuous derivatives).

[7 marks]

e. The ISTA algorithm estimates $f^{(k)}$ based only on the previous point $f^{(k-1)}$; how can it be accelerated by combining the two previous points $f^{(k-1)}$ and $f^{(k-2)}$?

[7 marks]

[Total 33 marks]

END OF PAPER