2 Fourier Transforms and Sampling

2.1 The Fourier Transform

The Fourier Transform is an integral operator that transforms a continuous function into a continuous function

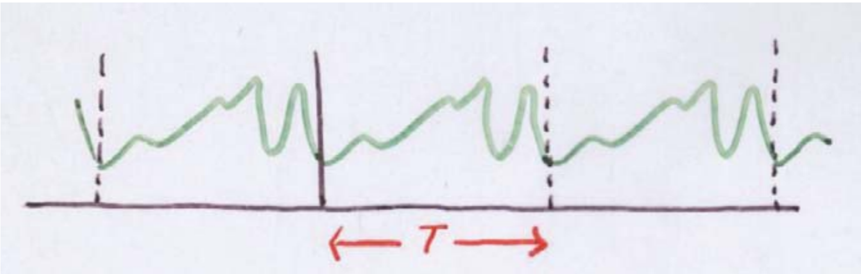
$$H(\omega) = \mathcal{F}_{t\to\omega}\left[h(t)\right] := \int_{-\infty}^{\infty} h(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t \tag{1}$$

The inverse Fourier Transform recovers the original function

$$h(t) = \mathcal{F}_{\omega\to t}\left[H(\omega)\right] := \frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega \tag{2}$$

2.1.1 The Fourier Series

If $h(t)$ is periodic with period T (i.e. $h(t+kT) = h(t)\,\forall\, k \in \mathbb{N}$)



then it can be expanded

$$h(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t) \tag{3}$$

where $\omega+0 = \frac{2\pi}{T}$. The coefficients $\{a_i, b_i\}$ are obtained by integration and invoking the trigonometric relation

$$\begin{aligned}
\int_{-T/2}^{T/2} \cos\left(\frac{2n\pi t}{T}\right)\cos\left(\frac{2m\pi t}{T}\right)\mathrm{d}t &= \frac{T}{2}\delta_{nm} \quad = \int_{-T/2}^{T/2} \sin\left(\frac{2n\pi t}{T}\right)\sin\left(\frac{2m\pi t}{T}\right)\mathrm{d}t \\
\int_{-T/2}^{T/2} \cos\left(\frac{2n\pi t}{T}\right)\sin\left(\frac{2m\pi t}{T}\right)\mathrm{d}t &= 0 \quad \text{where}\, \delta_{nm} = \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}
\end{aligned}$$

to give

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} h(t)\cos\left(n\omega_0 t\right)\mathrm{d}t\,, \quad b_n = \frac{2}{T}\int_{-T/2}^{T/2} h(t)\sin\left(n\omega_0 t\right)\mathrm{d}t \tag{4}$$

If we define complex coefficients :-

$$\begin{aligned} c_n = \frac{1}{2}(a_n - \mathrm{i}b_n) &= \frac{1}{T}\int_{-T/2}^{T/2} h(t)\mathrm{e}^{-\mathrm{i}n\omega_0 t}\mathrm{d}t \quad n > 0 \\ c_{-n} = \frac{1}{2}(a_n + \mathrm{i}b_n) &= \frac{1}{T}\int_{-T/2}^{T/2} h(t)\mathrm{e}^{\mathrm{i}n\omega_0 t}\mathrm{d}t \quad n > 0 \end{aligned}$$

then we reconsider the Fourier Series as

$$h(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}n\omega_0 t} \tag{5}$$

with c_n given (for all integer n)

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} h(t)\mathrm{e}^{-\mathrm{i}n\omega_0 t}\mathrm{d}t\,. \tag{6}$$

Thus the Fourier Series expansion transforms a continuous function to a descrete series. The relation Eq. 6 is also known as the *Discrete Time Fourier Transform*, and Eq. 5 as the *Inverse Discrete Time Fourier Transform*.

2.1.2 Discrete Fourier Transform (DFT)

The DFT is a transform from a discrete series to a discrete series.

Given a list of length N : $\{h_0, h_1, \ldots h_{N-1}\}$ then we can derive another list : $\{H_0, H_1, \ldots H_{N-1}\}$ by the following relations

$$H_m = \sum_{k=0}^{N-1} h_k \mathrm{e}^{-\frac{2\pi \mathrm{i} mk}{N}} \tag{7}$$

$$h_k = \sum_{k=0}^{N-1} H_k \mathrm{e}^{\frac{2\pi \mathrm{i} mk}{N}} \tag{8}$$

2.1.3 Relationship Between Transforms

Clear the transform relations (Eq. 1,Eq. 2), (Eq. 5,Eq. 6), (Eq. 7,Eq. 8) are related, but the precise relationships contains some subtleties.

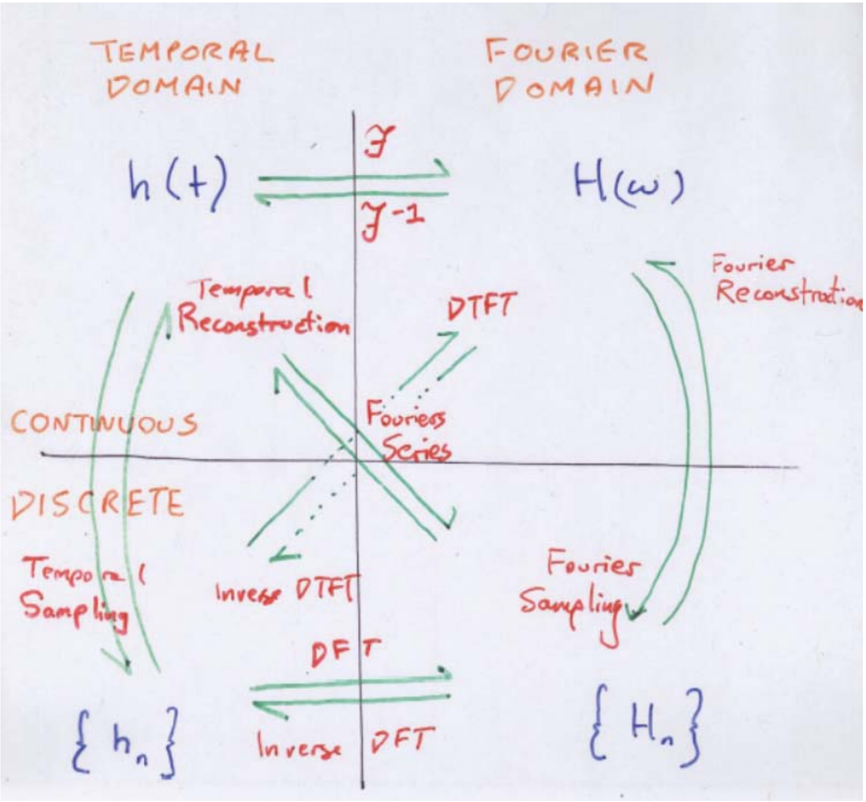
Often the Fourier Series is used to define the Fourier Transform as a limit $T \to \infty$ from Eq. 6

$$Tc_n = \int_{-T/2}^{T/2} h(t)\mathrm{e}^{-\mathrm{i}n\omega_0 t}\mathrm{d}t \quad \Rightarrow \lim_{T\to\infty}\int_{-\infty}^{\infty} h(t)\mathrm{e}^{-\mathrm{i}n\omega_0 t}\mathrm{d}t =: H(nw_0) \tag{9}$$

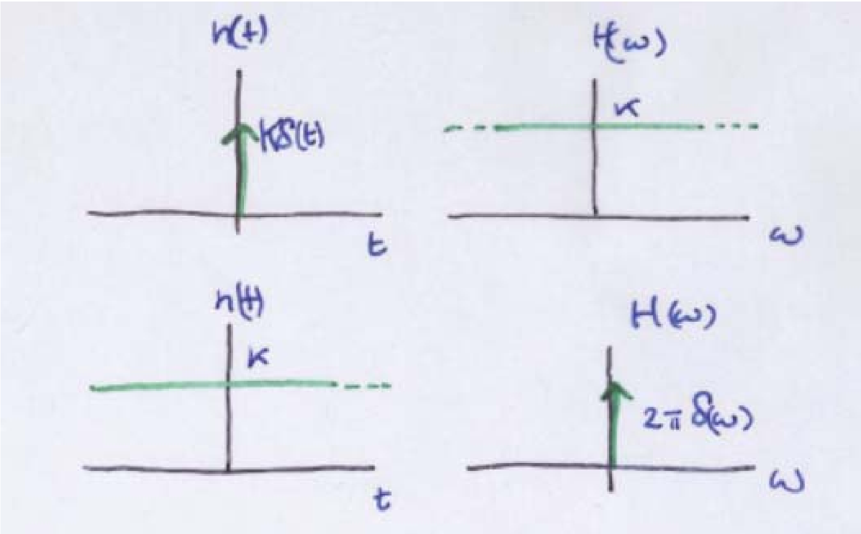
From Eq. 5 and Eq. 9 and using $\omega_0 = \frac{2\pi}{T} \to 0$ gives

$$h(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} Tc_n \mathrm{e}^{\mathrm{i}n\omega_0 t}\omega_0 \quad \lim_{T\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} H(\omega)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega \tag{10}$$

This depends on accepting that a sum in a limit becomes an integral which is mathematically a bit flakey. Furthermore it does not lead easily to an interpretation of the DFT. It is preferable to use the FT as the starting point for all other forms.



2.1.4 Fourier Transforms Involving δ-functions



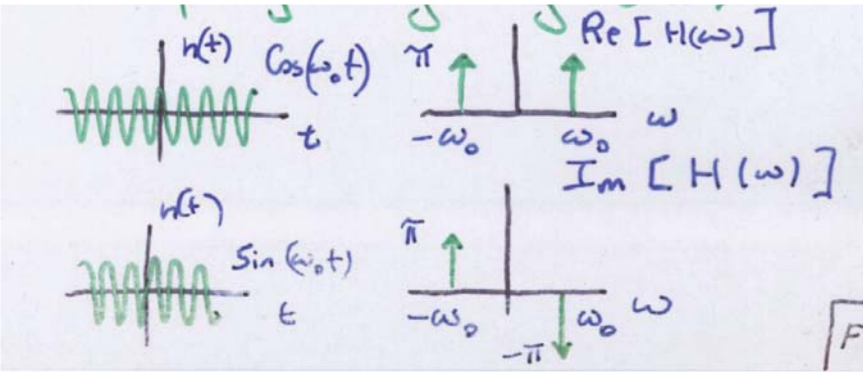
To find the FT of periodic functions consider $\cos(t), \sin(t)$.

$$\mathcal{F}\left[\cos(\omega_0 t)\right] = \mathcal{F}\left[\frac{1}{2}\left(\mathrm{e}^{\mathrm{i}\omega_0 t} + \mathrm{e}^{-\mathrm{i}\omega_0 t}\right)\right] \quad = \pi\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right] \tag{11}$$

[Note that the result is purely real]

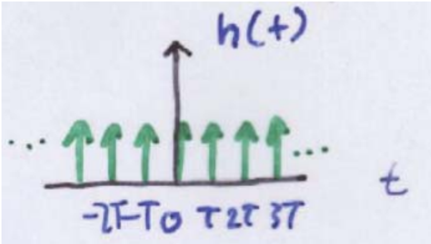
$$\mathcal{F}\left[\sin(\omega_0 t)\right] = \mathcal{F}\left[\frac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}\omega_0 t} - \mathrm{e}^{-\mathrm{i}\omega_0 t}\right)\right] \quad = \mathrm{i}\pi\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right] \tag{12}$$

[Note that the result is purely imaginary]



Fourier Transform of a Pulse Train

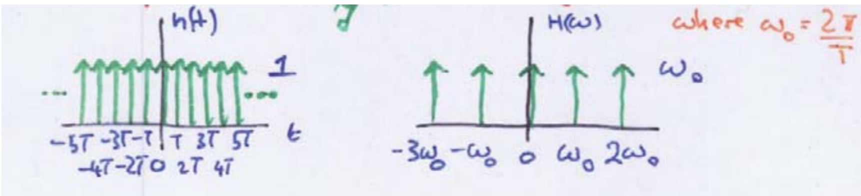
An extremely important function in sampling theory is a pulse train of "Comb" function



$$\mathsf{Comb}_T(t) := \sum_{n=-\infty}^{\infty} \delta(t - nT) \tag{13}$$

consisting of an infinite sequence of equidistant δ-functions. We will prove the following :

$$\mathsf{Comb}_T(t) \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\Leftrightarrow}} \omega_0 \mathsf{Comb}_{\omega_0}(\omega) \tag{14}$$



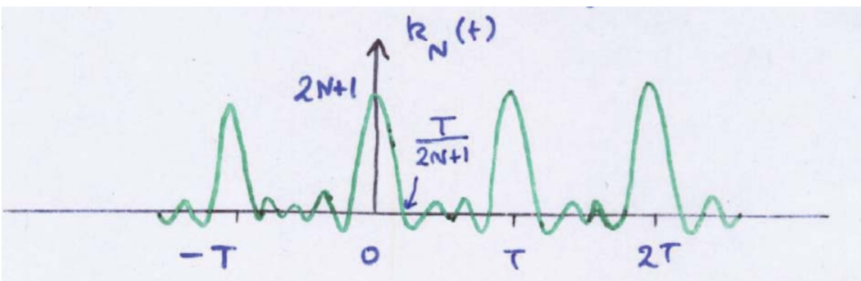
It suffices to show that the inverse transform is true (since Fourier Transforms are symmetric). First

$$\mathcal{F}^{-1}\left[\mathsf{Comb}_{\omega_0}(\omega)\right] = \frac{1}{2\pi}\frac{2\pi}{T}\int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_0)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t \quad = \frac{1}{T}\sum_{n=-\infty}^{\infty}\mathrm{e}^{\mathrm{i}n\omega_0 t} \tag{15}$$

We need to show that the sum on the right in Eq. 15 is equivalent to a sequence of δ-functions in the time domain. Consider the partial sum

$$k_N(t) = \frac{1}{T}\sum_{n=-N}^{N}\mathrm{e}^{\mathrm{i}n\omega_0 t} = \frac{\mathrm{e}^{\mathrm{i}(N+1)\omega_0 t} - \mathrm{e}^{\mathrm{i}N\omega_0 t}}{T(\mathrm{e}^{\mathrm{i}\omega_0 t} - 1)} = \frac{\sin\left((N+1/2)\omega_0 t\right)}{T\sin\left(\frac{\omega_0 t}{2}\right)}$$

This function is known as the *Fourier Series Kernel* with period $\frac{2\pi}{\omega_0} = T$.



We need to prove that in the interval $[-T/2, T/2]$ $k_N(t)$ tends to $\delta(t)$ since then

$$\lim_{N\to\infty} k_N(t) = \mathsf{Comb}_T(t).$$

Consider

$$k_N(t) = \frac{\sin\left((N+1/2)\omega_0 t\right)}{T\sin\left(\frac{\omega_0 t}{2}\right)} = \frac{\sin\left((N+1/2)\omega_0 t\right)}{Tt}\frac{t}{\sin\left(\frac{\omega_0 t}{2}\right)}$$

We know, from the definitiion of a δ-function that

$$\lim_{N\to\infty}\frac{\sin\left((N+1/2)\omega_0 t\right)}{Tt} = \frac{\pi}{T}\delta(t)$$

and, since $\frac{t}{\sin\left(\frac{\omega_0 t}{2}\right)}$ is bounded in $[-T/2, T/2]$, we have

$$\lim_{N\to\infty} k_N(t) = \frac{\pi}{T}\frac{t}{\sin\left(\frac{\omega_0 t}{2}\right)}\delta(t) = \frac{\pi}{T}\frac{t}{\sin\left(\frac{\omega_0 t}{2}\right)}\delta(t)\Bigg|_{t=0} = \delta(t)$$

as required. In the final step, note that from the definition of the δ-function that $f(t)\delta(t) = f(0)\delta(t)$ and $\frac{t}{\sin\left(\frac{\omega_0 t}{2}\right)} = \frac{T}{\pi\mathsf{sinc}(\omega_0 t/2)}$ whic has value $\frac{T}{\pi}$ at $t = 0$

Inversion Formula Proof

Using properties of the δ-function we can now prove the inversion formula

$$\begin{aligned}
\mathcal{F}^{-1}\left[\mathcal{F}\left[h(t)\right]\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t}\int_{-\infty}^{\infty}\mathrm{e}^{-\mathrm{i}\omega t'}h(t')\mathrm{d}t'\mathrm{d}t \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\omega(t-t')}h(t')\mathrm{d}t'\mathrm{d}t \\
&= \int_{-\infty}^{\infty} h(t')\delta(t-t')\mathrm{d}t' \\
&= h(t)
\end{aligned}$$

as required.

2.2 Convolution

Next to the inversion formula Eq. 2, the convolution theorem is the most important tool in Fourier Analysis.

2.2.1 The Convolution Integral

given two functions $f(t), g(t)$, their convolution produces a third function :

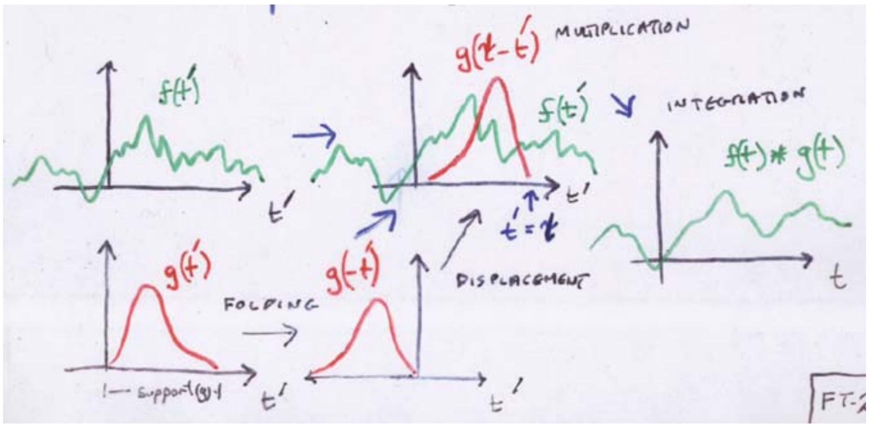
$$h(t) = \int_{-\infty}^{\infty} f(t')g(t-t')dt' := f(t) * g(t) \tag{16}$$

It is trivially seen by substitution of variables that convolution is commutative, i.e.

$$f(t) * g(t) = g(t) * f(t)$$

2.2.2 Interpretation of The Convolution Integral

The convolution of $f(t)$ by $g(t)$) is often interpreted as a weighted averaging of $f(t)$ by a *filter* $g(t)$ where the extent of the average is the *support* of $g(t)$.



2.2.3 The Convolution Theorem

We state the convolution theorem as

The Fourier Transform of the convolution of two functions equals the product of the Fourier Transform of the functions individually

$$\mathcal{F}[f(t) * g(t)] = \mathcal{F}[f(t)]\,\mathcal{F}[g(t)] \tag{17}$$

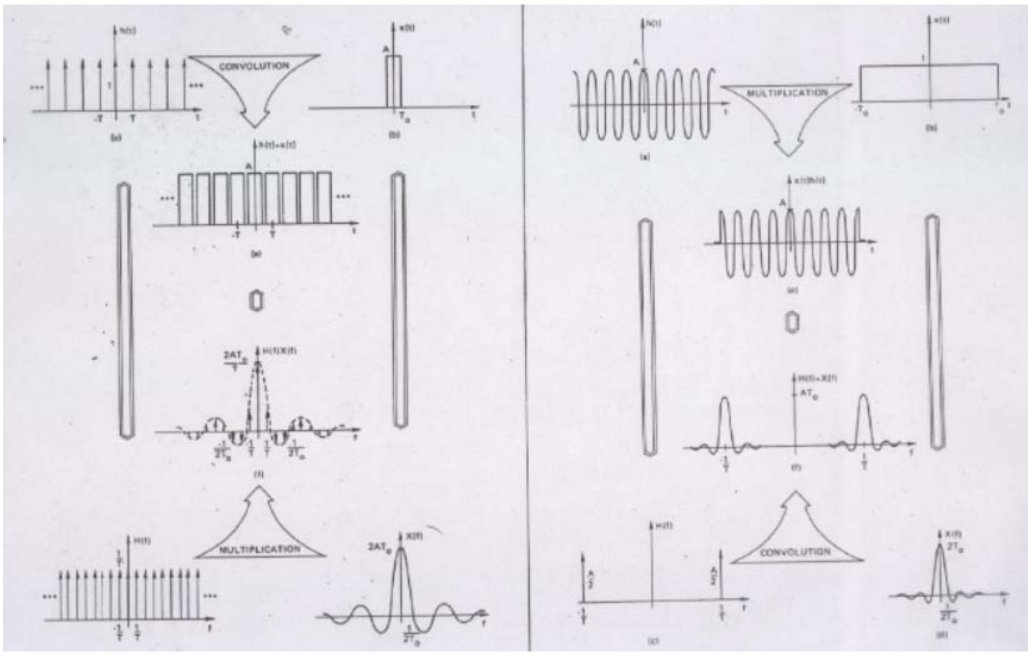
The proof is straightforward :

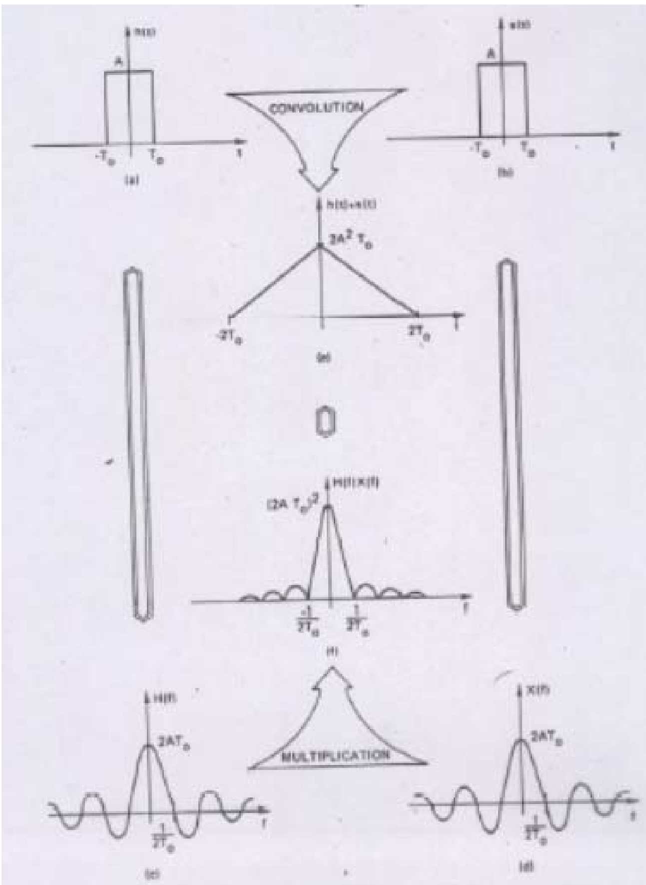
$$\begin{aligned}
\mathcal{F}[f(t) * g(t)] &= \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega t} \int_{-\infty}^{\infty} f(\tau) g(t-\tau) \mathrm{d}t \mathrm{d}\tau \\
\text{(substitute } z = t - \tau) \quad &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega(z+\tau)} f(\tau) g(z) \mathrm{d}\tau \mathrm{d}z \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega\tau} f(\tau) \mathrm{d}\tau \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}\omega z} g(z) \mathrm{d}z \\
&= \mathcal{F}[f(t)]\,\mathcal{F}[g(t)]
\end{aligned}$$

The complementray form of Eq. 17 is the *Frequency Convolution Theorem*

The Fourier Transform of the product of two functions equals $(\frac{1}{2\pi})$ *times the convolution of the Fourier Transform of the functions individually*

$$\mathcal{F}\left[f(t)g(t)\right] = \frac{1}{2\pi}\mathcal{F}\left[f(t)\right] * \mathcal{F}\left[g(t)\right] \tag{18}$$





2.2.4 Parseval's Theorem

there is a relation between the area under the function curve in the temporal and Fourier domains:

$$\int_{-\infty}^{\infty} |h(t)|^2 \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 \mathrm{d}\omega \tag{19}$$

First we require a result regarding *conjugation*

If $F(\omega) = \mathcal{F}\left[f(t)\right]$ *then* $\overline{F}(-\omega) = \mathcal{F}\left[\overline{f}(t)\right]$ *where* $\overline{a}$ *is the complex conjugate of* a.

Proof

$$\begin{aligned}
F(\omega) &= \int_{-\infty}^{\infty} \left(\mathsf{Re}\left[f(t)\right] + \mathrm{i}\,\mathsf{Im}\left[f(t)\right]\right) \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t \\
\Rightarrow \quad \overline{F}(\omega) &= \int_{-\infty}^{\infty} \left(\mathsf{Re}\left[f(t)\right] - \mathrm{i}\,\mathsf{Im}\left[f(t)\right]\right) \mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t \\
\Rightarrow \quad \overline{F}(-\omega) &= \int_{-\infty}^{\infty} \underbrace{\left(\mathsf{Re}\left[f(t)\right] - \mathrm{i}\,\mathsf{Im}\left[f(t)\right]\right)}_{\overline{f}(t)} \mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t
\end{aligned}$$

From this

$$\begin{aligned}
\mathcal{F}\left[|h(t)|^2\right] = \mathcal{F}\left[h(t)\overline{h}(t)\right] &= \frac{1}{2\pi}\Big(H(\omega) * \bar{(}H)(-\omega)\Big) \\
(\text{set } \omega = 0) \quad \int_{-\infty}^{\infty} |h(t)|^2 \mathrm{d}t &= \frac{1}{2\pi}\int_{-\infty}^{\infty} H(y)\overline{H}(-(\omega - y))\mathrm{d}y\Big|_{\omega=0} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(y)|^2 \mathrm{d}y
\end{aligned}$$

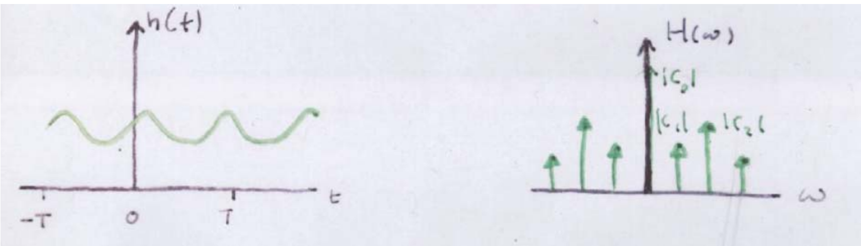
2.3 Fourier Series and Sampled Waveforms

In the technical literature, Fourier Series is usually developed independently of the Fourier Integral. We already saw (Eq. 9, Eq. 10) that the Fourier Integral could be developed by taking the limit $T \to \infty$ of the Fourier Series. Now we derive it in the other direction. We want to show that

$$\begin{aligned}
\text{if} \quad h(t+T) &= h(t) \qquad \text{i.e. } h(t) \text{ is periodic, period } T \\
\text{then} \quad h(t) &= \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}n\omega_0 t} \quad \text{where } \omega_0 = \frac{2\pi}{T} \quad \text{(same as Eq. 5)} \\
\text{where} \quad c_n &= \frac{1}{T}\int_{-T/2}^{T/2} h(t)\mathrm{e}^{-\mathrm{i}n\omega_0 t}\mathrm{d}t \qquad \text{(same as Eq. 6).}
\end{aligned}$$

It follows from Eq. 5 and the properties of δ-functions (*viz.* $\delta(t) = \mathcal{F}^{-1}[1] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}\omega$) that the FT of a periodic function is a sequence of equidistant δ-functions:

$$H(\omega) = \mathcal{F}\left[h(t)\right] = 2\pi \sum_{n=-\infty}^{\infty} c_n \delta(\omega - \omega_0 n) \tag{20}$$



[Note : at this point we say nothing about whether or not $|H(\omega)|$ is a finite or infinite series.]

Firstly. it is simple to prove the inverse, i.e. that the inverse FT of a sequence of equidistant δ-functions is a periodic function :

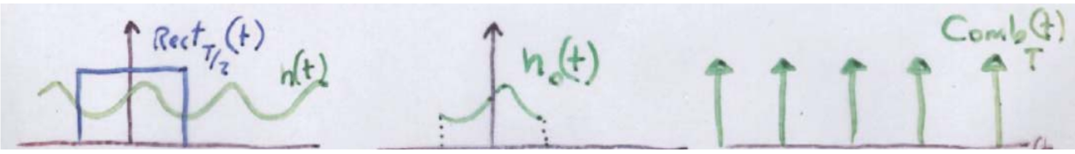
$$\mathcal{F}^{-1}\left[\sum_{n=-\infty}^{\infty} A_n \delta(\omega - n\omega_0)\right] = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} A_n \mathrm{e}^{\mathrm{i}n\omega_0 t} =: \phi(t) \tag{21}$$

and since $\mathrm{e}^{\mathrm{i}n\omega_0\left(t+\frac{2\pi}{\omega_0}\right)} = \mathrm{e}^{\mathrm{i}n\omega_0 t}$ we must have $\phi(t+\frac{2\pi}{\omega_0}) = \phi(t)$. To prove 20 we make use of the Fourier Transform pair Eq. 14. Consider the function

$$h_0(t) = \begin{cases} h(t) & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} = h(t)\mathsf{Rect}_{T/2}(t)$$

which is $h(t)$ masked ot zero except in a single period $(-T/2, T/2)$. The original, periodic, function is therefore

$$h(t) = \sum_{n=-\infty}^{\infty} h_0(t+nT) = h_0(t) * \mathsf{Comb}_T(t)$$



Then we have

$$\mathcal{F}\left[h(t)\right] = \mathcal{F}\left[h(t) * \mathsf{Comb}_T(t)\right] = \mathcal{F}\left[h_0(t)\right]\omega_0 \mathsf{Comb}_{\omega_0}(\omega)$$

but

$$H_0(\omega) = \int_{-\infty}^{\infty} h_0(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t = \int_{-T/2}^{T/2} h(t)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{d}t$$

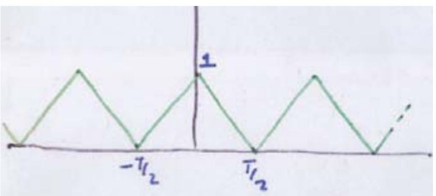
therefore

$$H(\omega) = H(\omega_0)\frac{\pi}{T}\sum_{n=-\infty}^{\infty}\delta(\omega - n\omega_0) = 2\pi\sum_{n=-\infty}^{\infty}\frac{H_0(n\omega_0)}{T}\delta(\omega - n\omega_0)$$

and since c_n from Eq. 5 is equal to $\frac{H_0(n\omega_0)}{T}$ we have proved Eq. 20.

The coefficients c_n of the Fourier Series expansion of a periodic function $h(t)$ are equal to $1/T$ times the value of the Fourier Transform of $h_0(t)$ sampled at the frequencies $\omega = n\omega_0 = \frac{2\pi n}{T}$

Example Triangular wave



$$a_n = \frac{1}{T}\sum_{n=-T/2}^{T/2} h(t)\cos(n\omega_0 t)\mathrm{d}t = \begin{cases} \frac{1}{2} & n = 0 \\ \frac{2}{n^2\pi^2} & n \text{ even}, n > 0 \\ 0 & n \text{ odd} \end{cases}$$

and

$$H_0(\omega) = \frac{8\sin^2(\omega\frac{T}{4})}{T\omega^2}$$

Thus $a_n = H_0(\omega)|_{\omega=\frac{n2\pi}{T}}$

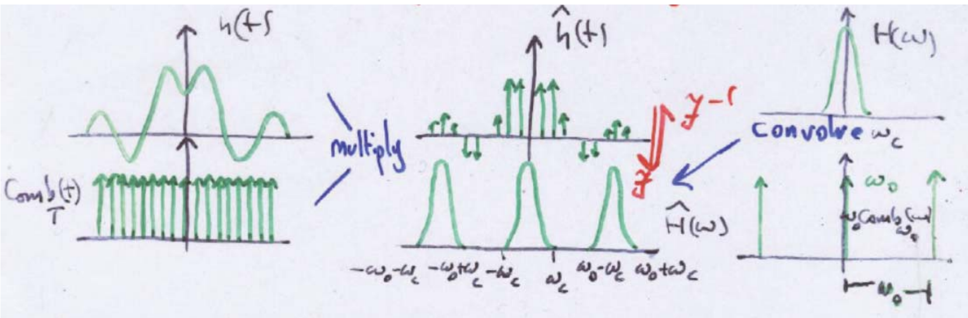
2.3.1 Waveform Sampling

The fourier Series represents a continuous, periodic function in time by a discrete sequence of equidistant δ-functions in the fruency domain. Consider now the effect of *sampling* in the time domain

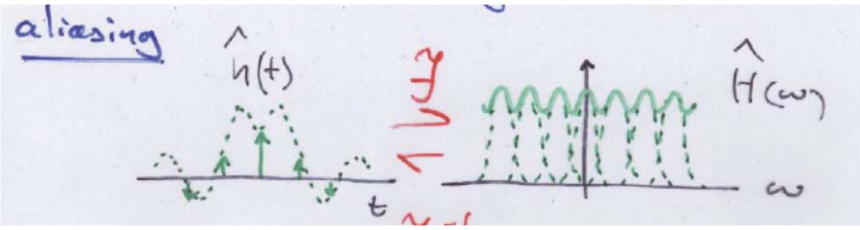
$$\{h_n\} = \hat{h}(t) = \sum_{n=-\infty}^{\infty} h(nT)\delta(t - nT) = h(t)\mathsf{Comb}_T(t)$$

Clearly, in the Fourier Domain, the sampling process is a convolution

$$\hat{H}(\omega) = \mathcal{F}\left[\hat{h}(t)\right] = \frac{\omega_0}{2\pi} H(\omega) * \mathsf{Comb}_{\omega_0}(\omega)$$



As the sampling interval T *increases*, the seperation of the δ-functions in the frequency domain *decreases*. Eventually the replicated functions in the frequency domain *overlap* leading to *aliasing*.



2.3.2 The Sampling Theorem

The Shannon Sampling Theorem [Shannon, 1949] is fundamental in information theory. It states

If the Frequency Transform of a function $h(t)$ is zero above a certain frequency ω_c, i.e.

$$H(\omega) = 0 \quad \text{for } |\omega| \geq \omega_c \tag{22}$$

then $h(t)$ can be uniquely determined from its values

$$h_n = h\left(\frac{n\pi}{\omega_c}\right)$$

i.e. samples of $h(t)$ at a sequence of equidistant points, $\frac{\pi}{\omega_c}$ apart. In fact $h(t)$ is given by

$$h(t) = \sum_{n=-\infty}^{\infty} h_n \frac{\sin(\omega_c t - n\pi)}{(\omega_c t - n\pi)} \tag{23}$$

Proof

Applying the Inverse Fourier Transform to Eq. 22 gives

$$h(t) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} H(w) \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}\omega$$

so that the sampled function is given by

$$h_n = h\left(\frac{n\pi}{\omega_c}\right) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} H(w) \mathrm{e}^{\frac{\mathrm{i}n\pi\omega}{\omega_c}} \mathrm{d}\omega$$

If we expand $H(\omega)$ into a Fourier Series in the interval $(-\omega_c, \omega_c)$

$$H(\omega) = \sum_{n=-\infty}^{\infty} A_n \mathrm{e}^{\frac{\mathrm{i}n2\pi\omega}{2\omega_c}} \qquad -\omega_c < \omega < \omega_c$$

where

$$A_n = \frac{1}{2\omega_c} \int_{-\omega_c}^{\omega_c} H(w) \mathrm{e}^{\frac{\mathrm{i}n\pi\omega}{\omega_c}} \mathrm{d}\omega$$

which gives immediately that $A_n = \frac{\pi}{\omega_c} h_n$. However, the Fourier Series represents a replicated function

$$\tilde{H}(\omega) = \sum_{n=-\infty}^{\infty} \frac{\pi}{\omega_c} h_n \mathrm{e}^{-\frac{\mathrm{i}n\pi\omega}{\omega_c}}$$

where $\tilde{H}(\omega) = H(\omega)$ within the interval $(-\omega_c, \omega_c)$. To recover $H(\omega)$ fully requires masking it outside the $(-\omega_c, \omega_c)$ interval

$$H(\omega) = \mathsf{Rect}_{\omega_c}(\omega) \sum_{n=-\infty}^{\infty} \frac{\pi}{\omega_c} h_n \mathrm{e}^{-\frac{\mathrm{i}n\pi\omega}{\omega_c}}$$

given the Fourier Transform pair

$$\frac{\omega_c}{\pi} \frac{\sin(\omega_c t - n\pi)}{(\omega_c t - n\pi)} \underset{\mathcal{F}^{-1}}{\overset{\mathcal{F}}{\Leftrightarrow}} \mathsf{Rect}_{\omega_c}(\omega) \mathrm{e}^{-\frac{\mathrm{i}n\pi\omega}{\omega_c}}$$

we have the result Eq. 23.

The frequency $2\omega_c$ is called the *Nyquist Sampling Rate*. If time-domain samples are taken at an interval less than $\Delta T_c = \frac{\pi}{\omega_c}$ then the original function is *exactly* reproduced by *Sinc interpolation* Eq. 23. If $\Delta T > \Delta T_c$ then aliasing occurs.

2.3.3 Frequency Sampling Theorem

The corresponding sampling theorem in the frequency domain is as follows

If a function $h(t)$ is zero outside a certain period T, i.e.

$$h(t) = 0 \quad \text{for } |t| \geq T \tag{24}$$

then its Fourier Transform $H(\omega)$ can be uniquely determined from its values $H\left(\frac{n\pi}{T}\right)$ at a sequence of equidistant points, $\frac{\pi}{T}$ apart in frequency. In fact $H(\omega)$ is given by

$$H(\omega) = \sum_{n=-\infty}^{\infty} H\left(\frac{n\pi}{T}\right) \frac{\sin(\omega T - n\pi)}{(\omega T - n\pi)} \tag{25}$$

3 Interpolation of Functions

3.1 Interpolation through Shannon Reconstruction

The Shannon Sampling Theorem Eq. 23 tells us that a function can be exactly reconstructed by convolution with a Sinc function and summing to infinity, provided that it is sampled above the Nyquist rate. Let's look at an example

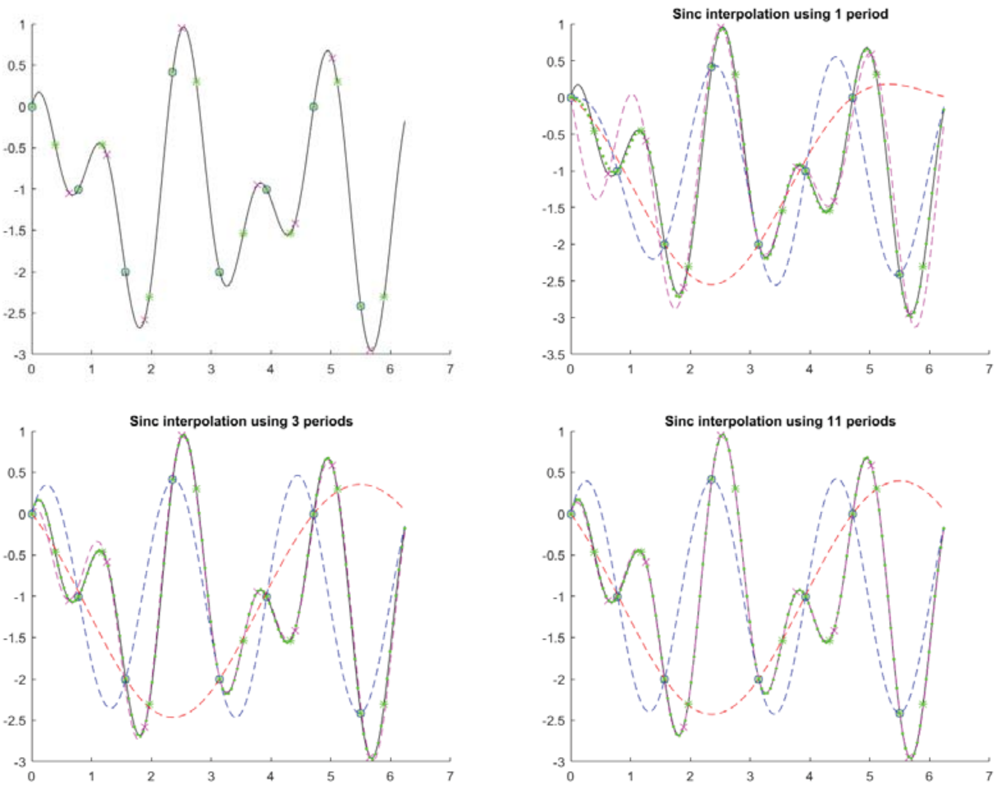


Figure 1: Top Left : The function $\cos(5t) + \sin 3t - 1$ sampled at different rates : $+$ 4 samples in period, $\circ$ 8 samples per period, $\times$ 10 samples per period (i.e. the Nyquist rate), $*$ 16 samples per period. Top Right : Reconstruction of the function at 128 sampling points from Eq. 23 using $n = 1$; $--$ reconstruction from 4 samples in period, $--$ reconstruction from 8 samples per period, $--$ reconstruction from 10 samples per period (i.e. the Nyquist rate), $--$ reconstruction from 16 samples per period. Bottom Left : As Top Right with $n = 3$. Bottom Right : As Top Right with $n = 11$.

In figure 1 we show the function

$$\cos(5t) + \sin 3t - 1$$

with different sampling rates. By construction the maximum frequency in the signal is 5 (units of radians) which means that 10 samples per period are required to meet the Nyquist sampling rate. We show samples with $\{4, 8, 10, 16\}$ per period. When using these samples in Eq. 23 we see two things

1. When sampling *below* the Nyquist rate, the function cannot be reconstructed adequately
2. When sampling *at or above* the Nyquist rate, the function can be well reconstructed, **but** the accuracy still depends on how large a number of periods we reconstruct from; i.e. how close n is to the 'ideal' value of infinity.

3.2 Other Interpolation Schemes

The Shannon reconstruction is the "ideal interpolant" because convolving with the Sinc ($\text{Sinc}(t) := \frac{\sin(t)}{t}$) corresponds to truncation in the Fourier domain of all the higher harmonic copies of the original function which were induced by the sampling process. However, in the spatial domain, it is interpolation with a Sinc function. Since the Sinc function has *infinite support*, to interpolate to between-sample-points amounts to finding a weighted sum of all the other samples, not just those taken, but those implicitly taken by replicating the sampled period out to infinity in both directions; (recall this is an *assumption* of the Fourier Transform : the function being sampled is periodic and so the samples repeat infinitely). Clearly this becomes computationally heavy, especially in multiple dimensions.

Other interpolation schemes use interpolants of *finite support* which means that the summation of weighted samples is limited to a small number.

- **Nearest Neighbour Interpolation** simply takes the between-sample value to be that of the nearest value. This is equivalent to convolution with $\text{Rect}_{\Delta t}(t)$: a rectangle function of width Δt. In the frequency domain this is therefore multiplication with a Sinc with period $N\omega_0$, i.e. the one whose zeroes correspond to the maximum sampling frequency and multiples thereof. We can see that the resultant function is not the original one but one with multiple harmonics, going out to infinity, decreasing at a rate of $1/\omega$. Thus high frequencies are generated in the original function which should not be there
- **Linear Interpolation** is the next simplest scheme. Here the between-sample value is the linear interpolation of the two nearest samples. This is equivalent to convolution with $\text{Triangle}_{2\Delta t}(t)$: a triangle function of width $2\Delta t$. The Triangle function is in fact the convolution of two Rect functions. From this we see that its Fourier Transform is Sinc^2. The Sinc^2 function has zeros at the same places as the Sinc function, but its decay rate is $1/\omega^2$. Thus the generation of false high frequency is less intrusive than for Nearest Neighbor interpolation.
- **Cubic Interpolation**. By convolving a Triangle with itself we get a cubic function. This weights four sample points to give a between-sample value. Its Fourier Transform is Sinc^4.

Thus the false high frequencies are killed off even faster; but so too are the true high frequencies up to the Nyquist. So this interpolant has a smoothing effect but might still not be all that is desired.

3.3 Fourier Zero-Fill Method

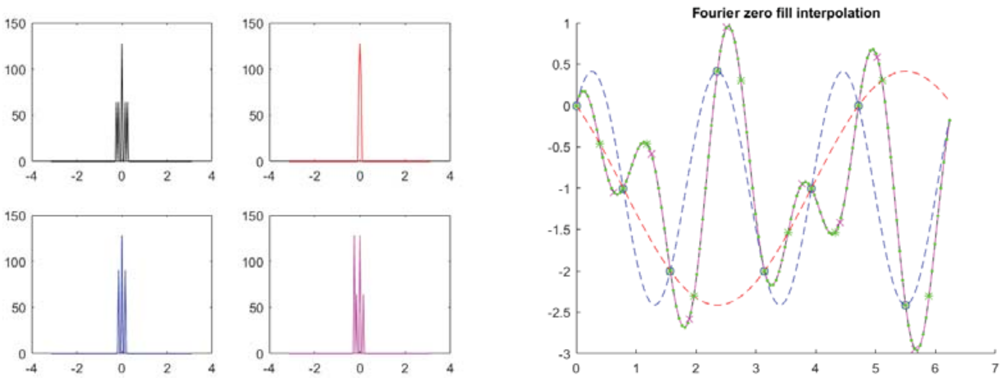


Figure 2: Left : The Fourier Transform of the original function and the samples from figure 1. Right : the result of adding $N - M$ zeros to the DFT of the sampled data followed by IDFT (Fourier zero-fill interpolation).

The above interpolation schemes give an analytic function from a set of discrete samples, i.e. they are discrete-to-continuous schemes. If we only require an increased number of samples, e.g. from the original N samples to a larger number M, but still in the same period, we can instead exploit the Fourier Transform as a consequence of the Sampling Theorem. Recall that the Sampling Theorem not only tells us that we can reconstruct exactly given samples taken at or above the Nyquist rate, but, in fact, that the Nyquist Rate is all that is needed and going at a higher rate adds *no extra information.* To see this recall that the original function is band-limited at maximum frequency ω_c. This means that its Fourier Transform is zero for $|\omega| > \omega_c$. When sampling in an interval of length T, the *lowest* frequency obtained is $\omega_0 = \frac{2\pi}{T}$, known as the fundamental frequency. If we take N samples in the time domain they lie at the points

$$\{-N/2\omega_0, (1 - N/2)\omega_0 \ldots, -\omega_0, 0, \omega_0, \ldots (N/2 - 1)\omega_0\}$$

where $\frac{N\omega_0}{2} = \frac{N\pi}{T} = \frac{\pi}{\Delta t}$. Thus if $\frac{N\omega_0}{2} > \omega_c$ it will have a value of zero, because we are just taking samples of a function which is zero above the critical frequency. We therefore see that in order to synthesise samples in the time domain that are at a higher rate, we only need to add the extra frquencies in the Fourier domain - which are all zero. This is the basis of *Fourier Zero-Fill interpolation* which is summarised like this

Take N samples of the function $h(t)$ at spacing ΔT : $\{h_n = h(n\Delta T); n = 0, \ldots, N-1\}$

Take the DFT : $\{H_n\} = \mathsf{DFT}(\{h_n\})$

Add $M - N$ zeros into the list $\{H_n\}$ at the most positive and most negative frequencies $\{\tilde{H}_n\} = \{0, 0, \ldots, 0, \{\tilde{H}_n\}, 0, \ldots 0\}$.

Take the inverse DFT : $\{\tilde{h}_n\} = \mathsf{DFT}^{-1}(\{\tilde{H}_n\})$

Now $\tilde{h}_n = h\left(\frac{N}{M} n\Delta T\right)$; i.e. the sample interval has decreased from ΔT to $\frac{N}{M}\Delta T$