Taylor Series for Functions of One Variable

22.1. $$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^{(n-1)}(a)(x-a)^{n-1}}{(n-1)!} + R_n$$

where R_n, the remainder after n terms, is given by either of the following forms:

22.2. Lagrange's form: $$R_n = \frac{f^{(n)}(\xi)(x-a)^n}{n!}$$

22.3. Cauchy's form: $$R_n = \frac{f^{(n)}(\xi)(x-\xi)^{n-1}(x-a)}{(n-1)!}$$

The value ξ, which may be different in the two forms, lies between a and x. The result holds if $f(x)$ has continuous derivatives of order n at least.

If $\lim_{n\to\infty} R_n = 0$, the infinite series obtained is called the *Taylor series* for $f(x)$ about $x = a$. If $a = 0$, the series is often called a *Maclaurin series*. These series, often called power series, generally converge for all values of x in some interval called the *interval of convergence* and diverge for all x outside this interval.

Some series contain the Bernoulli numbers B_n and the Euler numbers E_n defined in Chapter 23, pages 142–143.

Binomial Series

22.4. $$\begin{aligned}(a+x)^n &= a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \cdots \\ &= a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \binom{n}{3}a^{n-3}x^3 + \cdots\end{aligned}$$

Special cases are

22.5. $$(a+x)^2 = a^2 + 2ax + x^2$$

22.6. $$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$$

22.7. $$(a+x)^4 = a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4$$

22.8. $$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \cdots \qquad -1 < x < 1$$

22.9. $$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \cdots \qquad -1 < x < 1$$

22.10. $$(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + 15x^4 - \cdots \qquad -1 < x < 1$$

22.11. $(1+x)^{-1/2} = 1 - \dfrac{1}{2}x + \dfrac{1 \cdot 3}{2 \cdot 4}x^2 - \dfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \cdots \qquad -1 < x \leqq 1$

22.12. $(1+x)^{1/2} = 1 + \dfrac{1}{2}x - \dfrac{1}{2 \cdot 4}x^2 + \dfrac{1 \cdot 3}{2 \cdot 4 \cdot 6}x^3 - \cdots \qquad -1 < x \leqq 1$

22.13. $(1+x)^{-1/3} = 1 - \dfrac{1}{3}x + \dfrac{1 \cdot 4}{3 \cdot 6}x^2 - \dfrac{1 \cdot 4 \cdot 7}{3 \cdot 6 \cdot 9}x^3 + \cdots \qquad -1 < x \leqq 1$

22.14. $(1+x)^{1/3} = 1 + \dfrac{1}{3}x - \dfrac{2}{3 \cdot 6}x^2 + \dfrac{2 \cdot 5}{3 \cdot 6 \cdot 9}x^3 - \cdots \qquad -1 < x \leqq 1$

Series for Exponential and Logarithmic Functions

22.15. $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots \qquad -\infty < x < \infty$

22.16. $a^x = e^{x \ln a} = 1 + x \ln a + \dfrac{(x \ln a)^2}{2!} + \dfrac{(x \ln a)^3}{3!} + \cdots \qquad -\infty < x < \infty$

22.17. $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots \qquad -1 < x \leqq 1$

22.18. $\dfrac{1}{2}\ln\left(\dfrac{1+x}{1-x}\right) = x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \dfrac{x^7}{7} + \cdots \qquad -1 < x < 1$

22.19. $\ln x = 2\left\{\left(\dfrac{x-1}{x+1}\right) + \dfrac{1}{3}\left(\dfrac{x-1}{x+1}\right)^3 + \dfrac{1}{5}\left(\dfrac{x-1}{x+1}\right)^5 + \cdots\right\} \qquad x > 0$

22.20. $\ln x = \left(\dfrac{x-1}{x}\right) + \dfrac{1}{2}\left(\dfrac{x-1}{x}\right)^2 + \dfrac{1}{3}\left(\dfrac{x-1}{x}\right)^3 + \cdots \qquad x \geqq \dfrac{1}{2}$

Series for Trigonometric Functions

22.21. $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots \qquad -\infty < x < \infty$

22.22. $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \cdots \qquad -\infty < x < \infty$

22.23. $\tan x = x + \dfrac{x^3}{3} + \dfrac{2x^5}{15} + \dfrac{17x^7}{315} + \cdots + \dfrac{2^{2n}(2^{2n}-1)B_n x^{2n-1}}{(2n)!} + \cdots \qquad |x| < \dfrac{\pi}{2}$

22.24. $\cot x = \dfrac{1}{x} - \dfrac{x}{3} - \dfrac{x^3}{45} - \dfrac{2x^5}{945} - \cdots - \dfrac{2^{2n}B_n x^{2n-1}}{(2n)!} - \cdots \qquad 0 < |x| < \pi$

22.25. $\sec x = 1 + \dfrac{x^2}{2} + \dfrac{5x^4}{24} + \dfrac{61x^6}{720} + \cdots + \dfrac{E_n x^{2n}}{(2n)!} + \cdots \qquad |x| < \dfrac{\pi}{2}$

22.26. $\csc x = \dfrac{1}{x} + \dfrac{x}{6} + \dfrac{7x^3}{360} + \dfrac{31x^5}{15{,}120} + \cdots + \dfrac{2(2^{2n-1}-1)B_n x^{2n-1}}{(2n)!} + \cdots \qquad 0 < |x| < \pi$

22.27. $\sin^{-1} x = x + \dfrac{1}{2}\dfrac{x^3}{3} + \dfrac{1 \cdot 3}{2 \cdot 4}\dfrac{x^5}{5} + \dfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\dfrac{x^7}{7} + \cdots \qquad |x| < 1$

22.28. $\cos^{-1} x = \dfrac{\pi}{2} - \sin^{-1} x = \dfrac{\pi}{2} - \left(x + \dfrac{1}{2}\dfrac{x^3}{3} + \dfrac{1 \cdot 3}{2 \cdot 4}\dfrac{x^5}{5} + \cdots\right) \qquad |x| < 1$

22.29. $\tan^{-1} x = \begin{cases} x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \cdots & |x| < 1 \\ \pm\dfrac{\pi}{2} - \dfrac{1}{x} + \dfrac{1}{3x^3} - \dfrac{1}{5x^5} + \cdots & (+ \text{ if } x \geqq 1, - \text{ if } x \leqq -1) \end{cases}$

22.30. $\cot^{-1} x = \dfrac{\pi}{2} - \tan^{-1} x = \begin{cases} \dfrac{\pi}{2} - \left(x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \cdots\right) & |x| < 1 \\ p\pi + \dfrac{1}{x} - \dfrac{1}{3x^3} + \dfrac{1}{5x^5} - \cdots & (p = 0 \text{ if } x > 1,\ p = 1 \text{ if } x < -1) \end{cases}$

22.31. $\sec^{-1} x = \cos^{-1}(1/x) = \dfrac{\pi}{2} - \left(\dfrac{1}{x} + \dfrac{1}{2 \bullet 3x^3} + \dfrac{1 \cdot 3}{2 \bullet 4 \bullet 5x^5} + \cdots\right)$ $\qquad |x| > 1$

22.32. $\csc^{-1} x = \sin^{-1}(1/x) = \dfrac{1}{x} + \dfrac{1}{2 \bullet 3x^3} + \dfrac{1 \cdot 3}{2 \bullet 4 \bullet 5x^5} + \cdots$ $\qquad |x| > 1$

Series for Hyperbolic Functions

22.33. $\sinh x = x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \dfrac{x^7}{7!} + \cdots$ $\qquad -\infty < x < \infty$

22.34. $\cosh x = 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \cdots$ $\qquad -\infty < x < \infty$

22.35. $\tanh x = x - \dfrac{x^3}{3} + \dfrac{2x^5}{15} - \dfrac{17x^7}{315} + \cdots \dfrac{(-1)^{n-1} 2^{2n}(2^{2n} - 1)B_n x^{2n-1}}{(2n)!} + \cdots$ $\qquad |x| < \dfrac{\pi}{2}$

22.36. $\coth x = \dfrac{1}{x} + \dfrac{x}{3} - \dfrac{x^3}{45} + \dfrac{2x^5}{945} + \cdots \dfrac{(-1)^{n-1} 2^{2n} B_n x^{2n-1}}{(2n)!} + \cdots$ $\qquad 0 < |x| < \pi$

22.37. $\operatorname{sech} x = 1 - \dfrac{x^2}{2} + \dfrac{5x^4}{24} - \dfrac{61x^6}{720} + \cdots \dfrac{(-1)^n E_n x^{2n}}{(2n)!} + \cdots$ $\qquad |x| < \dfrac{\pi}{2}$

22.38. $\operatorname{csch} x = \dfrac{1}{x} - \dfrac{x}{6} + \dfrac{7x^3}{360} - \dfrac{31x^5}{15{,}120} + \cdots \dfrac{(-1)^n 2(2^{2n-1} - 1)B_n x^{2n-1}}{(2n)!} + \cdots$ $\qquad 0 < |x| < \pi$

22.39. $\sinh^{-1} x = \begin{cases} x - \dfrac{x^3}{2 \bullet 3} + \dfrac{1 \bullet 3x^5}{2 \bullet 4 \bullet 5} - \dfrac{1 \bullet 3 \bullet 5x^7}{2 \bullet 4 \bullet 6 \bullet 7} + \cdots & |x| < 1 \\ \pm\left(\ln|2x| + \dfrac{1}{2 \bullet 2x^2} - \dfrac{1 \bullet 3}{2 \bullet 4 \bullet 4x^4} + \dfrac{1 \bullet 3 \bullet 5}{2 \bullet 4 \bullet 6 \bullet 6x^6} - \cdots\right) & \begin{bmatrix} + \text{ if } x \geqq 1 \\ - \text{ if } x \leqq -1 \end{bmatrix} \end{cases}$

22.40. $\cosh^{-1} x = \pm\left\{\ln(2x) - \left(\dfrac{1}{2 \bullet 2x^2} + \dfrac{1 \bullet 3}{2 \bullet 4 \bullet 4x^4} + \dfrac{1 \bullet 3 \bullet 5}{2 \bullet 4 \bullet 6 \bullet 6x^6} + \cdots\right)\right\}$ $\qquad \begin{bmatrix} + \text{ if } \cosh^{-1} x > 0,\ x \geqq 1 \\ - \text{ if } \cosh^{-1} x < 0,\ x \geqq 1 \end{bmatrix}$

22.41. $\tanh^{-1} x = x + \dfrac{x^3}{3} + \dfrac{x^5}{5} + \dfrac{x^7}{7} + \cdots$ $\qquad |x| < 1$

22.42. $\coth^{-1} x = \dfrac{1}{x} + \dfrac{1}{3x^3} + \dfrac{1}{5x^5} + \dfrac{1}{7x^7} + \cdots$ $\qquad |x| > 1$

Miscellaneous Series

22.43. $e^{\sin x} = 1 + x + \dfrac{x^2}{2} - \dfrac{x^4}{8} - \dfrac{x^5}{15} + \cdots$ $\qquad -\infty < x < \infty$

22.44. $e^{\cos x} = e\left(1 - \dfrac{x^2}{2} + \dfrac{x^4}{6} - \dfrac{31x^6}{720} + \cdots\right)$ $\qquad -\infty < x < \infty$

22.45. $e^{\tan x} = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{2} + \dfrac{3x^4}{8} + \cdots \qquad |x| < \dfrac{\pi}{2}$

22.46. $e^x \sin x = x + x^2 + \dfrac{x^3}{3} - \dfrac{x^5}{30} - \dfrac{x^6}{90} + \cdots + \dfrac{2^{n/2}\sin(n\pi/4)x^n}{n!} + \cdots \qquad -\infty < x < \infty$

22.47. $e^x \cos x = 1 + x - \dfrac{x^3}{3} - \dfrac{x^4}{6} + \cdots + \dfrac{2^{n/2}\cos(n\pi/4)x^n}{n!} + \cdots \qquad -\infty < x < \infty$

22.48. $\ln|\sin x| = \ln|x| - \dfrac{x^2}{6} - \dfrac{x^4}{180} - \dfrac{x^6}{2835} - \cdots - \dfrac{2^{2n-1}B_n x^{2n}}{n(2n)!} + \cdots \qquad 0 < |x| < \pi$

22.49. $\ln|\cos x| = -\dfrac{x^2}{2} - \dfrac{x^4}{12} - \dfrac{x^6}{45} - \dfrac{17x^8}{2520} - \cdots - \dfrac{2^{2n-1}(2^{2n}-1)B_n x^{2n}}{n(2n)!} + \cdots \qquad |x| < \dfrac{\pi}{2}$

22.50. $\ln|\tan x| = \ln|x| + \dfrac{x^2}{3} + \dfrac{7x^4}{90} + \dfrac{62x^6}{2835} + \cdots + \dfrac{2^{2n}(2^{2n-1}-1)B_n x^{2n}}{n(2n)!} + \cdots \qquad 0 < |x| < \dfrac{\pi}{2}$

22.51. $\dfrac{\ln(1+x)}{1+x} = x - (1+\tfrac{1}{2})x^2 + (1+\tfrac{1}{2}+\tfrac{1}{3})x^3 - \cdots \qquad |x| < 1$

Reversion of Power Series

Suppose

22.52. $y = C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + C_6x^6 + \cdots$

then

22.53. $x = C_1y + C_2y^2 + C_3y^3 + C_4y^4 + C_5y^5 + C_6y^6 + \cdots$

where

22.54. $c_1C_1 = 1$

22.55. $c_1^3C_2 = -c_2$

22.56. $c_1^5C_3 = 2c_2^2 - c_1c_3$

22.57. $c_1^7C_4 = 5c_1c_2c_3 - 5c_2^3 - c_1^2c_4$

22.58. $c_1^9C_5 = 6c_1^2c_2c_4 + 3c_1^2c_3^2 - c_1^3c_5 + 14c_2^4 - 21c_1c_2^2c_3$

22.59. $c_1^{11}C_6 = 7c_1^3c_2c_5 + 84c_1c_2^3c_3 + 7c_1^3c_3c_4 - 28c_1^2c_2c_3^2 - c_1^4c_6 - 28c_1^2c_2^2c_4 - 42c_2^5$

Taylor Series for Functions of Two Variables

22.60. $$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) + \frac{1}{2!}\{(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\} + \cdots$$

where $f_x(a,b)$, $f_y(a,b), \ldots$ denote partial derivatives with respect to $x, y, \ldots$ evaluated at $x = a, y = b$.