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# On Gerstner's Water Wave

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**Abstract**

We present a simple approach showing that Gerstner's flow is dynamically possible: each particle moves on a circle, but the particles never collide and fill out the entire region below the surface wave.

## 1 Introduction

The motion of a body of water as waves propagate along the surface is a ubiquitous, easily observable, yet eminently perplexing phenomenon. The undulations on the water surface often adopt inherently beautiful forms, such as the concentric circles observed when ripples spread out from a source of disturbance, or the smooth and regular features of steady periodic waves. Yet despite mankind's innate curiosity concerning water wave motion, it was really only in the $18^{th}$ century that efforts to provide a rigorous mathematical description of fluid motion began in earnest [31]. It was during this time that Daniel Bernoulli introduced the term hydrodynamics to comprise the two sciences of hydrostatics and hydraulics, and this was the era when the foundations of the modern theory of hydrodynamics were laid down by Bernoulli, Euler, Lagrange and d'Alembert, among others. The $19^{th}$ century produced many major advances in this nascent science, notably from such luminaries as Navier, and especially Stokes, and from the time of these seminal works to this present day such vast development and progress has been made in the subject that fluid dynamics now comprises several branches of mathematics and indeed science.

Yet, and given how hydrodynamics has developed symbiotically with so many branches of mathematics, it is startling how many gaps remain in the basic mathematical theory of fluid dynamics, even in the idealised case of a so-called perfect fluid (homogeneous and inviscid). Nowhere is this shortfall encapsulated better than the fact that there are very few explicit solutions known for the full governing equations of a perfect fluid with a non-flat free surface [19, 4, 16, 24]. The first such solution, which will be presented below, was described by Gerstner as far back as 1802 [19], c.f. [31], and was independently re-discovered later by Rankine [32]. Furthermore, Gerstner's solution is restrictive insofar as it exists only for deep water, and it describes a rotational wave. In recent years there has been growing interest in free surface water waves (see [1, 2, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15, 18, 20, 21, 22, 25, 35, 36]), but the current state of affairs regarding explicit solutions has not advanced much since the last century.

A mathematical analysis of Gerstner's solution was performed by Constantin in [3] which investigated rigorously, for the first time, the evolution of the fluid domain under the propagation of Gerstner's wave. It was shown, using a mixture of analytical and topological methods, that it is indeed possible to have a motion of the whole fluid body where all the particles describe circles with a depth-dependant radius— the fluid domain as a whole evolves in a manner which is consistent with the full governing equations. It is interesting to note that the analysis presented here was adapted by the same author in [4] to present an explicit solution of the governing equations which describes the propagation of edge-waves along a sloping beach. It is our aim in this paper to present an alternative approach to the analysis of Gerstner's wave, mainly the use of degree theory, and in the process hope to simplify some of the proofs contained in [3].

Notice that approaches concentrating on the analysis of particle paths within the fluid were recently shown to be of interest in the context of tsunamis [10]. It is thus of interest to broaden our understanding of the few cases where we can analyse the particle paths for explicit solutions to the governing equations for water waves.

## 2 Governing equations for a perfect fluid

In many commonly occurring circumstances the motion of water waves appears to be two-dimensional. For instance, individual waves which are generated far offshore by disturbances such as the wind can be observed grouping into wave-trains, which then propagate along the surface of the ocean in the horizontal direction perpendicular to the crest-line [23]. Assuming the waves to be two dimensional, that is, the motion is identical in any direction parallel to the crest line, we need only consider a cross section of the flow in the direction perpendicular to the crest line, using the Cartesian coordinates $(x, y)$ where the $x-$axis is the direction of wave propagation while the $y-$axis points vertically upwards. Furthermore, for water waves of non-negligible amplitude, such as those to be found on the ocean, it is reasonable to neglect the effects of surface tension [23] and so from now on we assume that the only restorative force being applied to the fluid is gravity— we are dealing with gravity waves.

The free surface is described by $y = \eta(x)$, with mean water level $y = 0$. Often in water wave problems it is the determination of the free-surface which is of most interest to us, and it is undetermined *a priori*. To describe deep water waves we assume the fluid domain to be the semi-infinite region $\{(x, y) : x \in \mathbb{R}, y < \eta(t, x)\}$ which is bounded above by $\eta$. Let $(u(t, x, y), v(t, x, y))$ be the velocity field of the flow in the fluid domain. We adopt the assumption of homogeneity (constant density) in the fluid— this is physically reasonable for the case of surface waves propagating on a deep ocean [17]— which implies the equation of mass conservation

$$u_x + v_y = 0. \tag{2.1}$$

Making the further assumption of inviscid flow, the governing equations for the motion of the waves are given by Euler's equation

$$\begin{cases} u_t + uu_x + vu_y = -P_x, \\ v_t + uv_x + vv_y = -P_y - g, \end{cases} \tag{2.2}$$

where $P(t,x,y)$ denotes the pressure and $g$ is the gravitational constant of acceleration. To decouple the motion of the air from that of the free surface particles [23] we introduce the dynamic boundary condition

$$P = P_0 \qquad \text{on } y = \eta(t,x), \tag{2.3}$$

where $P_0$ is the constant atmospheric pressure. Since the free-surface is always composed of the same particles we have the kinematic boundary condition

$$v = \eta_t + u\eta_x \qquad \text{on } y = \eta(t,x). \tag{2.4}$$

The boundary condition at the bottom, expressing the fact that at great depths there is practically no motion, is given by

$$(u,v) \to (0,0) \text{ as } y \to -\infty, \text{ uniformly for } x \in \mathbb{R}, t \geq 0. \tag{2.5}$$

The governing equations for the deep water gravity wave problem are encompassed by the nonlinear free boundary problem (2.1)–(2.5), cf. [23].

## 3 Gerstner's wave

An in-depth description of Gerstner's solution is performed in [3]: we will repeat some of the main points here. Gerstner's wave is a two-dimensional wave which adopts the Lagrangian viewpoint, describing the evolution of individual water particles. A description of Gerstner's wave requires two parameters, $a \in \mathbb{R}$ and $b \leq b_0$ for a fixed $b_0 \leq 0$, and so the lower half-plane represents the still water body. Suppose we choose a particular particle, by fixing $a$ and $b$. Gerstner's wave is given by

$$\begin{aligned} x &= a + \tfrac{e^{mb}}{m} \sin m(a + \sqrt{\tfrac{g}{m}}t) \\ y &= b - \tfrac{e^{mb}}{m} \cos m(a + \sqrt{\tfrac{g}{m}}t) \end{aligned} \tag{3.1}$$

where $m > 0$ is fixed. It follows that the path of the particle is a circle centred at $(a,b)$ with radius $\frac{e^{mb}}{m}$, with the particle moving anti-clockwise with angular velocity $\sqrt{\frac{g}{m}}$. We can obtain a description of the motion of another particle simply by changing the values of $a$ and $b$ in (3.1).

Let us call the set prescribed by $\{(a,b_0) : a \in \mathbb{R}\}$ the still water surface. Then the profile of the surface wave as time evolves is given by a smooth curve, obtained by setting $b = b_0$ in (3.1):

$$\begin{aligned} x &= a + \tfrac{e^{mb_0}}{m} \sin m(a + \sqrt{\tfrac{g}{m}}t) \\ y &= b_0 - \tfrac{e^{mb_0}}{m} \cos m(a + \sqrt{\tfrac{g}{m}}t), \end{aligned} \tag{3.2}$$

which is the equation of a troichoid for $b \leq b_0$. The extreme case with the still water surface being at $\{(a,0) : a \in \mathbb{R}\}$ leads to a surface wave having the profile of a cycloid, a continuous curve with upward cusps. We have used the term "still water surface" somewhat loosely. We will see that Gerstner's wave describes rotational motion and this fact rules out the possibility that the motion could have developed from a water body initially at rest [34].

In order to show that (3.1) provides a solution to the governing equations, we must prove that it defines a motion of the whole fluid body for which:

1. the equation of continuity (2.1) holds;
2. a suitable pressure exists which enables Eulers equation (2.2) and the boundary condition (2.3) to be satisfied;
3. a particle on the free surface stays on the free surface, as governed by condition (2.4);
4. the limiting boundary condition (2.5) is satisfied.

In the literature [31] discussions of Gerstner's wave typically focus on showing the compatibility of Gerstner's transformation (3.1) with the full governing equations for two dimensional waves. The paper [3] was the first work on this subject containing a rigorous examination of the motion of the entire fluid domain under Gerstner's wave motion (3.1). The author proved that it was possible to have a motion of the fluid where all particles describe circles with a depth-dependant radius. Our aim in this paper is to present an alternative approach to this analysis, using degree theory, and in the process to simplify some of the steps of [3].

We first note that it suffices to analyse Gerstner's map (3.1) at the time $t = 0$, where it is of the form

$$\begin{aligned} x &= a + \frac{e^{mb}}{m} \sin ma \\ y &= b - \frac{e^{mb}}{m} \cos ma, \end{aligned} \tag{3.3}$$

since the general case follows by changing variables $(a, b) \to (a + t\sqrt{\frac{g}{m}}, b)$, performing the map (3.3), and then shifting the horizontal variable by $t\sqrt{\frac{g}{m}}$. Now, the right hand side of (3.3) has a straightforward $a-$dependence. As $a$ varies by the amount $2\pi/m$, the $y$ value reoccurs, while $x$ is shifted linearly by $2\pi/m$. Therefore it suffices in what follows to analyse the map (3.3) on the restricted interval $a \in [0, 2\pi/m]$.

For fixed $b \leq b_0$, we can plot the image of $\{(a, b) : a \in \mathbb{R}\}$ under (3.3) by regarding the horizontal coordinate as a function of the vertical coordinate. It is here that the restriction $b_0 \leq 0$ is important, for otherwise we would obtain the graph of a self-intersecting curve [3]. For fixed $b$ and $a \in [0, \pi/m]$ we have

$$\begin{aligned} &y \mapsto \frac{1}{m} \arccos\left[\frac{m(b-y)}{e^{mb}}\right] + \frac{e^{mb}}{m}\sqrt{1 - \frac{m^2}{e^{2mb}}(b-y)^2} \\ &y \in \left[b - \frac{e^{mb}}{m}, b + \frac{e^{mb}}{m}\right], \end{aligned}$$

while for $a \in [\pi/m, 2\pi/m]$ we obtain

$$\begin{aligned} &y \mapsto \frac{2\pi}{m} - \frac{1}{m} \arccos\left[\frac{m(b-y)}{e^{mb}}\right] - \frac{e^{mb}}{m}\sqrt{1 - \frac{m^2}{e^{2mb}}(b-y)^2} \\ &y \in \left[b - \frac{e^{mb}}{m}, b + \frac{e^{mb}}{m}\right]. \end{aligned}$$

We see in Figure 1 that for $b < 0$ the image of the line $y = b$ is a smooth curve— a troichoid— while in the extreme case $b = 0$ we get a piecewise smooth curve with sharp peaks occurring at the points $\{((2n + 1)\pi/m, 1) : n \in \mathbb{Z}\}$— a cycloid. Notice too that the vertical half-lines $\{(2n\pi/m, b) : b \leq b_0\}$ and $\{((2n + 1)\pi/m, b) : b \leq b_0\}$ are mapped into $\{(2n\pi/m, y) : y \leq b_0 - e^{mb_0}/m\}$ and $\{((2n + 1)\pi/m, y) : y \leq b_0 + e^{mb_0}/m\}$ respectively.

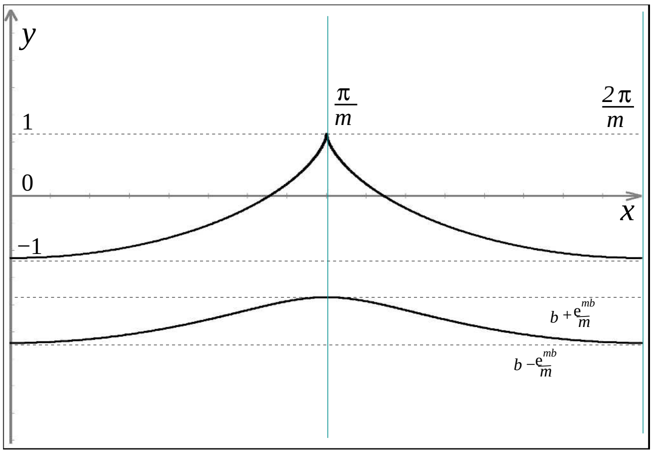


Figure 1: Graph of $\{(a, b) : a \in [0, \pi/m]\}$ under (3.3). For a typical value $b \le b_0 < 0$ we get a smooth curve; while the extreme case $b = b_0 = 0$ results in a cusped curve along the surface.

For the sake of completeness we present the proof of a result from [3] stating that Gerstner's transformation is locally diffeomorphic and injective. A result on the global nature of the transformation will soon follow.

**Lemma 1.** *[3] For every fixed $t \ge 0$, the map* (3.1) *defines a local diffeomorphism from the domain $\mathcal{D} = \{(a, b) : a \in \mathbb{R}, b < 0\}$ in the $(x, y)-$plane injectively into its image.*

**Proof.** It follows from our previous comments that we can simplify our analysis by restricting ourselves to the time independent map (3.3), and to the domain $\mathcal{D}' = \mathcal{D} \cap \{(a, b) : a \in (0, 2\pi/m)\}$.

The differential of (3.3) at a fixed point $(a, b)$ with $b < 0$ takes the form

$$\begin{pmatrix} 1 + e^{mb} \cos ma & e^{mb} \sin ma \\ e^{mb} \sin ma & 1 - e^{mb} \cos ma \end{pmatrix}$$

with determinant $1 - e^{2mb} > 0$. Applying the inverse function theorem proves that (3.3) is a local diffeomorphism from $\mathcal{D}$ onto its image.

Regarding injectivity, let us adopt the notation of [3] and write the transformation (3.3) in the form

$$F(\mathbf{x}) = \mathbf{x} + f(\mathbf{x}),$$

where $\mathbf{x} = (x, y)$. If $|\cdot|$ denotes the usual Euclidean norm in $\mathbb{R}^2$ we have from the mean value theorem

$$\begin{aligned} |F(\mathbf{x}_1) - F(\mathbf{x}_2)| &\ge |\mathbf{x}_1 - \mathbf{x}_2| - |f(\mathbf{x}_1) - f(\mathbf{x}_2)| \\ &\ge |\mathbf{x}_1 - \mathbf{x}_2| - \max_{s \in [0,1]} (\|Df_{sx_1 + (1-s)x_2}\|) |\mathbf{x}_1 - \mathbf{x}_2|. \end{aligned}$$

Since the differential is given by

$$Df_{(a,b)} = e^{mb} \begin{pmatrix} \cos ma & \sin ma \\ \sin ma & \cos ma \end{pmatrix}$$

we have $\|Df_{(a,b)}\| = e^{mb}$ and so

$$|F(\mathbf{x}_1) - F(\mathbf{x}_2)| \geq (1 - e^{mb})|\mathbf{x}_1 - \mathbf{x}_2|$$

where $\mathbf{x}_1 = (a_1, b_1), \mathbf{x}_2 = (a_2, b_2)$ and $b = \max\{b_1, b_2\}$ in the relation above. This proves the injectivity of the map. ■

The above result shows that Gerstner's wave gives us a local diffeomorphism that is globally injective on $\mathcal{D}$. We will show that it is in fact a global diffeomorphism from $\mathcal{D}$ onto the fluid domain below the surface wave.

**Lemma 2.** *For every fixed $t \geq 0$, the map* (3.1) *defines a diffeomorphism from the domain below $\{(a, b_0) : a \in \mathbb{R}\}$ in the $(x, y)-$plane, where $b_0 \leq 0$, to the domain bounded above by the surface wave $y = h(t, x)$, where the profile of the surface wave is smooth if $b_0 < 0$, while if $b_0 = 0$ we have a continuous, piecewise smooth curve with upward cusps.*

**Proof.** Once again, we may focus on the restricted domain $\mathcal{D}' = \mathcal{D} \cap \{(a, b) : a \in (0, 2\pi/m)\}$ and the time independent map (3.3). The properties regarding the shape of the surface wave follow from our earlier discussions. As a result of Lemma (1), we will have proven our result if we can show that the map (3.3) is surjective onto the domain below the water surface.

It turns out that degree theory is tailor-made for supplying an answer to this question. Intuitively, one can regard the degree of a function on a set as measuring the existence or otherwise of zeros of that function on the set, and as such it is the ideal recourse when one is seeking answers concerning surjectivity, fixed points etc with respect to a certain mapping. For details regarding the construction of various types of degree c.f. [26, 28, 33]. We now recall the classical "Invariance of the Domain" theorem, in a form suitable for our purposes: *Let $\Omega$ be an open subset of $\mathbb{R}^n$ (not necessarily bounded). If $f : \Omega \to \mathbb{R}^n$ is one-to-one and continuous, then $f(\Omega)$ is open and $f(\partial\Omega) = \partial f(\Omega)$*; see [33]. It now follows that Gerstner's wave is surjective, using the following argument. If we define the domain $\Omega$ to be the open semi-infinite set $\mathcal{D}'$ with boundary $\{(0, b) : b \leq b_0\} \cup \{(a, b_0) : a \in [0, 2\pi/m]\} \cup \{((2\pi/m, b) : b \leq b_0\}$, then we have seen that $F$ maps the boundary of $\mathcal{D}'$ injectively into $\{(0, y) : y \leq b_0 - e^{mb_0}/m\} \cup \{troichoid/cycloid\} \cup \{(2\pi/m, y) : y \leq b_0 - e^{mb_0}/m\}$. The theorem asserts that the image of the boundary is the boundary of the image, and this, together with the image of $\mathcal{D}'$ being open, imply that the domain below the line $\{(a, b_0) : a \in \mathbb{R}\}$ in the $(x, y)-$plane, where $b_0 \leq 0$, is mapped surjectively by (3.3) onto the domain bounded above by the troichoid, or cycloid, given by setting $b = b_0$ in (3.3). Together with earlier results, it now follows that (3.3) is a global diffeomorphism. ■

We refer the reader to [3] for proof that Gerstner's wave satisfies the governing equations (2.1)–(2.3). Regarding the boundary conditions, we have seen above that particles on the free surface remain on it at all times, and therefore the kinematic surface condition (2.4) is satisfied. The velocity of a particle is given by

$$\mathbf{u}(t, x(t), y(t)) = \sqrt{\frac{g}{m}} \left( e^{mb} \cos m \left( a + \sqrt{\frac{g}{m}} t \right), e^{mb} \sin m \left( a + \sqrt{\frac{g}{m}} t \right) \right), \tag{3.4}$$

and since $y \to -\infty \implies b \to -\infty$, we have the absolute value of the velocity $\sqrt{\frac{g}{m}} e^{mb} \to 0$ as $y \to -\infty$, thereby satisfying (2.5).

Finally, we show by direct calculation that Gerstner's wave is rotational. For a two-dimensional flow the vorticity of each individual water particle is conserved as the particle moves about [30], and so it suffices to calculate the vorticity of the water particles at time $t = 0$. For each particle, represented in (3.3) by the parameters $(a, b)$ with $b < b_0$, the vorticity is calculated as $v_x - u_y$, and we use the relation

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} x_a & y_a \\ x_b & y_b \end{pmatrix}^{-1} \begin{pmatrix} \partial_a \\ \partial_b \end{pmatrix} = \frac{1}{1 - e^{2mb}} \begin{pmatrix} 1 - e^{mb}\cos ma & -e^{mb}\sin ma \\ -e^{mb}\sin ma & 1 + e^{mb}\cos ma \end{pmatrix} \begin{pmatrix} \partial_a \\ \partial_b \end{pmatrix},$$

together with

$$v_a = \sqrt{mg}e^{mb}\cos ma = u_b, \quad v_b = \sqrt{mg}e^{mb}\sin ma = -u_a,$$

to obtain

$$v_x - u_y = \frac{-2\sqrt{mg}e^{2mb}}{1 - e^{2mb}}.$$

Therefore the flow under Gerstner's wave is rotational, with vorticity only depending on the $b$−value of the particle's parameterisation and rapidly decreasing as we descend. Furthermore, the negative value of the vorticity shows that it is in the opposite sense to the motion of the particles.

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