Kosterlitz-Thouless phase transition

Simulation of the KT transition on basis of the xy model.

Ludwig Hendl, Isacco Gobbi

'Long range order in 2D systems with continuous symmetry?'

12 June 2020

Outline

1 Introduction

2 Code implementation

3 Results & Discussion
   - Observables

4 Performance

5 Conclusion

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- Berezinskii-Kosterlitz-Thouhless proposed topological phase transition 1971/1973 - Nobel prize 2016

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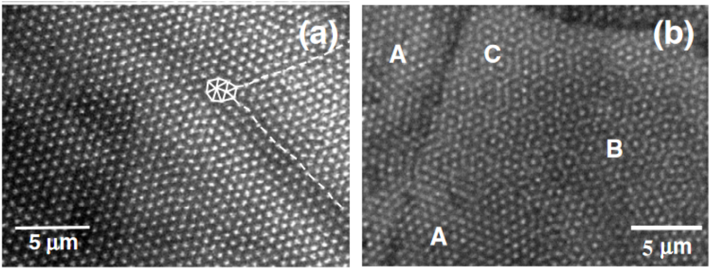

[0] Image from [3]

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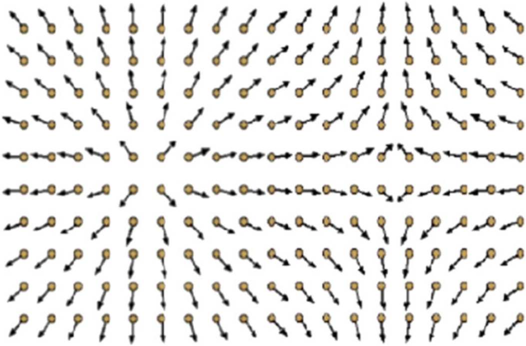


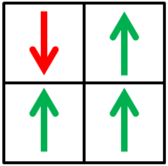
Fig. 2. Schematic view of pairs vortices anti-vortices.

[0] Image from [1]

2D Lattice Spin model

Ising model

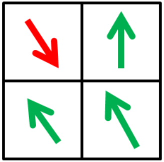
- Spin up, Spin down



XY-model

- continuous degree of freedom

$-\pi$ to π



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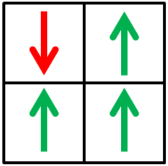

- $H = -J \sum_{\langle i,j \rangle} s_i s_j$

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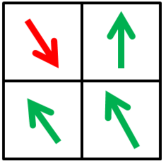

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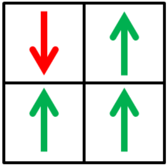

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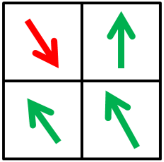

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Wolff's Cluster algorithm

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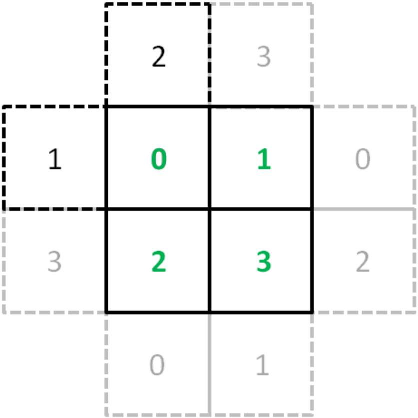

Wolff Cluster algorithm

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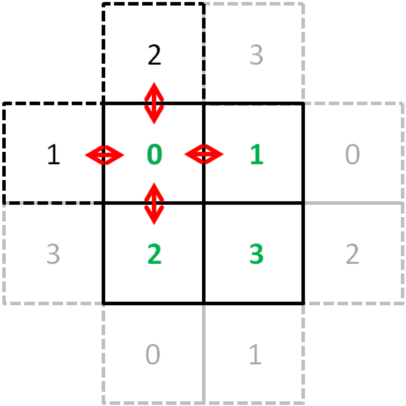
- Neighbor dependent cluster growth
- And to cluster if $\mathrm{rnd} < 1 - \exp^{-2J/k_b T}$



From continuous to discrete

- Draw random vector $\mathbf{u}$
- Project spin $\mathbf{s}$ onto $\mathbf{u}$
- Wolff algorithm onto ϵ_i

$$\mathbf{s}_i^{\parallel} = (\mathbf{s}_i \cdot \mathbf{u})\,\mathbf{u}$$
$$\mathbf{s}_i^{\perp} = \mathbf{s}_i - \mathbf{s}_i^{\parallel}$$
$$\mathbf{s}_i = \mathbf{s}_i^{\perp} + \epsilon_i \left|\mathbf{s}_i^{\parallel}\right| \mathbf{u}, \quad \epsilon_i = \pm 1$$

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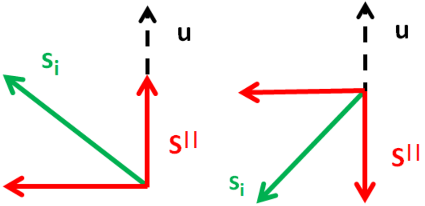

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$$\mathcal{H}\left[\epsilon_i\right] = \sum_{\langle ij \rangle} J_{ij}\epsilon_i\epsilon_j$$
$$J_{ij} = J \left|\mathbf{s}_i^{\parallel}\right| \mathbf{s}_j^{\parallel}|$$

Simulation

Lets look at an example animation for a lattice $L = 20$

- Magnetization, Susceptibility
- Energy, Specific heat
- Spin wave stiffness
- 20 by 20 spin lattice

Magnetization 20 by 20

- Magnetization: $\langle |m_{\hat{x},\hat{y}}| \rangle = \frac{1}{N} \langle | \sum_i s_{i\hat{x},\hat{y}} | \rangle$
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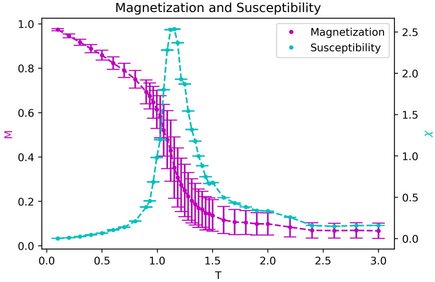
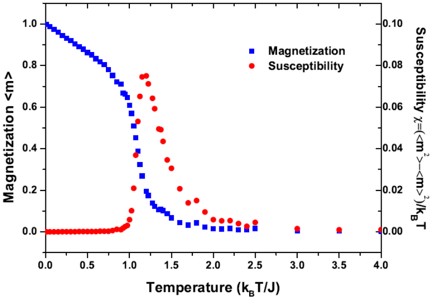
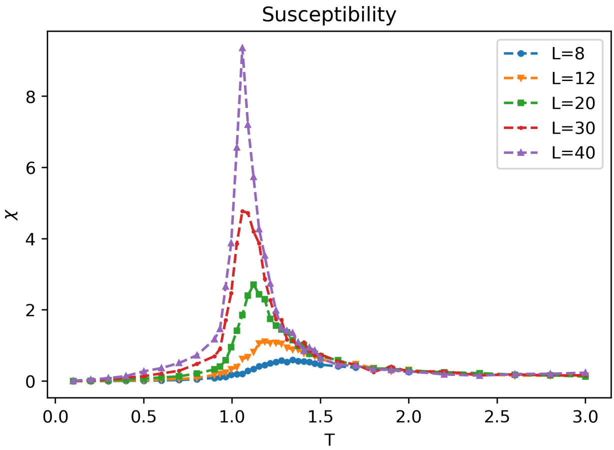




[0]Image from [2]

Finite lattice



Energy

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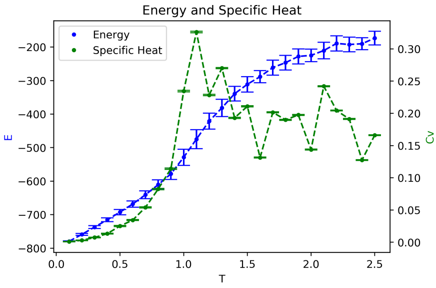
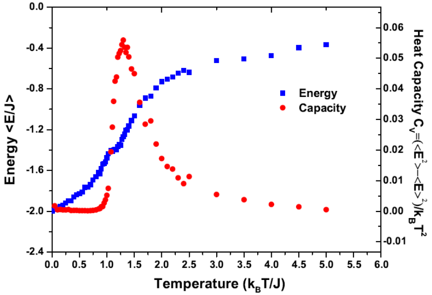




[0]Image from [2]

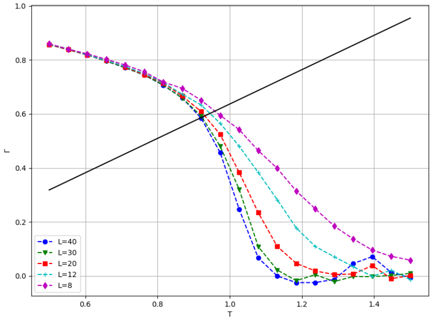
Helicity modulus

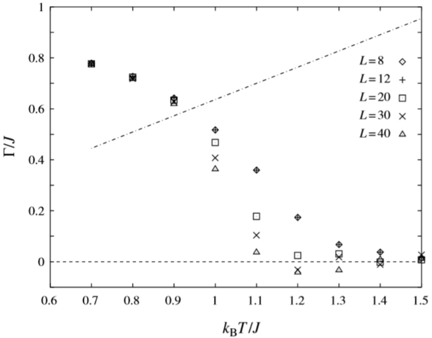
- Γ: Spin wave stiffness (or helicity modulus)
- At KT transition : $\Gamma = 2k_{\mathrm{B}}T_{\mathrm{KT}}/\pi$
- **Universal value**

$$\Gamma = \frac{J}{2L^2}\left\{\left\langle\sum_{\langle ij\rangle}\cos(\theta_i-\theta_j)\right\rangle - \frac{J}{k_{\mathrm{B}}T}\left\langle\left[\sum_i \sin(\theta_i-\theta_{i+\hat{e}_x})\right]^2\right\rangle - \frac{J}{k_{\mathrm{B}}T}\left\langle\left[\sum_i \sin(\theta_i-\theta_{i+\hat{e}_y})\right]^2\right\rangle\right\}$$

[0] Image from [5]

Helicity modulus





Extrapolation $T_{KT} = 0.9$

$T_{KT} = 0.89294(8)$ from literature

[0]Data from [5]
[0]Image from [4]
[0]Image from [5]

Performance

- Recursive algorithm $\rightarrow$ No parallelization :(

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- Some observables are expensive to track

Conclusion

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- Correlation function did not work

Bibliography

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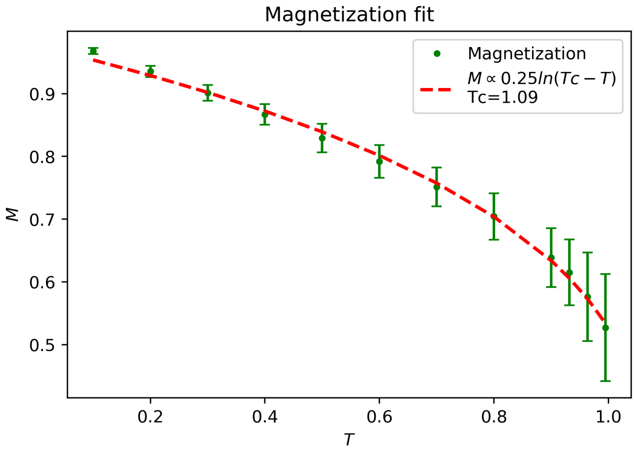
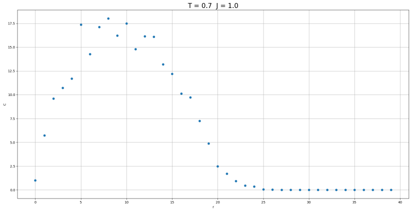
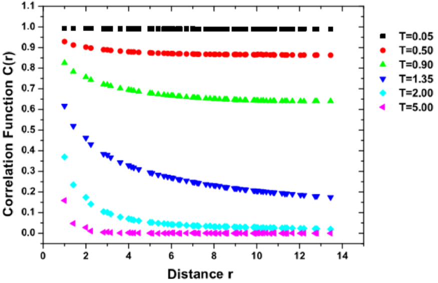
Backup Fit


Magnetization fit
Magnetization
$M \propto 0.25 ln(Tc - T)$
Tc=1.09
0.9
0.8
0.7
0.6
0.5
M
0.2
0.4
0.6
0.8
1.0
T

Correlation function





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