Grothendieck's Galois Theory

Gabriele Rastello

October 5, 2020

1 Section 1

Definition 1.1. In a category $\mathscr{C}$ an arrow $f\colon X \to Y$ is a **strict epimorphism** if it is the joint coequalizer of all the arrows it coequalizes. This means that any arrow $g\colon X \to Z$ such that $g \circ x = g \circ y$ for any $x, y\colon C \to X$ such that $f \circ x = f \circ y$ there exists a unique arrow $h\colon Y \to Z$ such that $h \circ f = g$. Refer to Figure 1.1.

$$\begin{tikzcd}
C \arrow[r, shift left, "y"] \arrow[r, shift right, "x"'] & X \arrow[r, "f"] \arrow[dr, "g"] & Y \arrow[d, dashed, "h"] \\
 & & Z
\end{tikzcd}$$

Figure 1.1

Remark 1.2. Strict epimorphisms are coequalizers, thus epimorphisms (as the name implies).

Remark 1.3. If an arrow is both a stric epimorphism and a monomorphism then it is an epimorphism.

Definition 1.4. Let H be a group, A an object of $\mathscr{C}$ and $G = \mathrm{Aut}(A)$ the group of automorphisms of A in $\mathscr{C}$ i.e. the group whose underlying set is the set of isomorphisms of type $A \to A$ of $\mathscr{C}$ and whose operation is composition in $\mathscr{C}$. An **action** of H on A is a group homomorphism $H \to G$.

Notation 1.5. Given an action of a group H on an object A of $\mathscr{C}$ we denote, with a slight abuse of notation, the automorphism of A associated to $h \in H$ by the same symbol h.

Definition 1.6. If H acts on A as defined in 1.4 we define the quotient of A by H in $\mathscr{C}$ to be an element A/H of $\mathscr{C}$ equipped with an arrow $q\colon A \to A/H$ such that:

(1) for all $h \in H$ $q \circ h = q$ holds,

(2) for any $x\colon A \to X$ such that $x \circ h = x$ for all $h \in H$ there exists a unique arrow $\varphi\colon A/H \to X$ such that $x = \varphi \circ q$.

See also Figure 1.2.

Remark 1.7. Quotients are defined by a universal property, thus are unique up to unique isomorphism and we can speak of "the" quotient of A by H instead of "a" quotient of A by H.

Notation 1.8. Sometimes we use the sentence "the quotient of A by H" to refer to the object A/H, some others to the arrow $q\colon A \to A/H$; the context should be enough to differentiate between the two cases.

$$\begin{tikzcd}
A \arrow[loop left, "h"] \arrow[r, "q"] \arrow[dr, "x"'] & A/H \arrow[d, "\varphi"] \\
& X
\end{tikzcd}$$

Figure 1.2

Remark 1.9. Consider a quotient $q\colon A \to A/H$; by condition (1) above $q \circ h = q = q \circ 1_A$ so q coequalizes all the pairs $(h, 1_A)$, for $h \in H$. If another arrow $x\colon A \to X$ coequalizes all the pairs that q does then this arrow is such that $x \circ h = x \circ 1_A = x$ for all $h \in H$ and thus, by condition (2), we have a unique factorization $x = \varphi \circ q$. This proves that all quotients are strict epimorphisms.

Remark 1.10. Let G be a group, **GSet** the category of G-sets and G-invariant maps and A an object of **GSet**. In this category Definition 1.6 yelds the familiar notion of the set of all orbits of an action: A/G is the set of orbits of A.