

Grothendieck's Galois Theory

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1 Section 1

Definition 1.1. In a category \mathcal{C} an arrow $f: X \rightarrow Y$ is a **strict epimorphism** if it is the joint coequalizer of all the arrows it coequalizes. This means that any arrow $g: X \rightarrow Z$ such that $g \circ x = g \circ y$ for any $x, y: C \rightarrow X$ such that $f \circ x = f \circ y$ there exists a unique arrow $h: Y \rightarrow Z$ such that $h \circ f = g$. Refer to Figure 1.1.

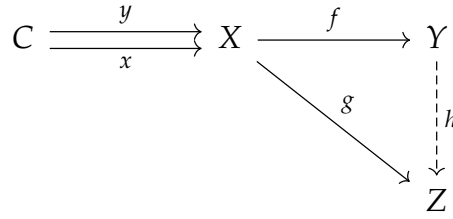


Figure 1.1

Remark 1.2. Strict epimorphisms are coequalizers, thus epimorphisms (as the name implies).

Remark 1.3. If an arrow is both a strict epimorphism and a monomorphism then it is an isomorphism.

Definition 1.4. Let H be a group, A an object of \mathcal{C} and $G = \text{Aut}(A)$ the group of automorphisms of A in \mathcal{C} i.e. the group whose underlying set is the set of isomorphisms of type $A \rightarrow A$ of \mathcal{C} and whose operation is composition in \mathcal{C} . An **action** of H on A is a group homomorphism $H \rightarrow G$.

Notation 1.5. Given an action of a group H on an object A of \mathcal{C} we denote, with a slight abuse of notation, the automorphism of A associated to $h \in H$ by the same symbol h .

Definition 1.6. If H acts on A as defined in 1.4 we define the quotient of A by H in \mathcal{C} to be an element A/H of \mathcal{C} equipped with an arrow $q: A \rightarrow A/H$ such that:

- (1) for all $h \in H$ $q \circ h = q$ holds,
- (2) for any $x: A \rightarrow X$ such that $x \circ h = x$ for all $h \in H$ there exists a unique arrow $\varphi: A/H \rightarrow X$ such that $x = \varphi \circ q$.

See also Figure 1.2.

Remark 1.7. Quotients are defined by a universal property, thus are unique up to unique isomorphism and we can speak of “the” quotient of A by H instead of “a” quotient of A by H .

Notation 1.8. Sometimes we use the sentence “the quotient of A by H ” to refer to the object A/H , some others to the arrow $q: A \rightarrow A/H$; the context should be enough to differentiate between the two cases.

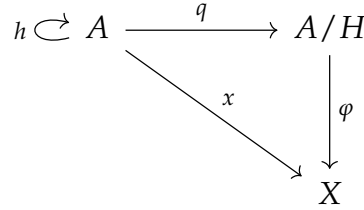


Figure 1.2

Remark 1.9. Consider a quotient $q: A \rightarrow A/H$; by condition (1) above $q \circ h = q = q \circ 1_A$ so q coequalizes all the pairs $(h, 1_A)$, for $h \in H$. If another arrow $x: A \rightarrow X$ coequalizes all the pairs that q does then this arrow is such that $x \circ h = x \circ 1_A = x$ for all $h \in H$ and thus, by condition (2), we have a unique factorization $x = \varphi \circ q$. This proves that all quotients are strict epimorphisms.

Remark 1.10. Let G be a group, \mathbf{GSet} the category of G -sets and G -invariant maps and A an object of \mathbf{GSet} . In this category Definition 1.6 yields the familiar notion of the set of all orbits of an action: A/G is the set of orbits of A .

Remark 1.11. Consider again \mathbf{GSet} . The underlying set of G (that we also denote as G) is a G -set with the action given by left multiplication in G ; we call this the **canonical action** of G on itself. Let $\varphi: G \rightarrow E$ be a G -invariant map; it is easy to see that such a φ , by virtue of being G -invariant, is determined uniquely by the value $\varphi(e)$, where e is the neutral element of G .

Let now E be a transitive G -set i.e. a set upon which the action of G is transitive i.e. such that $E/G = \{*\}$. Fix an $x \in E$ and let φ_x be the G -invariant map defined by $\varphi_x(e) = x$; we argue that $\varphi_x: G \rightarrow E$ makes E into a quotient of G by the subgroup

$$H = \text{Fix}(x) = \{g \in G: gx = x\}.$$

Indeed by using the definition of H and the fact that φ is G -invariant we have

$$(\varphi \circ h)(e) = \varphi(h(e)) = h(\varphi(e)) = h(x) = x.$$

for all $h \in H$. Moreover let $g: G \rightarrow F$ satisfy (1) of Definition 1.6; as we discussed above g is entirely determined by the image of e so we obtain (2) defining an arrow $f: E \rightarrow F$ by $f(x) = g(e)$. The situation is depicted in Figure 1.3.

Trivially G is a transitive G -set and for any $g \in G$ $G/\text{Fix}(g)$ is transitive as well so we have that an object $E \in \mathbf{GSet}$ is transitive if and only if it is isomorphic to G/H where $H = \text{Fix}(x)$ for (any) $x \in E$.

$$\begin{array}{ccc}
h \hookrightarrow G & \xrightarrow{\varphi} & E \cong G/H \\
& \searrow \scriptstyle g & \downarrow \scriptstyle f \\
& & F
\end{array}$$

Figure 1.3

For the rest of the section fix a category \mathcal{C} , an object $A \in \mathcal{C}$ and let $G = \text{Aut}(A)$.

Proposition 1.12. Consider a subgroup $H \leq G$ and an object $X \in \mathcal{C}$. H acts on the hom-set $[A, X]$ as follows¹:

$$\begin{aligned}
H \times [A, X] &\longrightarrow [A, X] \\
(h, x) &\longmapsto h \cdot x = x \circ h.
\end{aligned}$$

Remark 1.13. Assume that the action $G \times [A, X] \rightarrow [A, X]$ is transitive and let \mathbf{GSet}^t be the category of transitive G -sets (a subcategory of $G\text{set}$). Then we have a functor

$$\begin{array}{ccc}
[A, -]_G: \mathcal{C} & \longrightarrow & \mathbf{GSet}^t \\
X & & [A, X]_G \\
\downarrow f & \longmapsto & \downarrow f_* \\
Y & & [A, Y]_G
\end{array}$$

where we indicate with $[A, X]_G$ the hom-set $[A, X]$ upon which G acts as described in Proposition 1.12 and f_* is post-composition with f . It is easy to check that f_* is indeed G -invariant.

¹Since an action as of Definition 1.4 is a map that sends elements of a group to arrows it is, in this case, equivalent to give the definition of an action by uncurrying.