# Grothendieck's Galois Theory

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## 1 Group actions and adjoints

**Definition 1.1.** In a category  $\mathscr C$  an arrow  $f\colon X\to Y$  is a **strict epimorphism** if it is the joint coequalizer of all the arrows it coequalizes. This means that any arrow  $g\colon X\to Z$  such that  $g\circ x=g\circ y$  for any  $x,y\colon C\to X$  such that  $f\circ x=f\circ y$  there exists a unique arrow  $h\colon Y\to Z$  such that  $h\circ f=g$ . Refer to Figure 1.1.

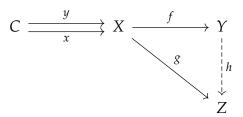


Figure 1.1

**Remark 1.2.** Strict epimorphisms are coequalizers, thus epimorphisms (as the name implies).

**Remark 1.3.** If an arrow is both a stric epimorphism and a monomorphism then it is an epimorphism.

**Definition 1.4.** Let H be a group, A an object of  $\mathscr C$  and  $G = \operatorname{Aut}(A)$  the group of automorphisms of A in  $\mathscr C$  i.e. the group whose underlying set is the set of isomorphisms of type  $A \to A$  of  $\mathscr C$  and whose operation is composition in  $\mathscr C$ . An **action** of H on A is a group homomorphism  $H \to G$ .

**Notation 1.5.** Given an action of a group H on an object A of  $\mathscr{C}$  we denote, with a slight abuse of notation, the automorphism of A associated to  $h \in H$  by the same symbol h.

**Definition 1.6.** If H acts on A as defined in 1.4 we define the quotient of A by H in  $\mathscr C$  to be an element A/H of  $\mathscr C$  equipped with an arrow  $g \colon A \to A/H$  such that:

- (1) for all  $h \in H$   $q \circ h = q$  holds,
- (2) for any  $x: A \to X$  such that  $x \circ h = x$  for all  $h \in H$  there exists a unique arrow  $\varphi: A/H \to X$  such that  $x = \varphi \circ q$ .

See also Figure 1.2.

**Remark 1.7.** Quotients are defined by a universal property, thus are unique up to unique isomorphism and we can speak of "the" quotient of *A* by *H* instead of "a" quotient of *A* by *H*.

**Notation 1.8.** Sometimes we use the sentence "the quotient of A by H" to refer to the object A/H, some others to the arrow  $q: A \to A/H$ ; the context should be enough to differentiate between the two cases.

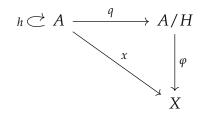


Figure 1.2

**Remark 1.9.** Consider a quotient  $q: A \to A/H$ ; by condition (1) above  $q \circ h = q = q \circ 1_A$  so q coequalizes all the pairs  $(h, 1_A)$ , for  $h \in H$ . If another arrow  $x: A \to X$  coequalizes all the pairs that q does then this arrow is such that  $x \circ h = x \circ 1_A = x$  for all  $h \in H$  and thus, by condition (2), we have a unique factorization  $x = \varphi \circ q$ . This proves that all quotients are strict epimorphisms.

**Remark 1.10.** Let G be a group, **GSet** the category of G-sets and G-invariant maps and A an object of **GSet**. In this category Definition 1.6 yelds the familiar notion of the set of all orbits of an action: A/G is the set of orbits of A.

**Remark 1.11.** Consider again **GSet**. The underlying set of G (that we also denote as G) is a G-set with the action given by left multiplication in G; we call this the **canonical action** of G on itself. Let  $\varphi: G \to E$  be a G-invariant map; it is easy to see that such a  $\varphi$ , by virtue of being G-invariant, is determined uniquely by the value  $\varphi(e)$ , where e is the neutral element of G.

Let now E be a transitive G-set i.e. a set upon which the action of G is transitive i.e. such that  $E/G = \{*\}$ . Fix an  $x \in E$  and let  $\varphi_x$  be the G-invarian map defined by  $\varphi_x(e) = x$ ; we argue that  $\varphi_x \colon G \to E$  makes E into a quotient of G by the subgroup

$$H = Fix(x) = \{g \in G \colon gx = x\}.$$

Indeed by using the definition of H and the fact that  $\phi$  is G-invariant we have

$$(\varphi \circ h)(e) = \varphi(h(e)) = h(\varphi(e)) = h(x) = x.$$

for all  $h \in H$ . Moreover let  $g: G \to F$  satisfy (1) of Definition 1.6; as we discussed above g is entirely determined by the image of e so we obtain (2) defining an arrow  $f: E \to F$  by f(x) = g(e). The situation is depicted in Figure 1.3.

Trivially *G* is a transitive G-set and for any  $g \in G$  G/Fix(g) is transitive as well so we have that an object  $E \in \mathbf{GSet}$  is transitive if and only if it is isomorphic to G/H where H = Fix(x) for (any)  $x \in E$ .

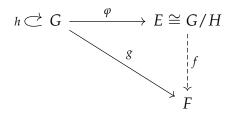


Figure 1.3

For the rest of the section fix a category  $\mathscr{C}$ , an object  $A \in \mathscr{C}$  and let  $G = \operatorname{Aut}(A)$ .

**Proposition 1.12.** Consider a subgroup  $H \leq G$  and an object  $X \in \mathcal{C}$ . H acts on the hom-set [A, X] as follows<sup>1</sup>:

$$H \times [A, X] \longrightarrow [A, X]$$
  
 $(h, x) \longmapsto h \cdot x = x \circ h.$ 

**Remark 1.13.** Assume that the action  $G \times [A, X] \to [A, X]$  is transitive and let **GSet**<sup>t</sup> be the category of transitive G-sets (a subcategory of Gset). Then we have a functor

$$[A,-]_G \colon \mathscr{C} \longrightarrow \mathbf{GSet}^t$$

$$\begin{matrix} X & [A,X]_G \\ \downarrow_f \longmapsto & \downarrow_{f_*} \\ Y & [A,Y]_G \end{matrix}$$

where we indicate with  $[A, X]_G$  the hom-set [A, X] upon which G acts as described in Proposition 1.12 and  $f_*$  is post-composition with f. It is easy to check that  $f_*$  is indeed G-invariant.

**Remark 1.14.** Consider an object  $E \in \mathbf{GSet}^t$ , pick an element  $x_0 \in E$  and let  $H = \mathrm{Fix}(x_0) \leq G$  (the choice of  $x_0$  is irrelevant as E is transitive). Moreover assume that  $\mathscr{C}$  has quotients of A by any subgroup of G.

By what we observed in Remark 1.11 we have a bijection between elements of E and arrows of type  $G \to E$ . Consider then  $f \in [A, X]$  and its corresponding arrow  $\varphi \colon G \to [A, X]_G$ ; we claim that f factors through A/H if and only if  $\varphi$  factors through  $E \cong G/H$  (see Figure 1.4). Indeed f factors if and only if  $f \circ h = f$  for all  $h \in H$ , by using the fact that  $\varphi$  is G-invariant we obtain

$$\varphi(h(e)) = h(\varphi(e)) = h(f) = f \circ h = f$$

and, since  $\varphi$  is uniquely determined by  $\varphi(e)$ ,  $\varphi \circ h = \varphi$  for all  $h \in H$ ; this happens if and only if  $\varphi$  factors through  $E \cong G/H$ .

<sup>&</sup>lt;sup>1</sup>Since an action as of Definition 1.4 is a map that sends elements of a group to arrows it is, in this case, equivalent to give the definition of an action by uncurrying.

This gives us, for each  $X \in \mathscr{C}$  and  $E \in \mathbf{GSet}^t$ , a bijection

$$[A/H, X] \cong \mathbf{GSet}^t(E, [A, X]_G)$$

natural in X in E (see Section 3). Thus, if  $\mathscr C$  has quotients of A by subgroups of G we have an adjunction  $A \times_G - \dashv [A, -]_G$  where  $A \times_G E = A/H$  for  $H = \operatorname{Fix}(x_0)$  and  $x_0 \in E$  as above<sup>2</sup>.

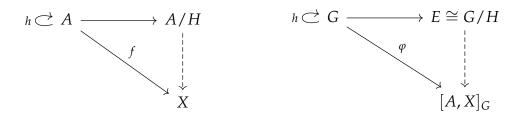


Figure 1.4

Our main problem will be that of finding conditions on  $\mathscr C$  that make this adjunction ineto an equivalence of categories.

## 2 Categorial axiomatization of Galois Theory

Through this section fix a category  $\mathscr C$  and an object  $A \in \mathscr C$ .

**Definition 2.1.** We define the following axioms.

- (1) For every  $X \in \mathcal{C}$  there is at least a map of type  $A \to X$  and all maps  $A \to X$  are strict epimorphisms.
- (2) For any subgroup  $H \leq \operatorname{Aut}(A)$  the quotient  $q: A \to A/H$  exists and is preserved by  $[A, -]: \mathscr{C} \to \mathbf{GSet}$ .
- (3) Every endomorphism of A is an isomorphism i.e. [A, A] = Aut(A).

**Remark 2.2.** It is known that if  $f \circ g$  is a strict epimorphism then so is f. Thus it follows from Axiom (1) that every arrow  $X \to Y$  in  $\mathscr C$  is a strict epimorphism.

**Proposition 2.3.** Axiom (1) implies that [A, -] is faithful, reflects monomorphisms and isomorphisms.

*Proof.* Consider arrows  $f, g: X \to Y \in \mathscr{C}$  such that [A, f] = [A, g]; that is  $f_* = g_*$ . By Axiom (1) let  $h: A \to X$  be a third arrow of  $\mathscr{C}$  then we have  $f_*(h) = g_*(h)$  i.e.  $f \circ h = g \circ h$ . Again by Axiom (1) h is an epimorphism (really, a strong one) and thus we obtain f = g; that is: [A, -] is faithful.

It is well known that every faithful functor reflects monomorphisms. Because of this if  $f \in \mathbf{GSet}$  is an isomorphism and  $g \in \mathscr{C}$  is such that [A,g] = f then g is a monomorphism too; but, as an arrow of  $\mathscr{C}$ , g is also a strict epimorphism and thus an isomorphism.

<sup>&</sup>lt;sup>2</sup>We use the notation  $A \times_G$  – for the left adjoint in "honor" of the tensor-hom adjunction.

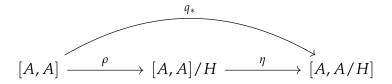


Figure 2.5

**Remark 2.4.** Consider  $H \leq G$  and the quotient  $q: A \to A/H$  in  $\mathscr{C}$ , then  $q_*: [A, A] \to [A, A/H]$  in **GSet** is a quotient too because quotients are preserved by [A, -] (Axiom (2)). Thus we have  $[A, A]/H \cong [A, A/H]$ , and the following diagram commutes (where  $\rho$  is a quotient arrow and  $\eta$  the isomorphism).

The fact that  $\eta$  is a bijection gives us the following:

- (i) for  $f,g \in [A,A]$  if  $g \circ f = g \circ g$  then there is some  $h \in H$  such that  $f = h \circ g$ ,
- (ii) for all  $x: A \to A/H$  there is an arrow  $f \in [A, A]$  such that  $q \circ f = x$ .

Moreover, under Axiom (3), (i) implies the following:

(iii)  $q \circ f = q$  implies  $f \in H$ .

Indeed by taking  $g = 1_A$  in (i) we obtain that f = h for some  $h \in H$ .

**Remark 2.5.** Consider an arrow  $x: A \to X$  and the epi-mono factorization  $x_* = \psi \circ \rho^3$ . With reference to the diagram below Axiom (iii) implies that I = [A, A]/H with  $H = \text{Fix}(x) \leq G$ .

$$[A,A] \xrightarrow{\rho} I \xrightarrow{\psi} [A,X]$$

Indeed by Axiom (iii)  $[A, A] = \operatorname{Aut}(A) = G$  so an  $h \in H$  acts on [A, A] by left multiplication. Given  $f \in [A, A]$  we have

$$(\psi \circ \rho \circ h)(f) = x_*(h \circ f) = x \circ h \circ f = x \circ f = x_*(f) = (\psi \circ \rho)(f)$$

that is  $\psi \circ \rho \circ h = \psi \circ \rho$  which, by monicness of  $\psi$ , implies  $\rho \circ h = \rho$ . Moreover as I is the image of  $x_*$  it is unique up to isomorphism and thus  $\rho$  is really a quotient arrow; I = [A, A]/H.

**Proposition 2.6.** Any arrow  $x: A \to X$  of  $\mathscr{C}$  is a quotient of A by  $H = \text{Fix}(x) \leq G$  (with respect to the action on [A, X] described in Proposition 1.12) i.e. X = A/H.

*Proof.* By choosing H = Fix(x) we get  $x \circ h = x$  for all  $h \in H$  and thus there is a unique arrow  $\varepsilon \colon A/H \to X$  of  $\mathscr C$  such that  $x = \varepsilon \circ q$ , where  $q \colon A \to A/H$  is the quotient of A by H. Graphically:

Now by applying [A, -] to the diagram above we obtain

<sup>&</sup>lt;sup>3</sup>We recall that **GSet** is a topos and, as such, has epi-mono factorization.

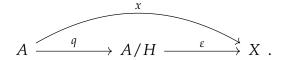


Figure 2.6

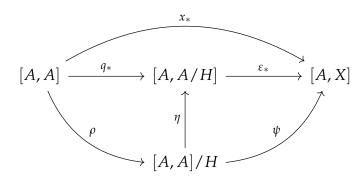


Figure 2.7

where  $\rho$ ,  $\psi$  and  $\eta$  are as discussed before. We now have

$$\varepsilon_* \circ \eta \circ \rho = \varepsilon_* \circ q_* = x_* = \psi \circ \rho$$

that by epicness of  $\rho$  implies  $\varepsilon_* \circ \eta = \psi$ . Now  $\varepsilon_*$  must be monic since  $\psi$  and  $\eta$  are; but this, by Proposition 2.3, implies that  $\varepsilon \in \mathscr{C}$  is monic too. Being an arrow of  $\mathscr{C}$ , by Axiom (i),  $\varepsilon$  is also a strict epimorphism and thus an isomorphism.

**Proposition 2.7.** The action of  $\operatorname{Aut}(A)$  on [A, X] is transitive for all  $X \in \mathscr{C}$ .

*Proof.* Consider again Figure 2.7. We proved that  $\varepsilon$  is an isomorphism and thus  $\varepsilon_*$  must be one too. Moreover from Remark 2.4 we know  $\eta$  is iso too and so we have  $[A, X] \cong [A, A]/H$ , but by Axiom (3)  $[A, A] = \operatorname{Aut}(A)$  and so we have that [A, X] is transitive.

**Theorem 2.8.** Given a category  $\mathscr{C}$  and an object  $A \in \mathscr{C}$  such that Axioms (1), (2) and (3) hold there exists an adjunction

$$A\times_G -\dashv [A,-]_G$$

where G = Aut(A) such that the maps

$$\eta: E \cong [A, A]/H \to [A, A/H]$$
  
 $\varepsilon: A/H \to X$ 

are isomorphisms. This enstablishes an equivalence of categories between  $\mathscr{C}$  and  $\mathbf{GSet}^t$ .

*Proof.* The existence of the adjunction follows from the discussion in Section 1, the fact that  $\eta$  and  $\varepsilon$  are (the components of) the unit and counit of the andjunction is a a textbook calculation and the fact that they are isomorphisms follows respectively from Remark 2.4 and Proposition 2.6.

## 3 Appendixes

#### **3.1** Naturality of $[A/H, X] \cong \mathbf{GSet}^t(E, [A, X]_G)$

Fix a category  $\mathscr{C}$ , an element  $A \in \mathscr{C}$  and let  $G = \operatorname{Aut}(A)$ . Let's indicate with  $\psi_{EX}$  the bijection between [A/H, X] and  $\operatorname{\mathbf{GSet}}^t(E, [A, X]_G)$  described in Remark 1.14; we shall prove that it is natural in both  $X \in \mathscr{C}$  and  $E \in \operatorname{\mathbf{GSet}}^t$ .

*Naturality in X.* Given an arrow  $f: X \to Y$  of  $\mathscr{C}$  we want to prove that  $\psi_{EX}$  are the components of a natural transformation  $[A/H, -] \Rightarrow \mathbf{GSet}^t(E, [A, -]_G)$  i.e that the following diagram commutes.

$$[A/H, X] \xrightarrow{\psi_{EX}} \mathbf{GSet}^{t}(E, [A, X]_{G})$$

$$\downarrow^{f_{*}} \qquad \qquad \downarrow^{(f_{*})_{*}}$$

$$[A/H, Y] \xrightarrow{\psi_{EY}} \mathbf{GSet}^{t}(E, [A, Y]_{G})$$

Recall that here  $H = \operatorname{Fix}(x_0)$  for  $x_0 \in E$ . Pick any  $x \in [A/H, X]$  and let  $q: A \to A/H$  be the quotient arrow (the quotient exists because  $\mathscr C$  is assumed to have all quotients by subgroups of G); then chasing x through the diagram down the two possible ways yields two arrows in  $\operatorname{\mathbf{GSet}}^t$  of type  $E \to [A, Y]_G$ . Since E is transitive we have  $E \cong G/H$  and this means that arrows of type  $E \to [A, Y]_G$  are determined uniquely by the image of  $e \in E \cong G/H$  (refer to the right part of Diagram 1.4). Keeping this in mind the following computations shows that the square commutes.

$$((f_*)_* \circ \psi_{EX})(x)(e) = (f_* \circ \psi_{EX}(x))(e) = f_*(\psi_{EX}(x)(e)) = f_*(x \circ q) = f \circ x \circ q,$$
$$(\psi_{EY} \circ f_*)(x)(e) = \psi_{EX}(f \circ x)(e) = f \circ x \circ q.$$

Naturality in E. We want to prove that  $\psi_{EX}$  are the components of a natural transformation  $[A/-,X]\Rightarrow \mathbf{GSet}^t(-,[A,X]_G)$ , but before drawing the naturality square as above we shall note that this case is slightly complicated by the fact that it is not immediately clear how the functor [A/-,X] acts on arrows. Indeed let  $f\colon E\cong G/H\to F\cong G/H'$  be an arrow of  $\mathbf{GSet}^t$  with  $H=\mathrm{Fix}(x_0),H'=\mathrm{Fix}(f(x_0))$  and  $x_0\in E$ ; these choices are justified by the fact that both E and F are transitive G-sets. Now notice that, for  $h\in H$ ,  $f(h\cdot x_0)=f(x_0)$  by definition of H and  $f(h\cdot x_0)=h\cdot f(x_0)$  by G-invariance of f; thus  $h\in H'$  and thus  $H\subset H'$ . This means that if we let  $g\colon A\to A/H, g'\colon A\to A/H'$  be the quotients in  $\mathscr C$  then there is a unique arrow  $\widetilde f\colon A/H\to A/H'$  such that  $\widetilde f\circ q=q'$ . The functor [A/-,X] then sends the arrow  $f\colon E\to F$  to  $\widetilde f^*\colon [A/H',X]\to [A/H,X]$ .

Now we want to prove the naturality of the following square.

$$\begin{array}{ccc} [A/H,X] & \xrightarrow{\psi_{EX}} & \mathbf{GSet}^t(E,[A,X]_G) \\ & & & & f^* \\ \hline [A/H',X] & \xrightarrow{\psi_{FX}} & \mathbf{GSet}(F,[A,X]_G) \end{array}$$

As before we chase a  $x \in [A/H', X]$  down the two possible paths to get two arrows of type  $E \to [A, X]_G$  and prove that they are the same by evaluating them on  $e \in E$ :

$$(\psi_{EX} \circ \widetilde{f}^*)(x)(e) = \psi_{EX}(x \circ \widetilde{f})(e) = x \circ \widetilde{f} \circ q = x \circ q',$$

$$(f^* \circ \psi_{FX})(x)(e) = (\psi_{FX}(x) \circ f)(e) = \psi_{FX}(x)(f(e)) = \psi_{FX}(x)(e) = x \circ q'.$$

Note that we used that f(e) = e, this holds because the action of G on  $G/H \cong E$  and  $G/H' \cong F$  is just left multiplication and f is G-invariant.

#### **3.2** $\eta$ and $\varepsilon$ are the unit and counit of $A \times_G - \exists [A, -]_G$

 $\eta$  *is the unit.* Our adjunction is given by the bijection

$$\psi \colon \mathscr{C}(A \times_G E, X) \cong \mathbf{GSet}^t(E, [A, X]_G)$$

natural in  $X \in \mathscr{C}$  and  $E \in \mathbf{GSet}^t$ . To obtain the unit we set  $X = A \times_G E$ :

$$\mathscr{C}(A \times_G E, A \times_G E) \cong \mathbf{GSet}^t(E, [A, A \times_G E]_G)$$

and so

$$\mathscr{C}(A/H, A/H) \cong \mathbf{GSet}^t(E, [A, A/H]_G)$$

where  $H = \operatorname{Fix}(x) \leq G$  ( $x \in E$ ) by definition of  $A \times_G E$ . Now (the component at E of) the unit is given by the image of  $1_{A/H}$  under this bijection. By the discussion in Section 1 if  $q: A \to A/H$  is the quotient arrow in  $\mathscr C$  then  $\psi(1_{A/H})$  is the map  $E \cong [A,A]_G/H \to [A,A/H]_G$  of  $\operatorname{\mathbf{GSet}}^t$  that factors the map  $G \to E$  that sends e to q. But this last map is  $q^*$  as in Figure 2.5.

 $\varepsilon$  is the counit. Keeping the proof above in mind we set E = [A, X] and obtain

$$\psi \colon \mathscr{C}(A \times_G [A, X], X) \cong \mathbf{GSet}^t([A, X]_G, [A, X]_G)$$

that becomes

$$\mathscr{C}(A/H,X) \cong \mathbf{GSet}^t([A,X]_G,[A,X]_G)$$

with  $H = \operatorname{Fix}(x)$  for some  $x \in [A, X]$ . Notice that since  $[A, X] \cong G/H$  by Proposition 2.7 we can consider  $1_{[A,X]}$  as a map of type  $G/H \to [A,X]$  such that  $1_{[A,X]}(e) = x$  (this can be obtained by chasing  $1_A$  around Figure 2.7 keeping in mind that  $\varepsilon_*$  there is iso). Now we have that  $\psi^{-1}(1_{[A,X]})$  is the arrow  $A/H \to X$  of  $\mathscr C$  that factorizes x i.e.  $\varphi^{-1}(1_{[A,X]}) = \varepsilon$  as in Figure 2.6.