

Grothendieck's Galois Theory

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1 Group actions and adjoints

Definition 1.1. In a category \mathcal{C} an arrow $f: X \rightarrow Y$ is a **strict epimorphism** if it is the joint coequalizer of all the pairs of arrows it coequalizes. This means that any arrow $g: X \rightarrow Z$ such that $g \circ x = g \circ y$ for all $x, y: C \rightarrow X$ such that $f \circ x = f \circ y$ there exists a unique arrow $h: Y \rightarrow Z$ such that $h \circ f = g$. Refer to Figure 1.1.

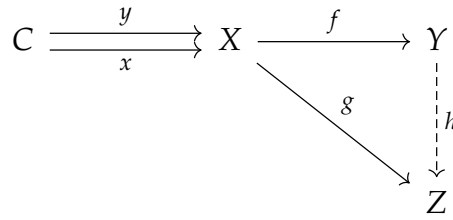


Figure 1.1

Remark 1.2. Strict epimorphisms are epimorphisms, as the name implies.

Remark 1.3. If an arrow is both a strict epimorphism and a monomorphism then it is an isomorphism.

Definition 1.4. Let H be a group, A an object of \mathcal{C} and $G = \text{Aut}(A)$ the group of automorphisms of A in \mathcal{C} i.e. the group whose underlying set is the set of isomorphisms of type $A \rightarrow A$ of \mathcal{C} and whose operation is composition in \mathcal{C} . An **action** of H on A is a group homomorphism $H \rightarrow G$.

Notation 1.5. Given an action of a group H on an object A of \mathcal{C} we denote, with a slight abuse of notation, the automorphism of A associated to $h \in H$ by the same symbol h .

Definition 1.6. If H acts on A as defined in 1.4 we define the quotient of A by H in \mathcal{C} to be an element A/H of \mathcal{C} equipped with an arrow $q: A \rightarrow A/H$ such that:

- (1) for all $h \in H$ we have $q \circ h = q$,
- (2) for any $x: A \rightarrow X$ such that $x \circ h = x$ for all $h \in H$ there exists a unique arrow $\varphi: A/H \rightarrow X$ such that $x = \varphi \circ q$.

See also Figure 1.2.

Remark 1.7. Quotients are defined by a universal property, thus are unique up to unique isomorphism and we can speak of “the” quotient of A by H instead of “a” quotient of A by H .

Notation 1.8. Sometimes we use the sentence “the quotient of A by H ” to refer to the object A/H , some others to the arrow $q: A \rightarrow A/H$; the context should be enough to differentiate between the two.

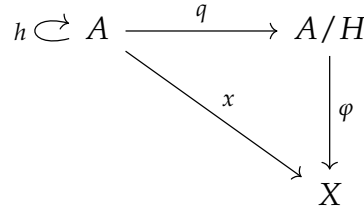


Figure 1.2

Remark 1.9. Consider a quotient $q: A \rightarrow A/H$; by condition (1) above $q \circ h = q = q \circ 1_A$ so q coequalizes all the pairs $(h, 1_A)$, for $h \in H$. If another arrow $x: A \rightarrow X$ coequalizes all the pairs that q does then this arrow is such that $x \circ h = x \circ 1_A = x$ for all $h \in H$ and thus, by condition (2), we have a unique factorization $x = \varphi \circ q$. This proves that all quotients are strict epimorphisms.

Notation 1.10. Let G be a group, then with **GSet** we denote the category of G -sets and G -invariant maps.

Remark 1.11. Consider **GSet**. The underlying set of G (that we also denote with G) is a G -set with the action given by left multiplication; we call this the **canonical action** of G on itself. Let $\varphi: G \rightarrow E$ be a G -invariant map; it is easy to see that such a φ , by virtue of being G -invariant, is determined uniquely by the value $\varphi(e)$, where e is the neutral element of G .

Let now E be a transitive G -set i.e. such that $E/G = \{*\}$ in **Set**¹. Fix an $x \in E$ and let φ be the G -invariant map defined by $\varphi(e) = x$; we argue that $\varphi: G \rightarrow E$ makes E into a quotient of G by the subgroup

$$H = \text{Fix}(x) = \{g \in G: g \cdot x = x\}.$$

Indeed by using the definition of H and the fact that φ is G -invariant we have

$$(\varphi \circ h)(e) = \varphi(h \cdot e) = h \cdot \varphi(e) = h \cdot x = x.$$

for all $h \in H$. Moreover let $g: G \rightarrow F$ satisfy (1) of Definition 1.6; as we discussed above g is entirely determined by the image of e so we obtain (2) defining an arrow $f: E \rightarrow F$ by $f(x) = g(e)$. The situation is depicted in Figure 1.3.

Finally: G is a transitive G -set and so is every G/H with $H \leq G$ and the action defined again by left multiplication (on cosets) so an object $E \in \mathbf{GSet}$ is transitive if and only if it is isomorphic to some G/H .

$$\begin{array}{ccc}
h \hookrightarrow G & \xrightarrow{\varphi} & E \cong G/H \\
& \searrow \scriptstyle g & \downarrow \scriptstyle f \\
& & F
\end{array}$$

Figure 1.3

For the rest of the section fix a category \mathcal{C} , an object $A \in \mathcal{C}$ and let $G = \text{Aut}(A)$.

Remark 1.12. Consider a subgroup $H \leq G$ and an object $X \in \mathcal{C}$. H acts on the hom-set $[A, X]$ as follows²:

$$\begin{aligned}
H \times [A, X] &\longrightarrow [A, X] \\
(h, x) &\longmapsto h \cdot x = x \circ h.
\end{aligned}$$

Remark 1.13. Assume that the action $G \times [A, X] \rightarrow [A, X]$ is transitive and let \mathbf{GSet}^t be the category of transitive G -sets (a subcategory of \mathbf{Gset}). Then we have a functor

$$\begin{array}{ccc}
[A, -]_G: \mathcal{C} & \longrightarrow & \mathbf{GSet}^t \\
X & & [A, X]_G \\
\downarrow f & \longmapsto & \downarrow f_* \\
Y & & [A, Y]_G
\end{array}$$

where we indicate with $[A, X]_G$ the hom-set $[A, X]$ upon which G acts as described in Proposition 1.12 and f_* is post-composition with f . It is easy to check that f_* is indeed G -invariant.

Remark 1.14. Consider an object $E \in \mathbf{GSet}^t$, pick an element $x_0 \in E$ and let $H = \text{Fix}(x_0) \leq G$ (the choice of x_0 is irrelevant as E is transitive). Moreover assume that \mathcal{C} has quotients of A by any subgroup of G .

By what we observed in Remark 1.11 we have a bijection between elements of E and arrows of type $G \rightarrow E$. Consider then $f \in [A, X]_G$ and its corresponding arrow $\varphi: G \rightarrow [A, X]_G$; we claim that f factors through A/H if and only if φ factors through $E \cong G/H$ (see Figure 1.4). Indeed f factors if and only if $f \circ h = f$ for all $h \in H$, by using the fact that φ is G -invariant we obtain

$$\varphi(h \cdot e) = h \cdot \varphi(e) = h \cdot f = f \circ h = f$$

and, since φ is uniquely determined by $\varphi(e)$, $\varphi \circ h = \varphi$ for all $h \in H$; this happens if and only if φ factors through $E \cong G/H$.

¹This states that the set of orbits of E is a singleton that is of course the case if and only if E is transitive.

²Since an action as of Definition 1.4 is a map that sends elements of a group to arrows it is, in this case, equivalent to give the definition of an action by uncurrying.

This gives us, for each $X \in \mathcal{C}$ and $E \in \mathbf{GSet}^t$, a bijection

$$[A/H, X] \cong \mathbf{GSet}^t(E, [A, X]_G) \quad (*)$$

natural in X in E (see Section 3).

Provided that \mathcal{C} has quotients of A by subgroups of G we can define a functor $A \times_G -$ from \mathbf{GSet}^t to \mathcal{C} by setting $A \times_G E = A/H$ for $H = \text{Fix}(x_0)$ and $x_0 \in E$ as above. The behaviour of $A \times_G -$ on arrows is defined as follows. Let $f: E \cong G/H \rightarrow F \cong G/H'$ be an arrow in \mathbf{GSet}^t with $H = \text{Fix}(x_0)$, $H' = \text{Fix}(f(x_0))$ and $x_0 \in E$. Notice that for all $h \in H$ we have $f(h \cdot x_0) = f(x_0)$ by definition of H and $f(h \cdot x_0) = h \cdot f(x_0)$ by G -invariance of f ; thus $H \leq H'$. This means that if $q: A \rightarrow A/H$ and $q': A \rightarrow A/H'$ are the quotients in \mathcal{C} there is a unique arrow $\tilde{f}: A/H \rightarrow A/H'$ such that $q' = \tilde{f} \circ q$; we set $A \times_G f = \tilde{f}$.

Finally the naturality of the bijection $(*)$ now indicates that we have an adjunction

$$A \times_G - \dashv [A, -]_G.$$

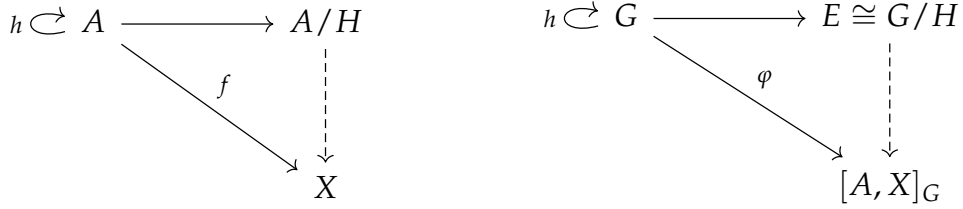


Figure 1.4

Our main problem will be that of finding conditions on \mathcal{C} that make this adjunction into an equivalence of categories.

2 Categorical axiomatization of Galois Theory

Through this section fix a category \mathcal{C} and an object $A \in \mathcal{C}$.

Definition 2.1. We define the following axioms.

- (1) For every $X \in \mathcal{C}$ there is at least a map of type $A \rightarrow X$ and all maps $A \rightarrow X$ are strict epimorphisms.
- (2) For any subgroup $H \leq \text{Aut}(A)$ the quotient $q: A \rightarrow A/H$ exists and is preserved by $[A, -]: \mathcal{C} \rightarrow \mathbf{GSet}$.
- (3) Every endomorphism of A is an isomorphism i.e. $[A, A] = \text{Aut}(A)$.

Remark 2.2. It is known that if $f \circ g$ is a strict epimorphism then so is f . Thus it follows from Axiom (1) that every arrow $X \rightarrow Y$ in \mathcal{C} is a strict epimorphism.

Proposition 2.3. Axiom (1) implies that $[A, -]$ is faithful, reflects monomorphisms and isomorphisms.

$$\begin{array}{ccccc}
& & q_* & & \\
& \nearrow & & \searrow & \\
[A, A] & \xrightarrow{\rho} & [A, A]/H & \xrightarrow{\eta} & [A, A/H]
\end{array}$$

Figure 2.5

Proof. Consider arrows $f, g: X \rightarrow Y \in \mathcal{C}$ such that $[A, f] = [A, g]$; that is $f_* = g_*$. By Axiom (1) let $h: A \rightarrow X$ be a third arrow of \mathcal{C} then we have $f_*(h) = g_*(h)$ i.e. $f \circ h = g \circ h$. Again by Axiom (1) h is an epimorphism (really, a strong one) and thus we obtain $f = g$; that is: $[A, -]$ is faithful.

It is well known that every faithful functor reflects monomorphisms. Because of this if $f \in \mathbf{GSet}$ is an isomorphism and $g \in \mathcal{C}$ is such that $[A, g] = f$ then g is a monomorphism too; but, as an arrow of \mathcal{C} , g is also a strict epimorphism and thus an isomorphism. \square

Remark 2.4. Consider $H \leq G$ and the quotient $q: A \rightarrow A/H$ in \mathcal{C} , then $q_*: [A, A] \rightarrow [A, A/H]$ in \mathbf{GSet} is a quotient too because quotients are preserved by $[A, -]$ (Axiom (2)). Thus we have $[A, A]/H \cong [A, A/H]$, and the following diagram commutes (where ρ is a quotient arrow and η the isomorphism).

The fact that η is a bijection gives us the following:

- (i) for $f, g \in [A, A]$ if $q \circ f = q \circ g$ then there is some $h \in H$ such that $f = h \circ g$,
- (ii) for all $x: A \rightarrow A/H$ there is an arrow $f \in [A, A]$ such that $q \circ f = x$.

Moreover, under Axiom (3), (i) implies the following:

- (iii) $q \circ f = q$ implies $f \in H$.

Indeed by taking $g = 1_A$ in (i) we obtain that $f = h$ for some $h \in H$.

Remark 2.5. Consider an arrow $x: A \rightarrow X$ and the epi-mono factorization $x_* = \psi \circ \rho^3$. With reference to the diagram below Axiom (iii) implies that $I = [A, A]/H$ with $H = \text{Fix}(x) \leq G$.

$$\begin{array}{ccccc}
& & x_* & & \\
& \nearrow & & \searrow & \\
[A, A] & \xrightarrow{\rho} & I & \xrightarrow{\psi} & [A, X]
\end{array}$$

Indeed by Axiom (iii) $[A, A] = \text{Aut}(A) = G$ so an $h \in H$ acts on $[A, A]$ by left multiplication. Given $f \in [A, A]$ we have

$$(\psi \circ \rho \circ h)(f) = x_*(h \circ f) = x \circ h \circ f = x \circ f = x_*(f) = (\psi \circ \rho)(f)$$

that is $\psi \circ \rho \circ h = \psi \circ \rho$ which, by monicness of ψ , implies $\rho \circ h = \rho$. Moreover as I is the image of x_* it is unique up to isomorphism and thus ρ is really a quotient arrow; $I = [A, A]/H$.

³We recall that \mathbf{GSet} is a topos and, as such, has epi-mono factorization.

$$\begin{array}{ccccc}
 & & x & & \\
 & \nearrow & & \searrow & \\
 A & \xrightarrow{q} & A/H & \xrightarrow{\varepsilon} & X .
 \end{array}$$

Figure 2.6

$$\begin{array}{ccccc}
 & & x_* & & \\
 & \nearrow & & \searrow & \\
 [A, A] & \xrightarrow{q_*} & [A, A/H] & \xrightarrow{\varepsilon_*} & [A, X] \\
 & \searrow \rho & \uparrow \eta & \nearrow \psi & \\
 & & [A, A]/H & &
 \end{array}$$

Figure 2.7

Proposition 2.6. Any arrow $x: A \rightarrow X$ of \mathcal{C} is a quotient of A by $H = \text{Fix}(x) \leq G$ (with respect to the action on $[A, X]$ described in Proposition 1.12) i.e. $X = A/H$.

Proof. By choosing $H = \text{Fix}(x)$ we get $x \circ h = x$ for all $h \in H$ and thus there is a unique arrow $\varepsilon: A/H \rightarrow X$ of \mathcal{C} such that $x = \varepsilon \circ q$, where $q: A \rightarrow A/H$ is the quotient of A by H . Graphically:

Now by applying $[A, -]$ to the diagram above we obtain where ρ, ψ and η are as discussed before. We now have

$$\varepsilon_* \circ \eta \circ \rho = \varepsilon_* \circ q_* = x_* = \psi \circ \rho$$

that by epicness of ρ implies $\varepsilon_* \circ \eta = \psi$. Now ε_* must be monic since ψ and η are; but this, by Proposition 2.3, implies that $\varepsilon \in \mathcal{C}$ is monic too. Being an arrow of \mathcal{C} , by Axiom (i), ε is also a strict epimorphism and thus an isomorphism. \square

Proposition 2.7. The action of $\text{Aut}(A)$ on $[A, X]$ is transitive for all $X \in \mathcal{C}$.

Proof. Consider again Figure 2.7. We proved that ε is an isomorphism and thus ε_* must be one too. Moreover from Remark 2.4 we know η is iso too and so we have $[A, X] \cong [A, A]/H$, but by Axiom (3) $[A, A] = \text{Aut}(A)$ and so we have that $[A, X]$ is transitive. \square

Theorem 2.8. Given a category \mathcal{C} and an object $A \in \mathcal{C}$ such that Axioms (1), (2) and (3) hold there exists an adjunction

$$A \times_G - \dashv [A, -]_G$$

where $G = \text{Aut}(A)$ such that the maps

$$\eta: E \cong [A, A]/H \rightarrow [A, A/H]$$

$$\varepsilon: A/H \rightarrow X$$

are isomorphisms. This enstablishes an equivalence of categories between \mathcal{C} and \mathbf{GSet}^t .

Proof. The existence of the adjunction follows from the discussion in Section 1, the fact that η and ε are (the components of) the unit and counit of the adjunction is a calculation that has been moved to the appendix and the fact that they are isomorphisms follows respectively from Remark 2.4 and Proposition 2.6. \square

3 Appendixes

3.1 Naturality of $[A/H, X] \cong \mathbf{GSet}^t(E, [A, X]_G)$

Fix a category \mathcal{C} , an element $A \in \mathcal{C}$ and let $G = \text{Aut}(A)$. Let's indicate with ψ_{EX} the bijection between $[A/H, X]$ and $\mathbf{GSet}^t(E, [A, X]_G)$ described in Remark 1.14; we shall prove that it is natural in both $X \in \mathcal{C}$ and $E \in \mathbf{GSet}^t$.

Naturality in X. Given an arrow $f: X \rightarrow Y$ of \mathcal{C} we want to prove that the ψ_{EX} are the components of a natural transformation $[A/H, -] \Rightarrow \mathbf{GSet}^t(E, [A, -]_G)$ i.e that the following diagram commutes.

$$\begin{array}{ccc} [A/H, X] & \xrightarrow{\psi_{EX}} & \mathbf{GSet}^t(E, [A, X]_G) \\ \downarrow f_* & & \downarrow (f_*)_* \\ [A/H, Y] & \xrightarrow{\psi_{EY}} & \mathbf{GSet}^t(E, [A, Y]_G) \end{array}$$

Recall that here $H = \text{Fix}(x_0)$ for $x_0 \in E$. Pick any $x \in [A/H, X]$ and let $q: A \rightarrow A/H$ be the quotient arrow (the quotient exists because \mathcal{C} is assumed to have all quotients by subgroups of G); then chasing x through the diagram down the two possible ways yields two arrows in \mathbf{GSet}^t of type $E \rightarrow [A, Y]_G$. Since E is transitive arrows out of E are determined uniquely by the image of x_0 ; keeping this in mind the following computations show that the square commutes.

$$\begin{aligned} ((f_*)_* \circ \psi_{EX})(x)(x_0) &= (f_* \circ \psi_{EX}(x))(x_0) = f_*(\psi_{EX}(x)(x_0)) = f_*(x \circ q) = f \circ x \circ q \\ (\psi_{EY} \circ f_*)(x)(x_0) &= \psi_{EY}(f \circ x)(x_0) = f \circ x \circ q \end{aligned}$$

\square

Naturality in E. We want to prove that the ψ_{EX} are the components of a natural transformation $[A/-, X] \Rightarrow \mathbf{GSet}^t(-, [A, X]_G)$. Given $f: E \cong G/H \rightarrow F \cong G/H'$ with $H = \text{Fix}(x_0), H' = \text{Fix}(f(x_0)), x_0 \in E$, using the notation $\tilde{f} = A \times_G f$ (see Remark 1.14), we show the naturality of the following square.

$$\begin{array}{ccc} [A/H, X] & \xrightarrow{\psi_{EX}} & \mathbf{GSet}^t(E, [A, X]_G) \\ \uparrow \tilde{f}^* & & \uparrow f^* \\ [A/H', X] & \xrightarrow{\psi_{FX}} & \mathbf{GSet}^t(F, [A, X]_G) \end{array}$$

As before we chase an $x \in [A/H', X]$ down the two possible paths to get two arrows of type $E \rightarrow [A, X]_G$ and prove that they are the same by evaluating them on $x_0 \in E$:

$$\begin{aligned} (\psi_{EX} \circ \tilde{f}^*)(x)(x_0) &= \psi_{EX}(x \circ \tilde{f})(x_0) = x \circ \tilde{f} \circ q = x \circ q', \\ (f^* \circ \psi_{FX})(x)(x_0) &= (\psi_{FX}(x) \circ f)(x_0) = \psi_{FX}(x)(f(x_0)) = x \circ q'. \end{aligned}$$

□

3.2 η and ε are the unit and counit of $A \times_G - \dashv [A, -]_G$

η is the unit. Our adjunction is given by the bijection

$$\psi: \mathcal{C}(A \times_G E, X) \cong \mathbf{GSet}^t(E, [A, X]_G)$$

natural in $X \in \mathcal{C}$ and $E \in \mathbf{GSet}^t$. To obtain the unit we set $X = A \times_G E$:

$$\mathcal{C}(A \times_G E, A \times_G E) \cong \mathbf{GSet}^t(E, [A, A \times_G E]_G)$$

and so

$$\mathcal{C}(A/H, A/H) \cong \mathbf{GSet}^t(E, [A, A/H]_G)$$

where $H = \text{Fix}(x) \leq G$ ($x \in E$) by definition of $A \times_G E$. Now (the component at E of) the unit is given by the image of $1_{A/H}$ under this bijection. By the discussion in Section 1 if $q: A \rightarrow A/H$ is the quotient arrow in \mathcal{C} then $\psi(1_{A/H})$ is the map $E \cong [A, A]_G/H \rightarrow [A, A/H]_G$ of \mathbf{GSet}^t that factors the map $G \rightarrow E$ that sends e to q . But this last map is q^* as in Figure 2.5. □

ε is the counit. Keeping the proof above in mind we set $E = [A, X]$ and obtain

$$\psi: \mathcal{C}(A \times_G [A, X], X) \cong \mathbf{GSet}^t([A, X]_G, [A, X]_G)$$

that becomes

$$\mathcal{C}(A/H, X) \cong \mathbf{GSet}^t([A, X]_G, [A, X]_G)$$

with $H = \text{Fix}(x)$ for some $x \in [A, X]$. Notice that since $[A, X] \cong G/H$ by Proposition 2.7 we can consider $1_{[A, X]}$ as a map of type $G/H \rightarrow [A, X]$ such that $1_{[A, X]}(e) = x$ (this can be obtained by chasing 1_A around Figure 2.7 keeping in mind that ε_* there is iso). Now we have that $\psi^{-1}(1_{[A, X]})$ is the arrow $A/H \rightarrow X$ of \mathcal{C} that factorizes x i.e. $\varphi^{-1}(1_{[A, X]}) = \varepsilon$ as in Figure 2.6. □