

# Grothendieck's Galois Theory

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## 1 Group actions and adjoints

**Definition 1.1.** In a category  $\mathcal{C}$  an arrow  $f: X \rightarrow Y$  is a **strict epimorphism** if it is the joint coequalizer of all the arrows it coequalizes. This means that any arrow  $g: X \rightarrow Z$  such that  $g \circ x = g \circ y$  for any  $x, y: C \rightarrow X$  such that  $f \circ x = f \circ y$  there exists a unique arrow  $h: Y \rightarrow Z$  such that  $h \circ f = g$ . Refer to Figure 1.1.

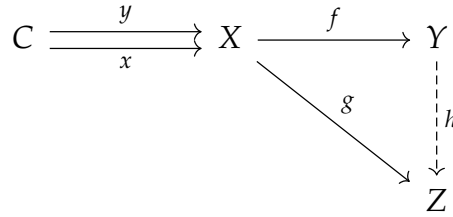


Figure 1.1

**Remark 1.2.** Strict epimorphisms are coequalizers, thus epimorphisms (as the name implies).

**Remark 1.3.** If an arrow is both a strict epimorphism and a monomorphism then it is an isomorphism.

**Definition 1.4.** Let  $H$  be a group,  $A$  an object of  $\mathcal{C}$  and  $G = \text{Aut}(A)$  the group of automorphisms of  $A$  in  $\mathcal{C}$  i.e. the group whose underlying set is the set of isomorphisms of type  $A \rightarrow A$  of  $\mathcal{C}$  and whose operation is composition in  $\mathcal{C}$ . An **action** of  $H$  on  $A$  is a group homomorphism  $H \rightarrow G$ .

**Notation 1.5.** Given an action of a group  $H$  on an object  $A$  of  $\mathcal{C}$  we denote, with a slight abuse of notation, the automorphism of  $A$  associated to  $h \in H$  by the same symbol  $h$ .

**Definition 1.6.** If  $H$  acts on  $A$  as defined in 1.4 we define the quotient of  $A$  by  $H$  in  $\mathcal{C}$  to be an element  $A/H$  of  $\mathcal{C}$  equipped with an arrow  $q: A \rightarrow A/H$  such that:

- (1) for all  $h \in H$   $q \circ h = q$  holds,
- (2) for any  $x: A \rightarrow X$  such that  $x \circ h = x$  for all  $h \in H$  there exists a unique arrow  $\varphi: A/H \rightarrow X$  such that  $x = \varphi \circ q$ .

See also Figure 1.2.

**Remark 1.7.** Quotients are defined by a universal property, thus are unique up to unique isomorphism and we can speak of “the” quotient of  $A$  by  $H$  instead of “a” quotient of  $A$  by  $H$ .

**Notation 1.8.** Sometimes we use the sentence “the quotient of  $A$  by  $H$ ” to refer to the object  $A/H$ , some others to the arrow  $q: A \rightarrow A/H$ ; the context should be enough to differentiate between the two cases.

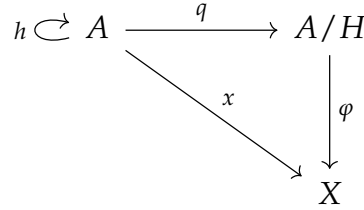


Figure 1.2

**Remark 1.9.** Consider a quotient  $q: A \rightarrow A/H$ ; by condition (1) above  $q \circ h = q = q \circ 1_A$  so  $q$  coequalizes all the pairs  $(h, 1_A)$ , for  $h \in H$ . If another arrow  $x: A \rightarrow X$  coequalizes all the pairs that  $q$  does then this arrow is such that  $x \circ h = x \circ 1_A = x$  for all  $h \in H$  and thus, by condition (2), we have a unique factorization  $x = \varphi \circ q$ . This proves that all quotients are strict epimorphisms.

**Remark 1.10.** Let  $G$  be a group, **GSet** the category of  $G$ -sets and  $G$ -invariant maps and  $A$  an object of **GSet**. In this category Definition 1.6 yields the familiar notion of the set of all orbits of an action:  $A/G$  is the set of orbits of  $A$ .

**Remark 1.11.** Consider again **GSet**. The underlying set of  $G$  (that we also denote as  $G$ ) is a  $G$ -set with the action given by left multiplication in  $G$ ; we call this the **canonical action** of  $G$  on itself. Let  $\varphi: G \rightarrow E$  be a  $G$ -invariant map; it is easy to see that such a  $\varphi$ , by virtue of being  $G$ -invariant, is determined uniquely by the value  $\varphi(e)$ , where  $e$  is the neutral element of  $G$ .

Let now  $E$  be a transitive  $G$ -set i.e. a set upon which the action of  $G$  is transitive i.e. such that  $E/G = \{*\}$ . Fix an  $x \in E$  and let  $\varphi_x$  be the  $G$ -invariant map defined by  $\varphi_x(e) = x$ ; we argue that  $\varphi_x: G \rightarrow E$  makes  $E$  into a quotient of  $G$  by the subgroup

$$H = \text{Fix}(x) = \{g \in G: gx = x\}.$$

Indeed by using the definition of  $H$  and the fact that  $\varphi$  is  $G$ -invariant we have

$$(\varphi \circ h)(e) = \varphi(h(e)) = h(\varphi(e)) = h(x) = x.$$

for all  $h \in H$ . Moreover let  $g: G \rightarrow F$  satisfy (1) of Definition 1.6; as we discussed above  $g$  is entirely determined by the image of  $e$  so we obtain (2) defining an arrow  $f: E \rightarrow F$  by  $f(x) = g(e)$ . The situation is depicted in Figure 1.3.

Trivially  $G$  is a transitive  $G$ -set and for any  $g \in G$   $G/\text{Fix}(g)$  is transitive as well so we have that an object  $E \in \mathbf{GSet}$  is transitive if and only if it is isomorphic to  $G/H$  where  $H = \text{Fix}(x)$  for (any)  $x \in E$ .

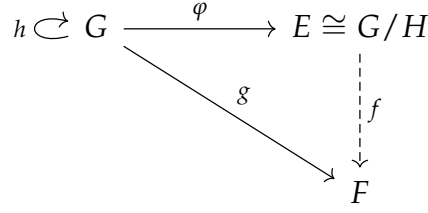


Figure 1.3

For the rest of the section fix a category  $\mathcal{C}$ , an object  $A \in \mathcal{C}$  and let  $G = \text{Aut}(A)$ .

**Proposition 1.12.** Consider a subgroup  $H \leq G$  and an object  $X \in \mathcal{C}$ .  $H$  acts on the hom-set  $[A, X]$  as follows<sup>1</sup>:

$$\begin{aligned}
H \times [A, X] &\longrightarrow [A, X] \\
(h, x) &\longmapsto h \cdot x = x \circ h.
\end{aligned}$$

**Remark 1.13.** Assume that the action  $G \times [A, X] \rightarrow [A, X]$  is transitive and let  $\mathbf{GSet}^t$  be the category of transitive  $G$ -sets (a subcategory of  $G\text{set}$ ). Then we have a functor

$$\begin{array}{ccc}
[A, -]_G: \mathcal{C} & \longrightarrow & \mathbf{GSet}^t \\
X & & [A, X]_G \\
\downarrow f & \longmapsto & \downarrow f_* \\
Y & & [A, Y]_G
\end{array}$$

where we indicate with  $[A, X]_G$  the hom-set  $[A, X]$  upon which  $G$  acts as described in Proposition 1.12 and  $f_*$  is post-composition with  $f$ . It is easy to check that  $f_*$  is indeed  $G$ -invariant.

**Remark 1.14.** Consider an object  $E \in \mathbf{GSet}^t$ , pick an element  $x_0 \in E$  and let  $H = \text{Fix}(x_0) \leq G$  (the choice of  $x_0$  is irrelevant as  $E$  is transitive). Moreover assume that  $\mathcal{C}$  has quotients of  $A$  by any subgroup of  $G$ .

By what we observed in Remark 1.11 we have a bijection between elements of  $E$  and arrows of type  $G \rightarrow E$ . Consider then  $f \in [A, X]$  and its corresponding arrow  $\varphi: G \rightarrow [A, X]_G$ ; we claim that  $f$  factors through  $A/H$  if and only if  $\varphi$  factors through  $E \cong G/H$  (see Figure 1.4). Indeed  $f$  factors if and only if  $f \circ h = f$  for all  $h \in H$ , by using the fact that  $\varphi$  is  $G$ -invariant we obtain

$$\varphi(h(e)) = h(\varphi(e)) = h(f) = f \circ h = f$$

and, since  $\varphi$  is uniquely determined by  $\varphi(e)$ ,  $\varphi \circ h = \varphi$  for all  $h \in H$ ; this happens if and only if  $\varphi$  factors through  $E \cong G/H$ .

<sup>1</sup>Since an action as of Definition 1.4 is a map that sends elements of a group to arrows it is, in this case, equivalent to give the definition of an action by uncurrying.

This gives us, for each  $X \in \mathcal{C}$  and  $E \in \mathbf{GSet}^t$ , a bijection

$$[A/H, X] \cong \mathbf{GSet}^t(E, [A, X]_G)$$

natural in  $X$  in  $E$  (see Section 2). Thus, if  $\mathcal{C}$  has quotients of  $A$  by subgroups of  $G$  we have an adjunction  $A \times_G - \dashv [A, -]_G$  where  $A \times_G E = A/H$  for  $H = \text{Fix}(x_0)$  and  $x_0 \in E$  as above<sup>2</sup>.

$$\begin{array}{ccc} h \hookrightarrow A & \xrightarrow{\quad} & A/H \\ & \searrow f & \downarrow \text{dashed} \\ & & X \end{array} \qquad \begin{array}{ccc} h \hookrightarrow G & \xrightarrow{\quad} & E \cong G/H \\ & \searrow \varphi & \downarrow \text{dashed} \\ & & [A, X]_G \end{array}$$

Figure 1.4

Our main problem will be that of finding conditions on  $\mathcal{C}$  that make this adjunction into an equivalence of categories.

## 2 Appendixes

### 2.1 Naturality of $[A/H, X] \cong \mathbf{GSet}^t(E, [A, X]_G)$

Fix a category  $\mathcal{C}$ , an element  $A \in \mathcal{C}$  and let  $G = \text{Aut}(A)$ . Let's indicate with  $\psi_{EX}$  the bijection between  $[A/H, X]$  and  $\mathbf{GSet}^t(E, [A, X]_G)$  described in Remark 1.14; we shall prove that it is natural in both  $X \in \mathcal{C}$  and  $E \in \mathbf{GSet}^t$ .

*Naturality in  $X$ .* Given an arrow  $f: X \rightarrow Y$  of  $\mathcal{C}$  we want to prove that  $\psi_{EX}$  is a natural transformation  $[A/H, -] \Rightarrow \mathbf{GSet}^t(E, [A, -]_G)$  i.e that the following diagram commutes.

$$\begin{array}{ccc} [A/H, X] & \xrightarrow{\psi_{EX}} & \mathbf{GSet}^t(E, [A, X]_G) \\ \downarrow f_* & & \downarrow (f_*)_* \\ [A/H, Y] & \xrightarrow{\psi_{EY}} & \mathbf{GSet}^t(E, [A, Y]_G) \end{array}$$

Recall that here  $H = \text{Fix}(x_0)$  for  $x_0 \in E$ . Pick any  $x \in [A/H, X]$  and let  $q: A \rightarrow A/H$  be the quotient arrow (the quotient exists because  $\mathcal{C}$  is assumed to have all quotients by subgroups of  $G$ ); then chasing  $x$  through the diagram down the two possible ways yields two arrows in  $\mathbf{GSet}^t$  of type  $E \rightarrow [A, Y]_G$ . Since  $E$  is transitive we have  $E \cong G/H$  and this means that arrows of type  $E \rightarrow [A, Y]_G$  are determined uniquely by the image of  $e \in E \cong G/H$  (refer to the right part of Diagram 1.4). Keeping this in mind the following computations shows that the square commutes.

$$((f_*)_* \circ \psi_{EX})(x)(e) = (f_* \circ \psi_{EX}(x))(e) = f_*(\psi_{EX}(x)(e)) = f_*(x \circ q) = f \circ x \circ q,$$

<sup>2</sup>We use the notation  $A \times_G -$  for the left adjoint in “honor” of the tensor-hom adjunction.

$$(\psi_{EY} \circ f_*)(x)(e) = \psi_{EX}(f \circ x)(e) = f \circ x \circ q.$$

□

*Naturality in E.*

□