Grothendieck's Galois Theory

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1 Section 1

Definition 1.1. In a category $\mathscr{C}$ an arrow $f\colon X \to Y$ is a **strict epimorphism** if it is the joint coequalizer of all the arrows it coequalizes. This means that any arrow $g\colon X \to Z$ such that $g \circ x = g \circ y$ for any $x, y\colon C \to X$ such that $f \circ x = f \circ y$ there exists a unique arrow $h\colon Y \to Z$ such that $h \circ f = g$. Refer to Figure 1.1.

$$
\begin{tikzcd}
C \arrow[r, shift left, "y"] \arrow[r, shift right, "x"'] & X \arrow[r, "f"] \arrow[dr, "g"] & Y \arrow[d, dashed, "h"] \\
& & Z
\end{tikzcd}
$$

Figure 1.1

Remark 1.2. Strict epimorphisms are coequalizers, thus epimorphisms (as the name implies).

Remark 1.3. If an arrow is both a stric epimorphism and a monomorphism then it is an epimorphism.

Definition 1.4. Let H be a group, A an object of $\mathscr{C}$ and $G = \mathrm{Aut}(A)$ the group of automorphisms of A in $\mathscr{C}$ i.e. the group whose underlying set is the set of isomorphisms of type $A \to A$ of $\mathscr{C}$ and whose operation is composition in $\mathscr{C}$. An **action** of H on A is a group homomorphism $H \to G$.

Notation 1.5. Given an action of a group H on an object A of $\mathscr{C}$ we denote, with a slight abuse of notation, the automorphism of A associated to $h \in H$ by the same symbol h.

Definition 1.6. If H acts on A as defined in 1.4 we define the quotient of A by H in $\mathscr{C}$ to be an element A/H of $\mathscr{C}$ equipped with an arrow $q\colon A \to A/H$ such that:

(1) for all $h \in H$ $q \circ h = q$ holds,

(2) for any $x\colon A \to X$ such that $x \circ h = x$ for all $h \in H$ there exists a unique arrow $\varphi\colon A/H \to X$ such that $x = \varphi \circ q$.

See also Figure 1.2.

Remark 1.7. Quotients are defined by a universal property, thus are unique up to unique isomorphism and we can speak of "the" quotient of A by H instead of "a" quotient of A by H.

Notation 1.8. Sometimes we use the sentence "the quotient of A by H" to refer to the object A/H, some others to the arrow $q\colon A \to A/H$; the context should be enough to differentiate between the two cases.

$$\begin{tikzcd}
A \arrow[loop left, "h"] \arrow[r, "q"] \arrow[dr, "x"] & A/H \arrow[d, "\varphi"] \\
& X
\end{tikzcd}$$

Figure 1.2

Remark 1.9. Consider a quotient $q\colon A \to A/H$; by condition (1) above $q \circ h = q = q \circ 1_A$ so q coequalizes all the pairs $(h, 1_A)$, for $h \in H$. If another arrow $x\colon A \to X$ coequalizes all the pairs that q does then this arrow is such that $x \circ h = x \circ 1_A = x$ for all $h \in H$ and thus, by condition (2), we have a unique factorization $x = \varphi \circ q$. This proves that all quotients are strict epimorphisms.

Remark 1.10. Let G be a group, **GSet** the category of G-sets and G-invariant maps and A an object of **GSet**. In this category Definition 1.6 yelds the familiar notion of the set of all orbits of an action: A/G is the set of orbits of A.

Remark 1.11. Consider again **GSet**. The underlying set of G (that we also denote as G) is a G-set with the action given by left multiplication in G; we call this the **canonical action** of G on itself. Let $\varphi\colon G \to E$ be a G-invariant map; it is easy to see that such a φ, by virtue of being G-invariant, is determined uniquely by the value $\varphi(e)$, where e is the neutral element of G.

Let now E be a transitive G-set i.e. a set upon which the action of G is transitive i.e. such that $E/G = \{*\}$. Fix an $x \in E$ and let φ_x be the G-invarian map defined by $\varphi_x(e) = x$; we argue that $\varphi_x\colon G \to E$ makes E into a quotient of G by the subgroup

$$H = \mathrm{Fix}(x) = \{g \in G\colon gx = x\}.$$

Indeed by using the definition of H and the fact that φ is G-invariant we have

$$(\varphi \circ h)(e) = \varphi(h(e)) = h(\varphi(e)) = h(x) = x.$$

for all $h \in H$. Moreover let $g\colon G \to F$ satisfy (1) of Definition 1.6; as we discussed above g is entirely determined by the image of e so we obtain (2) defining an arrow $f\colon E \to F$ by $f(x) = g(e)$. The situation is depicted in Figure 1.3.

Trivially G is a transitive G-set and for any $g \in G$ $G/\mathrm{Fix}(g)$ is transitive as well so we have that an object $E \in$ **GSet** is transitive if and only if it is isomorphic to G/H where $H = \mathrm{Fix}(x)$ for (any) $x \in E$.

$$
\begin{tikzcd}
h \circlearrowright G \arrow[r, "\varphi"] \arrow[dr, "g"'] & E \cong G/H \arrow[d, dashed, "f"] \\
& F
\end{tikzcd}
$$

Figure 1.3

For the rest of the section fix a category $\mathscr{C}$, an object $A \in \mathscr{C}$ and let $G = \mathrm{Aut}(A)$.

Proposition 1.12. Consider a subgroup $H \leq G$ and an object $X \in \mathscr{C}$. H acts on the hom-set $[A, X]$ as follows[1]:

$$
\begin{aligned}
H \times [A, X] &\longrightarrow [A, X] \\
(h, x) &\longmapsto h \cdot x = x \circ h.
\end{aligned}
$$

Remark 1.13. Assume that the action $G \times [A, X] \to [A, X]$ is transitive and let $\mathbf{GSet}^t$ be the category of transitive G-sets (a subcategory of $Gset$). Then we have a functor

$$
\begin{aligned}
[A, -]_G \colon \ \mathscr{C} &\longrightarrow \mathbf{GSet}^t \\
\begin{array}{c} X \\ \Big\downarrow f \\ Y \end{array} &\longmapsto \begin{array}{c} [A, X]_G \\ \Big\downarrow f_* \\ [A, Y]_G \end{array}
\end{aligned}
$$

where we indicate with $[A, X]_G$ the hom-set $[A, X]$ upon which G acts as described in Proposition 1.12 and f_* is post-composition with f. It is easy to check that f_* is indeed G-invariant.

[1]Since an action as of Definition 1.4 is a map that sends elements of a group to arrows it is, in this case, equivalent to give the definition of an action by uncurrying.