Grothendieck's Galois Theory

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1 Group actions and adjoints

Definition 1.1. In a category $\mathscr C$ an arrow $f\colon X\to Y$ is a **strict epimorphism** if it is the joint coequalizer of all the pairs of arrows it coequalizes. This means that any arrow $g\colon X\to Z$ such that $g\circ x=g\circ y$ for all $x,y\colon C\to X$ such that $f\circ x=f\circ y$ there exists a unique arrow $h\colon Y\to Z$ such that $h\circ f=g$. Refer to Figure 1.1.

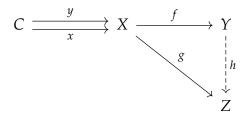


Figure 1.1

Remark 1.2. Strict epimorphisms are epimorphisms, as the name implies.

Remark 1.3. If an arrow is both a stric epimorphism and a monomorphism then it is an isomorphism.

Definition 1.4. Let H be a group, A an object of \mathscr{C} and $G = \operatorname{Aut}(A)$ the group of automorphisms of A in \mathscr{C} i.e. the group whose underlying set is the set of isomorphisms of type $A \to A$ of \mathscr{C} and whose operation is composition in \mathscr{C} . An **action** of H on A is a group homomorphism $H \to G$.

Notation 1.5. Given an action of a group H on an object A of \mathscr{C} we denote, with a slight abuse of notation, the automorphism of A associated to $h \in H$ by the same symbol h.

Definition 1.6. If H acts on A as defined in 1.4 we define the quotient of A by H in $\mathscr C$ to be an element A/H of $\mathscr C$ equipped with an arrow $q \colon A \to A/H$ such that:

- (1) for all $h \in H$ we have $q \circ h = q$,
- (2) for any $x: A \to X$ such that $x \circ h = x$ for all $h \in H$ there exists a unique arrow $\varphi: A/H \to X$ such that $x = \varphi \circ q$.

See also Figure 1.2.

Remark 1.7. Quotients are defined by a universal property, thus are unique up to unique isomorphism and we can speak of "the" quotient of *A* by *H* instead of "a" quotient of *A* by *H*.

Notation 1.8. Sometimes we use the sentence "the quotient of A by H" to refer to the object A/H, some others to the arrow $q: A \to A/H$; the context should be enough to differentiate between the two.

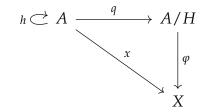


Figure 1.2

Remark 1.9. Consider a quotient $q: A \to A/H$; by condition (1) above $q \circ h = q = q \circ 1_A$ so q coequalizes all the pairs $(h, 1_A)$, for $h \in H$. If another arrow $x: A \to X$ coequalizes all the pairs that q does then this arrow is such that $x \circ h = x \circ 1_A = x$ for all $h \in H$ and thus, by condition (2), we have a unique factorization $x = \varphi \circ q$. This proves that all quotients are strict epimorphisms.

Notation 1.10. Let *G* be a group, then with **GSet** we denote the category of *G*-sets and *G*-invariant maps.

Remark 1.11. Consider **GSet**. The underlying set of G (that we also denote with G) is a G-set with the action given by left multiplication; we call this the **canonical action** of G on itself. Let $\varphi: G \to E$ be a G-invariant map; it is easy to see that such a φ , by virtue of being G-invariant, is determined uniquely by the value $\varphi(e)$, where e is the neutral element of G.

Let now *E* be a transitive G-set i.e. such that $E/G = \{*\}$ in **Set**¹. Fix an $x \in E$ and let φ be the G-invariant map defined by $\varphi(e) = x$; we argue that $\varphi \colon G \to E$ makes *E* into a quotient of *G* by the subgroup

$$H = Fix(x) = \{g \in G \colon gx = x\}.$$

Indeed by using the definition of H and the fact that φ is G-invariant we have

$$(\varphi \circ h)(e) = \varphi(h(e)) = h(\varphi(e)) = h(x) = x.$$

for all $h \in H$. Moreover let $g: G \to F$ satisfy (1) of Definition 1.6; as we discussed above g is entirely determined by the image of e so we obtain (2) defining an arrow $f: E \to F$ by f(x) = g(e). The situation is depicted in Figure 1.3.

Finally: G is a transitive G-set and so is every G/H with $H \leq G$ and the action defined again by left multiplication (on cosets) so an object $E \in \mathbf{GSet}$ is transitive if and only if it is isomorphic to some G/H.

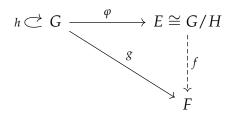


Figure 1.3

For the rest of the section fix a category \mathscr{C} , an object $A \in \mathscr{C}$ and let $G = \operatorname{Aut}(A)$.

Proposition 1.12. Consider a subgroup $H \leq G$ and an object $X \in \mathcal{C}$. H acts on the hom-set [A, X] as follows²:

$$H \times [A, X] \longrightarrow [A, X]$$

 $(h, x) \longmapsto h \cdot x = x \circ h.$

Remark 1.13. Assume that the action $G \times [A, X] \to [A, X]$ is transitive and let **GSet**^t be the category of transitive G-sets (a subcategory of Gset). Then we have a functor

$$[A,-]_G \colon \mathscr{C} \longrightarrow \mathbf{GSet}^t$$

$$\begin{matrix} X & [A,X]_G \\ \downarrow_f \longmapsto & \downarrow_{f_*} \\ Y & [A,Y]_G \end{matrix}$$

where we indicate with $[A, X]_G$ the hom-set [A, X] upon which G acts as described in Proposition 1.12 and f_* is post-composition with f. It is easy to check that f_* is indeed G-invariant.

Remark 1.14. Consider an object $E \in \mathbf{GSet}^t$, pick an element $x_0 \in E$ and let $H = \mathrm{Fix}(x_0) \leq G$ (the choice of x_0 is irrelevant as E is transitive). Moreover assume that \mathscr{C} has quotients of A by any subgroup of G.

By what we observed in Remark 1.11 we have a bijection between elements of E and arrows of type $G \to E$. Consider then $f \in [A, X]$ and its corresponding arrow $\varphi \colon G \to [A, X]_G$; we claim that f factors through A/H if and only if φ factors through $E \cong G/H$ (see Figure 1.4). Indeed f factors if and only if $f \circ h = f$ for all $h \in H$, by using the fact that φ is G-invariant we obtain

$$\varphi(h(e)) = h(\varphi(e)) = h(f) = f \circ h = f$$

and, since φ is uniquely determined by $\varphi(e)$, $\varphi \circ h = \varphi$ for all $h \in H$; this happens if and only if φ factors through $E \cong G/H$.

 $^{^{1}}$ This states that the set of orbits of E is a singleton that is of course the case if and only if E is transitive.

²Since an action as of Definition 1.4 is a map that sends elements of a group to arrows it is, in this case, equivalent to give the definition of an action by uncurrying.

This gives us, for each $X \in \mathscr{C}$ and $E \in \mathbf{GSet}^t$, a bijection

$$[A/H, X] \cong \mathbf{GSet}^t(E, [A, X]_G)$$

natural in X in E (see Section 3). Thus, if $\mathscr C$ has quotients of A by subgroups of G we have an adjunction $A \times_G - \dashv [A, -]_G$ where $A \times_G E = A/H$ for $H = \operatorname{Fix}(x_0)$ and $x_0 \in E$ as above³.

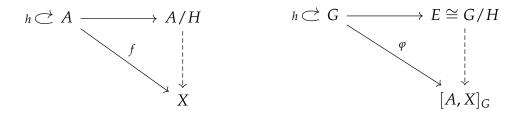


Figure 1.4

Our main problem will be that of finding conditions on $\mathscr C$ that make this adjunction ineto an equivalence of categories.

2 Categorial axiomatization of Galois Theory

Through this section fix a category $\mathscr C$ and an object $A \in \mathscr C$.

Definition 2.1. We define the following axioms.

- (1) For every $X \in \mathcal{C}$ there is at least a map of type $A \to X$ and all maps $A \to X$ are strict epimorphisms.
- (2) For any subgroup $H \leq \operatorname{Aut}(A)$ the quotient $q: A \to A/H$ exists and is preserved by $[A, -]: \mathscr{C} \to \mathbf{GSet}$.
- (3) Every endomorphism of A is an isomorphism i.e. [A, A] = Aut(A).

Remark 2.2. It is known that if $f \circ g$ is a strict epimorphism then so is f. Thus it follows from Axiom (1) that every arrow $X \to Y$ in $\mathscr C$ is a strict epimorphism.

Proposition 2.3. Axiom (1) implies that [A, -] is faithful, reflects monomorphisms and isomorphisms.

Proof. Consider arrows $f, g: X \to Y \in \mathscr{C}$ such that [A, f] = [A, g]; that is $f_* = g_*$. By Axiom (1) let $h: A \to X$ be a third arrow of \mathscr{C} then we have $f_*(h) = g_*(h)$ i.e. $f \circ h = g \circ h$. Again by Axiom (1) h is an epimorphism (really, a strong one) and thus we obtain f = g; that is: [A, -] is faithful.

It is well known that every faithful functor reflects monomorphisms. Because of this if $f \in \mathbf{GSet}$ is an isomorphism and $g \in \mathscr{C}$ is such that [A,g] = f then g is a monomorphism too; but, as an arrow of \mathscr{C} , g is also a strict epimorphism and thus an isomorphism.

³We use the notation $A \times_G$ – for the left adjoint in "honor" of the tensor-hom adjunction.

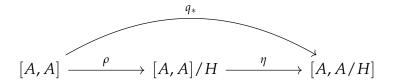


Figure 2.5

Remark 2.4. Consider $H \leq G$ and the quotient $q: A \to A/H$ in \mathscr{C} , then $q_*: [A, A] \to [A, A/H]$ in **GSet** is a quotient too because quotients are preserved by [A, -] (Axiom (2)). Thus we have $[A, A]/H \cong [A, A/H]$, and the following diagram commutes (where ρ is a quotient arrow and η the isomorphism).

The fact that η is a bijection gives us the following:

- (i) for $f,g \in [A,A]$ if $g \circ f = g \circ g$ then there is some $h \in H$ such that $f = h \circ g$,
- (ii) for all $x: A \to A/H$ there is an arrow $f \in [A, A]$ such that $q \circ f = x$.

Moreover, under Axiom (3), (i) implies the following:

(iii) $q \circ f = q$ implies $f \in H$.

Indeed by taking $g = 1_A$ in (i) we obtain that f = h for some $h \in H$.

Remark 2.5. Consider an arrow $x: A \to X$ and the epi-mono factorization $x_* = \psi \circ \rho^4$. With reference to the diagram below Axiom (iii) implies that I = [A, A]/H with $H = \text{Fix}(x) \leq G$.

$$[A,A] \xrightarrow{\rho} I \xrightarrow{\psi} [A,X]$$

Indeed by Axiom (iii) $[A, A] = \operatorname{Aut}(A) = G$ so an $h \in H$ acts on [A, A] by left multiplication. Given $f \in [A, A]$ we have

$$(\psi \circ \rho \circ h)(f) = x_*(h \circ f) = x \circ h \circ f = x \circ f = x_*(f) = (\psi \circ \rho)(f)$$

that is $\psi \circ \rho \circ h = \psi \circ \rho$ which, by monicness of ψ , implies $\rho \circ h = \rho$. Moreover as I is the image of x_* it is unique up to isomorphism and thus ρ is really a quotient arrow; I = [A, A]/H.

Proposition 2.6. Any arrow $x: A \to X$ of \mathscr{C} is a quotient of A by $H = \text{Fix}(x) \leq G$ (with respect to the action on [A, X] described in Proposition 1.12) i.e. X = A/H.

Proof. By choosing H = Fix(x) we get $x \circ h = x$ for all $h \in H$ and thus there is a unique arrow $\varepsilon \colon A/H \to X$ of $\mathscr C$ such that $x = \varepsilon \circ q$, where $q \colon A \to A/H$ is the quotient of A by H. Graphically:

Now by applying [A, -] to the diagram above we obtain

⁴We recall that **GSet** is a topos and, as such, has epi-mono factorization.

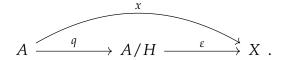


Figure 2.6

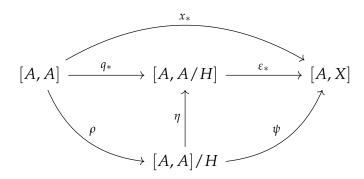


Figure 2.7

where ρ , ψ and η are as discussed before. We now have

$$\varepsilon_* \circ \eta \circ \rho = \varepsilon_* \circ q_* = x_* = \psi \circ \rho$$

that by epicness of ρ implies $\varepsilon_* \circ \eta = \psi$. Now ε_* must be monic since ψ and η are; but this, by Proposition 2.3, implies that $\varepsilon \in \mathscr{C}$ is monic too. Being an arrow of \mathscr{C} , by Axiom (i), ε is also a strict epimorphism and thus an isomorphism.

Proposition 2.7. The action of $\operatorname{Aut}(A)$ on [A, X] is transitive for all $X \in \mathscr{C}$.

Proof. Consider again Figure 2.7. We proved that ε is an isomorphism and thus ε_* must be one too. Moreover from Remark 2.4 we know η is iso too and so we have $[A, X] \cong [A, A]/H$, but by Axiom (3) $[A, A] = \operatorname{Aut}(A)$ and so we have that [A, X] is transitive.

Theorem 2.8. Given a category $\mathscr C$ and an object $A \in \mathscr C$ such that Axioms (1), (2) and (3) hold there exists an adjunction

$$A \times_G - \dashv [A, -]_G$$

where G = Aut(A) such that the maps

$$\eta: E \cong [A, A]/H \to [A, A/H]$$

 $\varepsilon: A/H \to X$

are isomorphisms. This enstablishes an equivalence of categories between \mathscr{C} and \mathbf{GSet}^t .

Proof. The existence of the adjunction follows from the discussion in Section 1, the fact that η and ε are (the components of) the unit and counit of the andjunction is a calculation that has been moved to the appendix and the fact that they are isomorphisms follows respectively from Remark 2.4 and Proposition 2.6.

3 Appendixes

3.1 Naturality of $[A/H, X] \cong \mathbf{GSet}^t(E, [A, X]_G)$

Fix a category \mathscr{C} , an element $A \in \mathscr{C}$ and let $G = \operatorname{Aut}(A)$. Let's indicate with ψ_{EX} the bijection between [A/H, X] and $\operatorname{\mathbf{GSet}}^t(E, [A, X]_G)$ described in Remark 1.14; we shall prove that it is natural in both $X \in \mathscr{C}$ and $E \in \operatorname{\mathbf{GSet}}^t$.

Naturality in X. Given an arrow $f: X \to Y$ of \mathscr{C} we want to prove that the ψ_{EX} are the components of a natural transformation $[A/H, -] \Rightarrow \mathbf{GSet}^t(E, [A, -]_G)$ i.e that the following diagram commutes.

$$[A/H, X] \xrightarrow{\psi_{EX}} \mathbf{GSet}^{t}(E, [A, X]_{G})$$

$$\downarrow^{f_{*}} \qquad \qquad \downarrow^{(f_{*})_{*}}$$

$$[A/H, Y] \xrightarrow{\psi_{EY}} \mathbf{GSet}^{t}(E, [A, Y]_{G})$$

Recall that here $H = \text{Fix}(x_0)$ for $x_0 \in E$. Pick any $x \in [A/H, X]$ and let $q: A \to A/H$ be the quotient arrow (the quotient exists because $\mathscr C$ is assumed to have all quotients by subgroups of G); then chasing x through the diagram down the two possible ways yields two arrows in \mathbf{GSet}^t of type $E \to [A,Y]_G$. Since E is transitive arrows out of E are determined uniquely by the image of x_0 ; keeping this in mind the following computations show that the square commutes.

$$((f_*)_* \circ \psi_{EX})(x)(x_0) = (f_* \circ \psi_{EX}(x))(x_0) = f_*(\psi_{EX}(x)(x_0)) = f_*(x \circ q) = f \circ x \circ q$$
$$(\psi_{EY} \circ f_*)(x)(x_0) = \psi_{EX}(f \circ x)(x_0) = f \circ x \circ q$$

Naturality in E. We want to prove that the ψ_{EX} are the components of a natural transformation $[A/-,X]\Rightarrow \mathbf{GSet}^t(-,[A,X]_G)$, but before drawing the naturality square as above we shall note that this case is slightly complicated by the fact that it is not immediately clear how the functor [A/-,X] acts on arrows. Indeed let $f\colon E\cong G/H\to F\cong G/H'$ be an arrow of \mathbf{GSet}^t with $H=\mathrm{Fix}(x_0),H'=\mathrm{Fix}(f(x_0))$ and $x_0\in E$; these choices are justified by the fact that both E and F are transitive G-sets. Now notice that, for $h\in H$, $f(h\cdot x_0)=f(x_0)$ by definition of H and $f(h\cdot x_0)=h\cdot f(x_0)$ by G-invariance of f; thus $h\in H'$ and $H\subseteq H'$. This means that if we let $g\colon A\to A/H, g'\colon A\to A/H'$ be the quotients in $\mathscr E$ then there is a unique arrow $\widetilde f\colon A/H\to A/H'$ such that $\widetilde f\circ q=g'$. The functor [A/-,X] then sends the arrow $f\colon E\to F$ to $\widetilde f^*\colon [A/H',X]\to [A/H,X]$.

Now we want to prove the naturality of the following square.

$$\begin{array}{ccc} [A/H,X] & \xrightarrow{& \psi_{EX} & } & \mathbf{GSet}^t(E,[A,X]_G) \\ & & & & & f^* \\ \hline [A/H',X] & \xrightarrow{& \psi_{FX} & } & \mathbf{GSet}(F,[A,X]_G) \end{array}$$

As before we chase an $x \in [A/H', X]$ down the two possible paths to get two arrows of type $E \to [A, X]_G$ and prove that they are the same by evaluating them on $x_0 \in E$:

$$(\psi_{EX} \circ \widetilde{f}^*)(x)(x_0) = \psi_{EX}(x \circ \widetilde{f})(x_0) = x \circ \widetilde{f} \circ q = x \circ q',$$

$$(f^* \circ \psi_{FX})(x)(x_0) = (\psi_{FX}(x) \circ f)(x_0) = \psi_{FX}(x)(f(x_0)) = x \circ q'.$$

3.2 η and ε are the unit and counit of $A \times_G - \exists [A, -]_G$

η is the unit. Our adjunction is given by the bijection

$$\psi \colon \mathscr{C}(A \times_G E, X) \cong \mathbf{GSet}^t(E, [A, X]_G)$$

natural in $X \in \mathscr{C}$ and $E \in \mathbf{GSet}^t$. To obtain the unit we set $X = A \times_G E$:

$$\mathscr{C}(A \times_G E, A \times_G E) \cong \mathbf{GSet}^t(E, [A, A \times_G E]_G)$$

and so

$$\mathscr{C}(A/H, A/H) \cong \mathbf{GSet}^t(E, [A, A/H]_G)$$

where $H = \operatorname{Fix}(x) \leq G$ ($x \in E$) by definition of $A \times_G E$. Now (the component at E of) the unit is given by the image of $1_{A/H}$ under this bijection. By the discussion in Section 1 if $q: A \to A/H$ is the quotient arrow in $\mathscr C$ then $\psi(1_{A/H})$ is the map $E \cong [A,A]_G/H \to [A,A/H]_G$ of $\operatorname{\mathbf{GSet}}^t$ that factors the map $G \to E$ that sends e to q. But this last map is q^* as in Figure 2.5.

 ε is the counit. Keeping the proof above in mind we set E = [A, X] and obtain

$$\psi \colon \mathscr{C}(A \times_G [A, X], X) \cong \mathbf{GSet}^t([A, X]_G, [A, X]_G)$$

that becomes

$$\mathscr{C}(A/H,X) \cong \mathbf{GSet}^t([A,X]_G,[A,X]_G)$$

with H = Fix(x) for some $x \in [A, X]$. Notice that since $[A, X] \cong G/H$ by Proposition 2.7 we can consider $1_{[A,X]}$ as a map of type $G/H \to [A,X]$ such that $1_{[A,X]}(e) = x$ (this can be obtained by chasing 1_A around Figure 2.7 keeping in mind that ε_* there is iso). Now we have that $\psi^{-1}(1_{[A,X]})$ is the arrow $A/H \to X$ of $\mathscr C$ that factorizes x i.e. $\varphi^{-1}(1_{[A,X]}) = \varepsilon$ as in Figure 2.6.