

# Grothendieck's Galois Theory

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## 1 Group actions and adjoints

**Definition 1.1.** In a category  $\mathcal{C}$  an arrow  $f: X \rightarrow Y$  is a **strict epimorphism** if it is the joint coequalizer of all the pairs of arrows it coequalizes. This means that any arrow  $g: X \rightarrow Z$  such that  $g \circ x = g \circ y$  for all  $x, y: C \rightarrow X$  such that  $f \circ x = f \circ y$  there exists a unique arrow  $h: Y \rightarrow Z$  such that  $h \circ f = g$ . Refer to Figure 1.1.

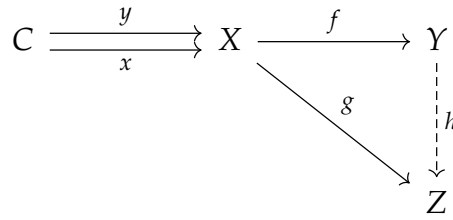


Figure 1.1

**Remark 1.2.** Strict epimorphisms are epimorphisms, as the name implies.

**Remark 1.3.** If an arrow is both a strict epimorphism and a monomorphism then it is an isomorphism.

**Definition 1.4.** Let  $H$  be a group,  $A$  an object of  $\mathcal{C}$  and  $G = \text{Aut}(A)$  the group of automorphisms of  $A$  in  $\mathcal{C}$  i.e. the group whose underlying set is the set of isomorphisms of type  $A \rightarrow A$  of  $\mathcal{C}$  and whose operation is composition in  $\mathcal{C}$ . An **action** of  $H$  on  $A$  is a group homomorphism  $H \rightarrow G$ .

**Notation 1.5.** Given an action of a group  $H$  on an object  $A$  of  $\mathcal{C}$  we denote, with a slight abuse of notation, the automorphism of  $A$  associated to  $h \in H$  by the same symbol  $h$ .

**Definition 1.6.** If  $H$  acts on  $A$  as defined in 1.4 we define the quotient of  $A$  by  $H$  in  $\mathcal{C}$  to be an element  $A/H$  of  $\mathcal{C}$  equipped with an arrow  $q: A \rightarrow A/H$  such that:

- (1) for all  $h \in H$  we have  $q \circ h = q$ ,
- (2) for any  $x: A \rightarrow X$  such that  $x \circ h = x$  for all  $h \in H$  there exists a unique arrow  $\varphi: A/H \rightarrow X$  such that  $x = \varphi \circ q$ .

See also Figure 1.2.

**Remark 1.7.** Quotients are defined by a universal property, thus are unique up to unique isomorphism and we can speak of “the” quotient of  $A$  by  $H$  instead of “a” quotient of  $A$  by  $H$ .

**Notation 1.8.** Sometimes we use the sentence “the quotient of  $A$  by  $H$ ” to refer to the object  $A/H$ , some others to the arrow  $q: A \rightarrow A/H$ ; the context should be enough to differentiate between the two.

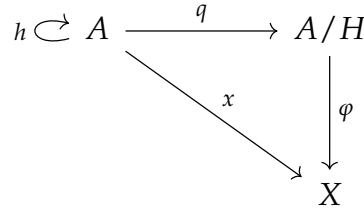


Figure 1.2

**Remark 1.9.** Consider a quotient  $q: A \rightarrow A/H$ ; by condition (1) above  $q \circ h = q = q \circ 1_A$  so  $q$  coequalizes all the pairs  $(h, 1_A)$ , for  $h \in H$ . If another arrow  $x: A \rightarrow X$  coequalizes all the pairs that  $q$  does then this arrow is such that  $x \circ h = x \circ 1_A = x$  for all  $h \in H$  and thus, by condition (2), we have a unique factorization  $x = \varphi \circ q$ . This proves that all quotients are strict epimorphisms.

**Notation 1.10.** Let  $G$  be a group, then with **GSet** we denote the category of  $G$ -sets and  $G$ -invariant maps.

**Remark 1.11.** Consider **GSet**. The underlying set of  $G$  (that we also denote with  $G$ ) is a  $G$ -set with the action given by left multiplication; we call this the **canonical action** of  $G$  on itself. Let  $\varphi: G \rightarrow E$  be a  $G$ -invariant map; it is easy to see that such a  $\varphi$ , by virtue of being  $G$ -invariant, is determined uniquely by the value  $\varphi(e)$ , where  $e$  is the neutral element of  $G$ .

Let now  $E$  be a transitive  $G$ -set i.e. such that  $E/G = \{*\}$  in **Set**<sup>1</sup>. Fix an  $x \in E$  and let  $\varphi$  be the  $G$ -invariant map defined by  $\varphi(e) = x$ ; we argue that  $\varphi: G \rightarrow E$  makes  $E$  into a quotient of  $G$  by the subgroup

$$H = \text{Fix}(x) = \{g \in G: gx = x\}.$$

Indeed by using the definition of  $H$  and the fact that  $\varphi$  is  $G$ -invariant we have

$$(\varphi \circ h)(e) = \varphi(h(e)) = h(\varphi(e)) = h(x) = x.$$

for all  $h \in H$ . Moreover let  $g: G \rightarrow F$  satisfy (1) of Definition 1.6; as we discussed above  $g$  is entirely determined by the image of  $e$  so we obtain (2) defining an arrow  $f: E \rightarrow F$  by  $f(x) = g(e)$ . The situation is depicted in Figure 1.3.

Finally:  $G$  is a transitive  $G$ -set and so is every  $G/H$  with  $H \leq G$  and the action defined again by left multiplication (on cosets) so an object  $E \in \mathbf{GSet}$  is transitive if and only if it is isomorphic to some  $G/H$ .

$$\begin{array}{ccc}
h \hookrightarrow G & \xrightarrow{\varphi} & E \cong G/H \\
& \searrow \scriptstyle g & \downarrow \scriptstyle f \\
& & F
\end{array}$$

Figure 1.3

For the rest of the section fix a category  $\mathcal{C}$ , an object  $A \in \mathcal{C}$  and let  $G = \text{Aut}(A)$ .

**Remark 1.12.** Consider a subgroup  $H \leq G$  and an object  $X \in \mathcal{C}$ .  $H$  acts on the hom-set  $[A, X]$  as follows<sup>2</sup>:

$$\begin{aligned}
H \times [A, X] &\longrightarrow [A, X] \\
(h, x) &\longmapsto h \cdot x = x \circ h.
\end{aligned}$$

**Remark 1.13.** Assume that the action  $G \times [A, X] \rightarrow [A, X]$  is transitive and let  $\mathbf{GSet}^t$  be the category of transitive  $G$ -sets (a subcategory of  $\mathbf{Gset}$ ). Then we have a functor

$$\begin{array}{ccc}
[A, -]_G: \mathcal{C} & \longrightarrow & \mathbf{GSet}^t \\
X & & [A, X]_G \\
\downarrow f & \longmapsto & \downarrow f_* \\
Y & & [A, Y]_G
\end{array}$$

where we indicate with  $[A, X]_G$  the hom-set  $[A, X]$  upon which  $G$  acts as described in Proposition 1.12 and  $f_*$  is post-composition with  $f$ . It is easy to check that  $f_*$  is indeed  $G$ -invariant.

**Remark 1.14.** Consider an object  $E \in \mathbf{GSet}^t$ , pick an element  $x_0 \in E$  and let  $H = \text{Fix}(x_0) \leq G$  (the choice of  $x_0$  is irrelevant as  $E$  is transitive). Moreover assume that  $\mathcal{C}$  has quotients of  $A$  by any subgroup of  $G$ .

By what we observed in Remark 1.11 we have a bijection between elements of  $E$  and arrows of type  $G \rightarrow E$ . Consider then  $f \in [A, X]$  and its corresponding arrow  $\varphi: G \rightarrow [A, X]_G$ ; we claim that  $f$  factors through  $A/H$  if and only if  $\varphi$  factors through  $E \cong G/H$  (see Figure 1.4). Indeed  $f$  factors if and only if  $f \circ h = f$  for all  $h \in H$ , by using the fact that  $\varphi$  is  $G$ -invariant we obtain

$$\varphi(h(e)) = h(\varphi(e)) = h(f) = f \circ h = f$$

and, since  $\varphi$  is uniquely determined by  $\varphi(e)$ ,  $\varphi \circ h = \varphi$  for all  $h \in H$ ; this happens if and only if  $\varphi$  factors through  $E \cong G/H$ .

<sup>1</sup>This states that the set of orbits of  $E$  is a singleton that is of course the case if and only if  $E$  is transitive.

<sup>2</sup>Since an action as of Definition 1.4 is a map that sends elements of a group to arrows it is, in this case, equivalent to give the definition of an action by uncurrying.

This gives us, for each  $X \in \mathcal{C}$  and  $E \in \mathbf{GSet}^t$ , a bijection

$$[A/H, X] \cong \mathbf{GSet}^t(E, [A, X]_G) \quad (*)$$

natural in  $X$  in  $E$  (see Section 3).

Provided that  $\mathcal{C}$  has quotients of  $A$  by subgroups of  $G$  we can define a functor  $A \times_G -$  from  $\mathbf{GSet}^t$  to  $\mathcal{C}$  by setting  $A \times_G E = A/H$  for  $H = \text{Fix}(x_0)$  and  $x_0 \in E$  as above. The behaviour of  $A \times_G -$  on arrows is defined as follows. Let  $f: E \cong G/H \rightarrow F \cong G/H'$  be an arrow in  $\mathbf{GSet}^t$  with  $H = \text{Fix}(x_0)$ ,  $H' = \text{Fix}(f(x_0))$  and  $x_0 \in E$ . Notice that for all  $h \in H$  we have  $f(h \cdot x_0) = f(x_0)$  by definition of  $H$  and  $f(h \cdot x_0) = h \cdot f(x_0)$  by  $G$ -invariance of  $f$ ; thus  $H \leq H'$ . This means that if  $q: A \rightarrow A/H$  and  $q': A \rightarrow A/H'$  are the quotients in  $\mathcal{C}$  there is a unique arrow  $\tilde{f}: A/H \rightarrow A/H'$  such that  $q' = \tilde{f} \circ q$ ; we set  $A \times_G f = \tilde{f}$ .

Finally the naturality of the bijection  $(*)$  now indicates that we have an adjunction

$$A \times_G - \dashv [A, -]_G.$$

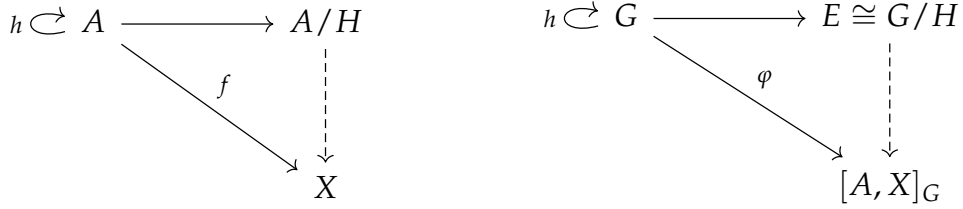


Figure 1.4

Our main problem will be that of finding conditions on  $\mathcal{C}$  that make this adjunction into an equivalence of categories.

## 2 Categorical axiomatization of Galois Theory

Through this section fix a category  $\mathcal{C}$  and an object  $A \in \mathcal{C}$ .

**Definition 2.1.** We define the following axioms.

- (1) For every  $X \in \mathcal{C}$  there is at least a map of type  $A \rightarrow X$  and all maps  $A \rightarrow X$  are strict epimorphisms.
- (2) For any subgroup  $H \leq \text{Aut}(A)$  the quotient  $q: A \rightarrow A/H$  exists and is preserved by  $[A, -]: \mathcal{C} \rightarrow \mathbf{GSet}$ .
- (3) Every endomorphism of  $A$  is an isomorphism i.e.  $[A, A] = \text{Aut}(A)$ .

**Remark 2.2.** It is known that if  $f \circ g$  is a strict epimorphism then so is  $f$ . Thus it follows from Axiom (1) that every arrow  $X \rightarrow Y$  in  $\mathcal{C}$  is a strict epimorphism.

**Proposition 2.3.** Axiom (1) implies that  $[A, -]$  is faithful, reflects monomorphisms and isomorphisms.

$$\begin{array}{ccccc}
& & q_* & & \\
& \nearrow & & \searrow & \\
[A, A] & \xrightarrow{\rho} & [A, A]/H & \xrightarrow{\eta} & [A, A/H]
\end{array}$$

Figure 2.5

*Proof.* Consider arrows  $f, g: X \rightarrow Y \in \mathcal{C}$  such that  $[A, f] = [A, g]$ ; that is  $f_* = g_*$ . By Axiom (1) let  $h: A \rightarrow X$  be a third arrow of  $\mathcal{C}$  then we have  $f_*(h) = g_*(h)$  i.e.  $f \circ h = g \circ h$ . Again by Axiom (1)  $h$  is an epimorphism (really, a strong one) and thus we obtain  $f = g$ ; that is:  $[A, -]$  is faithful.

It is well known that every faithful functor reflects monomorphisms. Because of this if  $f \in \mathbf{GSet}$  is an isomorphism and  $g \in \mathcal{C}$  is such that  $[A, g] = f$  then  $g$  is a monomorphism too; but, as an arrow of  $\mathcal{C}$ ,  $g$  is also a strict epimorphism and thus an isomorphism.  $\square$

**Remark 2.4.** Consider  $H \leq G$  and the quotient  $q: A \rightarrow A/H$  in  $\mathcal{C}$ , then  $q_*: [A, A] \rightarrow [A, A/H]$  in  $\mathbf{GSet}$  is a quotient too because quotients are preserved by  $[A, -]$  (Axiom (2)). Thus we have  $[A, A]/H \cong [A, A/H]$ , and the following diagram commutes (where  $\rho$  is a quotient arrow and  $\eta$  the isomorphism).

The fact that  $\eta$  is a bijection gives us the following:

- (i) for  $f, g \in [A, A]$  if  $q \circ f = q \circ g$  then there is some  $h \in H$  such that  $f = h \circ g$ ,
- (ii) for all  $x: A \rightarrow A/H$  there is an arrow  $f \in [A, A]$  such that  $q \circ f = x$ .

Moreover, under Axiom (3), (i) implies the following:

- (iii)  $q \circ f = q$  implies  $f \in H$ .

Indeed by taking  $g = 1_A$  in (i) we obtain that  $f = h$  for some  $h \in H$ .

**Remark 2.5.** Consider an arrow  $x: A \rightarrow X$  and the epi-mono factorization  $x_* = \psi \circ \rho^3$ . With reference to the diagram below Axiom (iii) implies that  $I = [A, A]/H$  with  $H = \text{Fix}(x) \leq G$ .

$$\begin{array}{ccccc}
& & x_* & & \\
& \nearrow & & \searrow & \\
[A, A] & \xrightarrow{\rho} & I & \xrightarrow{\psi} & [A, X]
\end{array}$$

Indeed by Axiom (iii)  $[A, A] = \text{Aut}(A) = G$  so an  $h \in H$  acts on  $[A, A]$  by left multiplication. Given  $f \in [A, A]$  we have

$$(\psi \circ \rho \circ h)(f) = x_*(h \circ f) = x \circ h \circ f = x \circ f = x_*(f) = (\psi \circ \rho)(f)$$

that is  $\psi \circ \rho \circ h = \psi \circ \rho$  which, by monicness of  $\psi$ , implies  $\rho \circ h = \rho$ . Moreover as  $I$  is the image of  $x_*$  it is unique up to isomorphism and thus  $\rho$  is really a quotient arrow;  $I = [A, A]/H$ .

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<sup>3</sup>We recall that  $\mathbf{GSet}$  is a topos and, as such, has epi-mono factorization.

$$\begin{array}{ccccc}
 & & x & & \\
 & \nearrow & & \searrow & \\
 A & \xrightarrow{q} & A/H & \xrightarrow{\varepsilon} & X .
 \end{array}$$

Figure 2.6

$$\begin{array}{ccccc}
 & & x_* & & \\
 & \nearrow & & \searrow & \\
 [A, A] & \xrightarrow{q_*} & [A, A/H] & \xrightarrow{\varepsilon_*} & [A, X] \\
 & \searrow \rho & \uparrow \eta & \nearrow \psi & \\
 & & [A, A]/H & & 
 \end{array}$$

Figure 2.7

**Proposition 2.6.** Any arrow  $x: A \rightarrow X$  of  $\mathcal{C}$  is a quotient of  $A$  by  $H = \text{Fix}(x) \leq G$  (with respect to the action on  $[A, X]$  described in Proposition 1.12) i.e.  $X = A/H$ .

*Proof.* By choosing  $H = \text{Fix}(x)$  we get  $x \circ h = x$  for all  $h \in H$  and thus there is a unique arrow  $\varepsilon: A/H \rightarrow X$  of  $\mathcal{C}$  such that  $x = \varepsilon \circ q$ , where  $q: A \rightarrow A/H$  is the quotient of  $A$  by  $H$ . Graphically:

Now by applying  $[A, -]$  to the diagram above we obtain where  $\rho, \psi$  and  $\eta$  are as discussed before. We now have

$$\varepsilon_* \circ \eta \circ \rho = \varepsilon_* \circ q_* = x_* = \psi \circ \rho$$

that by epicness of  $\rho$  implies  $\varepsilon_* \circ \eta = \psi$ . Now  $\varepsilon_*$  must be monic since  $\psi$  and  $\eta$  are; but this, by Proposition 2.3, implies that  $\varepsilon \in \mathcal{C}$  is monic too. Being an arrow of  $\mathcal{C}$ , by Axiom (i),  $\varepsilon$  is also a strict epimorphism and thus an isomorphism.  $\square$

**Proposition 2.7.** The action of  $\text{Aut}(A)$  on  $[A, X]$  is transitive for all  $X \in \mathcal{C}$ .

*Proof.* Consider again Figure 2.7. We proved that  $\varepsilon$  is an isomorphism and thus  $\varepsilon_*$  must be one too. Moreover from Remark 2.4 we know  $\eta$  is iso too and so we have  $[A, X] \cong [A, A]/H$ , but by Axiom (3)  $[A, A] = \text{Aut}(A)$  and so we have that  $[A, X]$  is transitive.  $\square$

**Theorem 2.8.** Given a category  $\mathcal{C}$  and an object  $A \in \mathcal{C}$  such that Axioms (1), (2) and (3) hold there exists an adjunction

$$A \times_G - \dashv [A, -]_G$$

where  $G = \text{Aut}(A)$  such that the maps

$$\eta: E \cong [A, A]/H \rightarrow [A, A/H]$$

$$\varepsilon: A/H \rightarrow X$$

are isomorphisms. This enstablishes an equivalence of categories between  $\mathcal{C}$  and  $\mathbf{GSet}^t$ .

*Proof.* The existence of the adjunction follows from the discussion in Section 1, the fact that  $\eta$  and  $\varepsilon$  are (the components of) the unit and counit of the adjunction is a calculation that has been moved to the appendix and the fact that they are isomorphisms follows respectively from Remark 2.4 and Proposition 2.6.  $\square$

### 3 Appendixes

#### 3.1 Naturality of $[A/H, X] \cong \mathbf{GSet}^t(E, [A, X]_G)$

Fix a category  $\mathcal{C}$ , an element  $A \in \mathcal{C}$  and let  $G = \text{Aut}(A)$ . Let's indicate with  $\psi_{EX}$  the bijection between  $[A/H, X]$  and  $\mathbf{GSet}^t(E, [A, X]_G)$  described in Remark 1.14; we shall prove that it is natural in both  $X \in \mathcal{C}$  and  $E \in \mathbf{GSet}^t$ .

*Naturality in X.* Given an arrow  $f: X \rightarrow Y$  of  $\mathcal{C}$  we want to prove that the  $\psi_{EX}$  are the components of a natural transformation  $[A/H, -] \Rightarrow \mathbf{GSet}^t(E, [A, -]_G)$  i.e that the following diagram commutes.

$$\begin{array}{ccc} [A/H, X] & \xrightarrow{\psi_{EX}} & \mathbf{GSet}^t(E, [A, X]_G) \\ \downarrow f_* & & \downarrow (f_*)_* \\ [A/H, Y] & \xrightarrow{\psi_{EY}} & \mathbf{GSet}^t(E, [A, Y]_G) \end{array}$$

Recall that here  $H = \text{Fix}(x_0)$  for  $x_0 \in E$ . Pick any  $x \in [A/H, X]$  and let  $q: A \rightarrow A/H$  be the quotient arrow (the quotient exists because  $\mathcal{C}$  is assumed to have all quotients by subgroups of  $G$ ); then chasing  $x$  through the diagram down the two possible ways yields two arrows in  $\mathbf{GSet}^t$  of type  $E \rightarrow [A, Y]_G$ . Since  $E$  is transitive arrows out of  $E$  are determined uniquely by the image of  $x_0$ ; keeping this in mind the following computations show that the square commutes.

$$\begin{aligned} ((f_*)_* \circ \psi_{EX})(x)(x_0) &= (f_* \circ \psi_{EX}(x))(x_0) = f_*(\psi_{EX}(x)(x_0)) = f_*(x \circ q) = f \circ x \circ q \\ (\psi_{EY} \circ f_*)(x)(x_0) &= \psi_{EY}(f \circ x)(x_0) = f \circ x \circ q \end{aligned}$$

$\square$

*Naturality in E.* We want to prove that the  $\psi_{EX}$  are the components of a natural transformation  $[A/-, X] \Rightarrow \mathbf{GSet}^t(-, [A, X]_G)$ . Given  $f: E \cong G/H \rightarrow F \cong G/H'$  with  $H = \text{Fix}(x_0), H' = \text{Fix}(f(x_0)), x_0 \in E$ , using the notation  $\tilde{f} = A \times_G f$  (see Remark 1.14), we show the naturality of the following square.

$$\begin{array}{ccc} [A/H, X] & \xrightarrow{\psi_{EX}} & \mathbf{GSet}^t(E, [A, X]_G) \\ \uparrow \tilde{f}^* & & \uparrow f^* \\ [A/H', X] & \xrightarrow{\psi_{FX}} & \mathbf{GSet}^t(F, [A, X]_G) \end{array}$$

As before we chase an  $x \in [A/H', X]$  down the two possible paths to get two arrows of type  $E \rightarrow [A, X]_G$  and prove that they are the same by evaluating them on  $x_0 \in E$ :

$$\begin{aligned} (\psi_{EX} \circ \tilde{f}^*)(x)(x_0) &= \psi_{EX}(x \circ \tilde{f})(x_0) = x \circ \tilde{f} \circ q = x \circ q', \\ (f^* \circ \psi_{FX})(x)(x_0) &= (\psi_{FX}(x) \circ f)(x_0) = \psi_{FX}(x)(f(x_0)) = x \circ q'. \end{aligned}$$

□

### 3.2 $\eta$ and $\varepsilon$ are the unit and counit of $A \times_G - \dashv [A, -]_G$

$\eta$  is the unit. Our adjunction is given by the bijection

$$\psi: \mathcal{C}(A \times_G E, X) \cong \mathbf{GSet}^t(E, [A, X]_G)$$

natural in  $X \in \mathcal{C}$  and  $E \in \mathbf{GSet}^t$ . To obtain the unit we set  $X = A \times_G E$ :

$$\mathcal{C}(A \times_G E, A \times_G E) \cong \mathbf{GSet}^t(E, [A, A \times_G E]_G)$$

and so

$$\mathcal{C}(A/H, A/H) \cong \mathbf{GSet}^t(E, [A, A/H]_G)$$

where  $H = \text{Fix}(x) \leq G$  ( $x \in E$ ) by definition of  $A \times_G E$ . Now (the component at  $E$  of) the unit is given by the image of  $1_{A/H}$  under this bijection. By the discussion in Section 1 if  $q: A \rightarrow A/H$  is the quotient arrow in  $\mathcal{C}$  then  $\psi(1_{A/H})$  is the map  $E \cong [A, A]_G/H \rightarrow [A, A/H]_G$  of  $\mathbf{GSet}^t$  that factors the map  $G \rightarrow E$  that sends  $e$  to  $q$ . But this last map is  $q^*$  as in Figure 2.5. □

$\varepsilon$  is the counit. Keeping the proof above in mind we set  $E = [A, X]$  and obtain

$$\psi: \mathcal{C}(A \times_G [A, X], X) \cong \mathbf{GSet}^t([A, X]_G, [A, X]_G)$$

that becomes

$$\mathcal{C}(A/H, X) \cong \mathbf{GSet}^t([A, X]_G, [A, X]_G)$$

with  $H = \text{Fix}(x)$  for some  $x \in [A, X]$ . Notice that since  $[A, X] \cong G/H$  by Proposition 2.7 we can consider  $1_{[A, X]}$  as a map of type  $G/H \rightarrow [A, X]$  such that  $1_{[A, X]}(e) = x$  (this can be obtained by chasing  $1_A$  around Figure 2.7 keeping in mind that  $\varepsilon_*$  there is iso). Now we have that  $\psi^{-1}(1_{[A, X]})$  is the arrow  $A/H \rightarrow X$  of  $\mathcal{C}$  that factorizes  $x$  i.e.  $\varphi^{-1}(1_{[A, X]}) = \varepsilon$  as in Figure 2.6. □