

Grothendieck's Galois Theory

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1 Section 1

Definition 1.1. In a category \mathcal{C} an arrow $f: X \rightarrow Y$ is a **strict epimorphism** if it is the joint coequalizer of all the arrows it coequalizes. This means that any arrow $g: X \rightarrow Z$ such that $g \circ x = g \circ y$ for any $x, y: C \rightarrow X$ such that $f \circ x = f \circ y$ there exists a unique arrow $h: Y \rightarrow Z$ such that $h \circ f = g$. Refer to Figure 1.1.

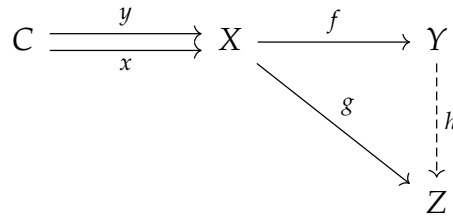


Figure 1.1

Remark 1.2. Strict epimorphisms are coequalizers, thus epimorphisms (as the name implies).

Remark 1.3. If an arrow is both a strict epimorphism and a monomorphism then it is an epimorphism.

Definition 1.4. Let H be a group, A an object of \mathcal{C} and $G = \text{Aut}(A)$ the group of automorphisms of A in \mathcal{C} i.e. the group whose underlying set is the set of isomorphisms of type $A \rightarrow A$ of \mathcal{C} and whose operation is composition in \mathcal{C} . An **action** of H on A is a group homomorphism $H \rightarrow G$.

Notation 1.5. Given an action of a group H on an object A of \mathcal{C} we denote, with a slight abuse of notation, the automorphism of A associated to $h \in H$ by the same symbol h .

Definition 1.6. If H acts on A as defined in 1.4 we define the quotient of A by H in \mathcal{C} to be an element A/H of \mathcal{C} equipped with an arrow $q: A \rightarrow A/H$ such that:

- (1) for all $h \in H$ $q \circ h = q$ holds,
- (2) for any $x: A \rightarrow X$ such that $x \circ h = x$ for all $h \in H$ there exists a unique arrow $\varphi: A/H \rightarrow X$ such that $x = \varphi \circ q$.

See also Figure 1.2.

Remark 1.7. Quotients are defined by a universal property, thus are unique up to unique isomorphism and we can speak of “the” quotient of A by H instead of “a” quotient of A by H .

Notation 1.8. Sometimes we use the sentence “the quotient of A by H ” to refer to the object A/H , some others to the arrow $q: A \rightarrow A/H$; the context should be enough to differentiate between the two cases.

$$\begin{array}{ccc}
 h \hookrightarrow A & \xrightarrow{q} & A/H \\
 & \searrow x & \downarrow \varphi \\
 & & X
 \end{array}$$

Figure 1.2

Remark 1.9. Consider a quotient $q: A \rightarrow A/H$; by condition (1) above $q \circ h = q = q \circ 1_A$ so q coequalizes all the pairs $(h, 1_A)$, for $h \in H$. If another arrow $x: A \rightarrow X$ coequalizes all the pairs that q does then this arrow is such that $x \circ h = x \circ 1_A = x$ for all $h \in H$ and thus, by condition (2), we have a unique factorization $x = \varphi \circ q$. This proves that all quotients are strict epimorphisms.

Remark 1.10. Let G be a group, **GSet** the category of G -sets and G -invariant maps and A an object of **GSet**. In this category Definition 1.6 yields the familiar notion of the set of all orbits of an action: A/G is the set of orbits of A .