

# Grothendieck's Galois Theory

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## 1 Section 1

**Definition 1.1.** In a category  $\mathcal{C}$  an arrow  $f: X \rightarrow Y$  is a **strict epimorphism** if it is the joint coequalizer of all the arrows it coequalizes. This means that any arrow  $g: X \rightarrow Z$  such that  $g \circ x = g \circ y$  for any  $x, y: C \rightarrow X$  such that  $f \circ x = f \circ y$  there exists a unique arrow  $h: Y \rightarrow Z$  such that  $h \circ f = g$ . Refer to Figure 1.1.

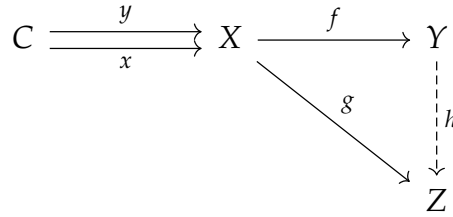


Figure 1.1

**Remark 1.2.** Strict epimorphisms are coequalizers, thus epimorphisms (as the name implies).

**Remark 1.3.** If an arrow is both a strict epimorphism and a monomorphism then it is an epimorphism.

**Definition 1.4.** Let  $H$  be a group,  $A$  an object of  $\mathcal{C}$  and  $G = \text{Aut}(A)$  the group of automorphisms of  $A$  in  $\mathcal{C}$  i.e. the group whose underlying set is the set of isomorphisms of type  $A \rightarrow A$  of  $\mathcal{C}$  and whose operation is composition in  $\mathcal{C}$ . An **action** of  $H$  on  $A$  is a group homomorphism  $H \rightarrow G$ .

**Notation 1.5.** Given an action of a group  $H$  on an object  $A$  of  $\mathcal{C}$  we denote, with a slight abuse of notation, the automorphism of  $A$  associated to  $h \in H$  by the same symbol  $h$ .

**Definition 1.6.** If  $H$  acts on  $A$  as defined in 1.4 we define the quotient of  $A$  by  $H$  in  $\mathcal{C}$  to be an element  $A/H$  of  $\mathcal{C}$  equipped with an arrow  $q: A \rightarrow A/H$  such that:

- (1) for all  $h \in H$   $q \circ h = q$  holds,
- (2) for any  $x: A \rightarrow X$  such that  $x \circ h = x$  for all  $h \in H$  there exists a unique arrow  $\varphi: A/H \rightarrow X$  such that  $x = \varphi \circ q$ .

See also Figure 1.2.

**Remark 1.7.** Quotients are defined by a universal property, thus are unique up to unique isomorphism and we can speak of “the” quotient of  $A$  by  $H$  instead of “a” quotient of  $A$  by  $H$ .

**Notation 1.8.** Sometimes we use the sentence “the quotient of  $A$  by  $H$ ” to refer to the object  $A/H$ , some others to the arrow  $q: A \rightarrow A/H$ ; the context should be enough to differentiate between the two cases.

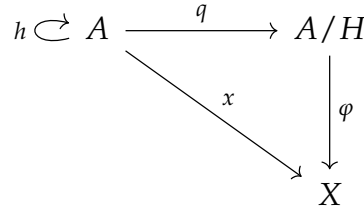


Figure 1.2

**Remark 1.9.** Consider a quotient  $q: A \rightarrow A/H$ ; by condition (1) above  $q \circ h = q = q \circ 1_A$  so  $q$  coequalizes all the pairs  $(h, 1_A)$ , for  $h \in H$ . If another arrow  $x: A \rightarrow X$  coequalizes all the pairs that  $q$  does then this arrow is such that  $x \circ h = x \circ 1_A = x$  for all  $h \in H$  and thus, by condition (2), we have a unique factorization  $x = \varphi \circ q$ . This proves that all quotients are strict epimorphisms.

**Remark 1.10.** Let  $G$  be a group, **GSet** the category of  $G$ -sets and  $G$ -invariant maps and  $A$  an object of **GSet**. In this category Definition 1.6 yields the familiar notion of the set of all orbits of an action:  $A/G$  is the set of orbits of  $A$ .

**Remark 1.11.** Consider again **GSet**. The underlying set of  $G$  (that we also denote as  $G$ ) is a  $G$ -set with the action given by left multiplication in  $G$ ; we call this the **canonical action** of  $G$  on itself. Let  $\varphi: G \rightarrow E$  be a  $G$ -invariant map; it is easy to see that such a  $\varphi$ , by virtue of being  $G$ -invariant, is determined uniquely by the value  $\varphi(e)$ , where  $e$  is the neutral element of  $G$ .

Let now  $E$  be a transitive  $G$ -set i.e. a set upon which the action of  $G$  is transitive i.e. such that  $E/G = \{*\}$ . Fix an  $x \in E$  and let  $\varphi_x$  be the  $G$ -invariant map defined by  $\varphi_x(e) = x$ ; we argue that  $\varphi_x: G \rightarrow E$  makes  $E$  into a quotient of  $G$  by the subgroup

$$H = \text{Fix}(x) = \{g \in G: gx = x\}.$$

Indeed by using the definition of  $H$  and the fact that  $\varphi$  is  $G$ -invariant we have

$$(\varphi \circ h)(e) = \varphi(h(e)) = h(\varphi(e)) = h(x) = x.$$

for all  $h \in H$ . Moreover let  $g: G \rightarrow F$  satisfy (1) of Definition 1.6; as we discussed above  $g$  is entirely determined by the image of  $e$  so we obtain (2) defining an arrow  $f: E \rightarrow F$  by  $f(x) = g(e)$ . The situation is depicted in Figure 1.3.

Trivially  $G$  is a transitive  $G$ -set and for any  $g \in G$   $G/\text{Fix}(g)$  is transitive as well so we have that an object  $E \in \mathbf{GSet}$  is transitive if and only if it is isomorphic to  $G/H$  where  $H = \text{Fix}(x)$  for (any)  $x \in E$ .

$$\begin{array}{ccc}
 {}_h\hookrightarrow G & \xrightarrow{\varphi} & E \cong G/H \\
 & \searrow g & \downarrow f \\
 & & F
 \end{array}$$

Figure 1.3