Grothendieck's Galois Theory

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1 Group actions and adjoints

Definition 1.1. In a category $\mathscr{C}$ an arrow $f\colon X \to Y$ is a **strict epimorphism** if it is the joint coequalizer of all the arrows it coequalizes. This means that any arrow $g\colon X \to Z$ such that $g \circ x = g \circ y$ for any $x, y\colon C \to X$ such that $f \circ x = f \circ y$ there exists a unique arrow $h\colon Y \to Z$ such that $h \circ f = g$. Refer to Figure 1.1.

$$\begin{tikzcd}
C \arrow[r, shift left, "y"] \arrow[r, shift right, "x"'] & X \arrow[r, "f"] \arrow[dr, "g"] & Y \arrow[d, dashed, "h"] \\
& & Z
\end{tikzcd}$$

Figure 1.1

Remark 1.2. Strict epimorphisms are coequalizers, thus epimorphisms (as the name implies).

Remark 1.3. If an arrow is both a stric epimorphism and a monomorphism then it is an epimorphism.

Definition 1.4. Let H be a group, A an object of $\mathscr{C}$ and $G = \mathrm{Aut}(A)$ the group of automorphisms of A in $\mathscr{C}$ i.e. the group whose underlying set is the set of isomorphisms of type $A \to A$ of $\mathscr{C}$ and whose operation is composition in $\mathscr{C}$. An **action** of H on A is a group homomorphism $H \to G$.

Notation 1.5. Given an action of a group H on an object A of $\mathscr{C}$ we denote, with a slight abuse of notation, the automorphism of A associated to $h \in H$ by the same symbol h.

Definition 1.6. If H acts on A as defined in 1.4 we define the quotient of A by H in $\mathscr{C}$ to be an element A/H of $\mathscr{C}$ equipped with an arrow $q\colon A \to A/H$ such that:

(1) for all $h \in H$ $q \circ h = q$ holds,

(2) for any $x\colon A \to X$ such that $x \circ h = x$ for all $h \in H$ there exists a unique arrow $\varphi\colon A/H \to X$ such that $x = \varphi \circ q$.

See also Figure 1.2.

Remark 1.7. Quotients are defined by a universal property, thus are unique up to unique isomorphism and we can speak of "the" quotient of A by H instead of "a" quotient of A by H.

Notation 1.8. Sometimes we use the sentence "the quotient of A by H" to refer to the object A/H, some others to the arrow $q\colon A \to A/H$; the context should be enough to differentiate between the two cases.

$$\begin{tikzcd} A \arrow[loop left, "h"] \arrow[r, "q"] \arrow[dr, "x"] & A/H \arrow[d, "\varphi"] \\ & \mathrm{X} \end{tikzcd}$$

Figure 1.2

Remark 1.9. Consider a quotient $q\colon A \to A/H$; by condition (1) above $q \circ h = q = q \circ 1_A$ so q coequalizes all the pairs $(h, 1_A)$, for $h \in H$. If another arrow $x\colon A \to X$ coequalizes all the pairs that q does then this arrow is such that $x \circ h = x \circ 1_A = x$ for all $h \in H$ and thus, by condition (2), we have a unique factorization $x = \varphi \circ q$. This proves that all quotients are strict epimorphisms.

Remark 1.10. Let G be a group, **GSet** the category of G-sets and G-invariant maps and A an object of **GSet**. In this category Definition 1.6 yelds the familiar notion of the set of all orbits of an action: A/G is the set of orbits of A.

Remark 1.11. Consider again **GSet**. The underlying set of G (that we also denote as G) is a G-set with the action given by left multiplication in G; we call this the **canonical action** of G on itself. Let $\varphi\colon G \to E$ be a G-invariant map; it is easy to see that such a φ, by virtue of being G-invariant, is determined uniquely by the value $\varphi(e)$, where e is the neutral element of G.

Let now E be a transitive G-set i.e. a set upon which the action of G is transitive i.e. such that $E/G = \{*\}$. Fix an $x \in E$ and let φ_x be the G-invarian map defined by $\varphi_x(e) = x$; we argue that $\varphi_x\colon G \to E$ makes E into a quotient of G by the subgroup

$$H = \mathrm{Fix}(x) = \{g \in G\colon gx = x\}.$$

Indeed by using the definition of H and the fact that φ is G-invariant we have

$$(\varphi \circ h)(e) = \varphi(h(e)) = h(\varphi(e)) = h(x) = x.$$

for all $h \in H$. Moreover let $g\colon G \to F$ satisfy (1) of Definition 1.6; as we discussed above g is entirely determined by the image of e so we obtain (2) defining an arrow $f\colon E \to F$ by $f(x) = g(e)$. The situation is depicted in Figure 1.3.

Trivially G is a transitive G-set and for any $g \in G$ $G/\mathrm{Fix}(g)$ is transitive as well so we have that an object $E \in$ **GSet** is transitive if and only if it is isomorphic to G/H where $H = \mathrm{Fix}(x)$ for (any) $x \in E$.

$$
\begin{tikzcd}
h \circlearrowright G \arrow[r, "\varphi"] \arrow[dr, "g"'] & E \cong G/H \arrow[d, dashed, "f"] \\
& F
\end{tikzcd}
$$

Figure 1.3

For the rest of the section fix a category $\mathscr{C}$, an object $A \in \mathscr{C}$ and let $G = \mathrm{Aut}(A)$.

Proposition 1.12. Consider a subgroup $H \leq G$ and an object $X \in \mathscr{C}$. H acts on the hom-set $[A, X]$ as follows[1]:

$$
\begin{aligned}
H \times [A, X] &\longrightarrow [A, X] \\
(h, x) &\longmapsto h \cdot x = x \circ h.
\end{aligned}
$$

Remark 1.13. Assume that the action $G \times [A, X] \to [A, X]$ is transitive and let $\mathbf{GSet}^t$ be the category of transitive G-sets (a subcategory of *Gset*). Then we have a functor

$$
\begin{tikzcd}
{[A,-]_G \colon \mathscr{C}} \arrow[r] & \mathbf{GSet}^t \\
X \arrow[d, "f"] \arrow[r, mapsto] & {[A,X]_G} \arrow[d, "f_*"] \\
Y & {[A,Y]_G}
\end{tikzcd}
$$

where we indicate with $[A, X]_G$ the hom-set $[A, X]$ upon which G acts as described in Proposition 1.12 and f_* is post-composition with f. It is easy to check that f_* is indeed G-invariant.

Remark 1.14. Consider an object $E \in \mathbf{GSet}^t$, pick an element $x_0 \in E$ and let $H = \mathrm{Fix}(x_0) \leq G$ (the choice of x_0 is irrelevant as E is transitive). Moreover assume that $\mathscr{C}$ has quotients of A by any subgroup of G.

By what we observed in Remark 1.11 we have a bijection between elements of E and arrows of type $G \to E$. Consider then $f \in [A, X]$ and its corresponding arrow $\varphi \colon G \to [A, X]_G$; we claim that f factors through A/H if and only if φ factors through $E \cong G/H$ (see Figure 1.4). Indeed f factors if and only if $f \circ h = f$ for all $h \in H$, by using the fact that φ is G-invariant we obtain

$$\varphi(h(e)) = h(\varphi(e)) = h(f) = f \circ h = f$$

and, since φ is uniquely determined by $\varphi(e)$, $\varphi \circ h = \varphi$ for all $h \in H$; this happens if and only if φ factors through $E \cong G/H$.

[1] Since an action as of Definition 1.4 is a map that sends elements of a group to arrows it is, in this case, equivalent to give the definition of an action by uncurrying.

This gives us, for each $X \in \mathscr{C}$ and $E \in \mathbf{GSet}^t$, a bijection

$$[A/H, X] \cong \mathbf{GSet}^t(E, [A, X]_G)$$

natural in X in E (see Section 3). Thus, if $\mathscr{C}$ has quotients of A by subgroups of G we have an adjunction $A \times_G - \dashv [A, -]_G$ where $A \times_G E = A/H$ for $H = \mathrm{Fix}(x_0)$ and $x_0 \in E$ as above[2].

$$\begin{tikzcd}
h \circlearrowright A \arrow[r] \arrow[dr, "f"] & A/H \arrow[d, dashed] \\
& X
\end{tikzcd}
\qquad
\begin{tikzcd}
h \circlearrowright G \arrow[r] \arrow[dr, "\varphi"] & E \cong G/H \arrow[d, dashed] \\
& [A, X]_G
\end{tikzcd}$$

Figure 1.4

Our main problem will be that of finding conditions on $\mathscr{C}$ that make this adjunction ineto an equivalence of categories.

2 Categorial axiomatization of Galois Theory

Through this section fix a category $\mathscr{C}$ and an object $A \in \mathscr{C}$.

Definition 2.1. We define the following axioms.

(1) For every $X \in \mathscr{C}$ there is at least a map of type $A \to X$ and all maps $A \to X$ are strict epimorphisms.

(2) For any subgroup $H \leq \mathrm{Aut}(A)$ the quotient $q\colon A \to A/H$ exists and is preserved by $[A, -]\colon \mathscr{C} \to \mathbf{GSet}$.

(3) Every endomorphism of A is an isomorphism i.e. $[A, A] = \mathrm{Aut}(A)$.

Remark 2.2. It is known that if $f \circ g$ is a strict epimorphism then so is f. Thus it follows from Axiom (1) that every arrow $X \to Y$ in $\mathscr{C}$ is a strict epimorphism.

Proposition 2.3. Axiom (1) implies that $[A, -]$ is faithful, reflects monomorphisms and isomorphisms.

Proof. Consider arrows $f, g\colon X \to Y \in \mathscr{C}$ such that $[A, f] = [A, g]$; that is $f_* = g_*$. By Axiom (1) let $h\colon A \to X$ be a third arrow of $\mathscr{C}$ then we have $f_*(h) = g_*(h)$ i.e. $f \circ h = g \circ h$. Again by Axiom (1) h is an epimorphism (really, a strong one) and thus we obtain $f = g$; that is: $[A, -]$ is faithful.

It is well known that every faithful functor reflects monomorphisms. Because of this if $f \in \mathbf{GSet}$ is an isomorphism and $g \in \mathscr{C}$ is such that $[A, g] = f$ then g is a monomorphism too; but, as an arrow of $\mathscr{C}$, g is also a strict epimorphism and thus an isomorphism. $\square$

[2]We use the notation $A \times_G -$ for the left adjoint in "honor" of the tensor-hom adjunction.

$$\begin{tikzcd}
{[A,A]} \arrow[r, "\rho"] \arrow[rr, bend left, "q_*"] & {[A,A]/H} \arrow[r, "\eta"] & {[A,A/H]}
\end{tikzcd}$$

Figure 2.5

Remark 2.4. Consider $H \leq G$ and the quotient $q\colon A \to A/H$ in $\mathscr{C}$, then $q_*\colon [A,A] \to [A,A/H]$ in **GSet** is a quotient too because quotients are preserved by $[A,-]$ (Axiom (2)). Thus we have $[A,A]/H \cong [A,A/H]$, and the following diagram commutes (where ρ is a quotient arrow and η the isomorphism).

The fact that η is a bijection gives us the following:

(i) for $f,g \in [A,A]$ if $q \circ f = q \circ g$ then there is some $h \in H$ such taht $f = h \circ g$,

(ii) for all $x\colon A \to A/H$ there is an arrow $f \in [A,A]$ such that $q \circ f = x$.

Moreover, under Axiom (3), (i) implies the following:

(iii) $q \circ f = q$ implies $f \in H$.

Indeed by taking $g = 1_A$ in (i) we obtain that $f = h$ for some $h \in H$.

Remark 2.5. Consider an arrow $x\colon A \to X$ and the epi-mono factorization $x_* = \psi \circ \rho$[3]. With reference to the diagram below Axiom (iii) implies that $I = [A,A]/H$ with $H = \mathrm{Fix}(x) \leq G$.

$$\begin{tikzcd}
{[A,A]} \arrow[r, "\rho"] \arrow[rr, bend left, "x_*"] & I \arrow[r, "\psi"] & {[A,X]}
\end{tikzcd}$$

Indeed by Axiom (iii) $[A,A] = \mathrm{Aut}(A) = G$ so an $h \in H$ acts on $[A,A]$ by left multiplication. Given $f \in [A,A]$ we have

$$(\psi \circ \rho \circ h)(f) = x_*(h \circ f) = x \circ h \circ f = x \circ f = x_*(f) = (\psi \circ \rho)(f)$$

that is $\psi \circ \rho \circ h = \psi \circ \rho$ which, by monicness of ψ, implies $\rho \circ h = \rho$. Moreover as I is the image of x_* it is unique up to isomorphism and thus ρ is really a quotient arrow; $I = [A,A]/H$.

Proposition 2.6. Any arrow $x\colon A \to X$ of $\mathscr{C}$ is a quotient of A by $H = \mathrm{Fix}(x) \leq G$ (with respect to the action on $[A,X]$ described in Proposition 1.12) i.e. $X = A/H$.

Proof. By choosing $H = \mathrm{Fix}(x)$ we get $x \circ h = x$ for all $h \in H$ and thus there is a unique arrow $\varepsilon\colon A/H \to X$ of $\mathscr{C}$ such that $x = \varepsilon \circ q$, where $q\colon A \to A/H$ is the quotient of A by H. Graphically:

Now by applying $[A,-]$ to the diagram above we obtain

[3] We recall that **GSet** is a topos and, as such, has epi-mono factorization.

$$\begin{tikzcd}
A \arrow[r, "q"] \arrow[rr, bend left, "x"] & A/H \arrow[r, "\varepsilon"] & X
\end{tikzcd} .$$

Figure 2.6

$$\begin{tikzcd}
{[A,A]} \arrow[r, "q_*"] \arrow[rr, bend left, "x_*"] \arrow[dr, bend right, "\rho"'] & {[A,A/H]} \arrow[r, "\varepsilon_*"] & {[A,X]} \\
& {[A,A]/H} \arrow[u, "\eta"] \arrow[ur, bend right, "\psi"'] &
\end{tikzcd}$$

Figure 2.7

where ρ, ψ and η are as discussed before. We now have

$$\varepsilon_* \circ \eta \circ \rho = \varepsilon_* \circ q_* = x_* = \psi \circ \rho$$

that by epicness of ρ implies $\varepsilon_* \circ \eta = \psi$. Now ε_* must be monic since ψ and η are; but this, by Proposition 2.3, implies that $\varepsilon \in \mathscr{C}$ is monic too. Being an arrow of $\mathscr{C}$, by Axiom (i), ε is also a strict epimorphism and thus an isomorphism. □

Proposition 2.7. The action of $\mathrm{Aut}(A)$ on $[A, X]$ is transitive for all $X \in \mathscr{C}$.

Proof. Consider again Figure 2.7. We proved that ε is an isomorphism and thus ε_* must be one too. Moreover from Remark 2.4 we know η is iso too and so we have $[A, X] \cong [A, A]/H$, but by Axiom (3) $[A, A] = \mathrm{Aut}(A)$ and so we have that $[A, X]$ is transitive. □

Theorem 2.8. Given a category $\mathscr{C}$ and an object $A \in \mathscr{C}$ such that Axioms (1), (2) and (3) hold there exists an adjunction

$$A \times_G - \dashv [A, -]_G$$

where $G = \mathrm{Aut}(A)$ such that the maps

$$\eta \colon E \cong [A, A]/H \to [A, A/H]$$

$$\varepsilon \colon A/H \to X$$

are isomorphisms. This enstablishes an equivalence of categories between $\mathscr{C}$ and $\mathbf{GSet}^t$.

Proof. The existence of the adjunction follows from the discussion in Section 1, the fact that η and ε are (the components of) the unit and counit of the andjunction is a a textbook calculation and the fact that they are isomorphisms follows respectively from Remark 2.4 and Proposition 2.6. □

3 Appendixes

3.1 Naturality of $[A/H, X] \cong \mathbf{GSet}^t(E, [A, X]_G)$

Fix a category $\mathscr{C}$, an element $A \in \mathscr{C}$ and let $G = \mathrm{Aut}(A)$. Let's indicate with ψ_{EX} the bijection between $[A/H, X]$ and $\mathbf{GSet}^t(E, [A, X]_G)$ described in Remark 1.14; we shall prove that it is natural in both $X \in \mathscr{C}$ and $E \in \mathbf{GSet}^t$.

Naturality in X. Given an arrow $f \colon X \to Y$ of $\mathscr{C}$ we want to prove that ψ_{EX} are the components of a natural transformation $[A/H, -] \Rightarrow \mathbf{GSet}^t(E, [A, -]_G)$ i.e that the following diagram commutes.

$$\begin{tikzcd}
{[A/H, X]} \arrow[r, "\psi_{EX}"] \arrow[d, "f_*"] & \mathbf{GSet}^t(E, [A, X]_G) \arrow[d, "(f_*)_*"] \\
{[A/H, Y]} \arrow[r, "\psi_{EY}"] & \mathbf{GSet}^t(E, [A, Y]_G)
\end{tikzcd}$$

Recall that here $H = \mathrm{Fix}(x_0)$ for $x_0 \in E$. Pick any $x \in [A/H, X]$ and let $q \colon A \to A/H$ be the quotient arrow (the quotient exists because $\mathscr{C}$ is assumed to have all quotients by subgroups of G); then chasing x through the diagram down the two possible ways yields two arrows in $\mathbf{GSet}^t$ of type $E \to [A, Y]_G$. Since E is transitive we have $E \cong G/H$ and this means that arrows of type $E \to [A, Y]_G$ are determined uniquely by the image of $e \in E \cong G/H$ (refer to the right part of Diagram 1.4). Keeping this in mind the following computations shows that the square commutes.

$$((f_*)_* \circ \psi_{EX})(x)(e) = (f_* \circ \psi_{EX}(x))(e) = f_*(\psi_{EX}(x)(e)) = f_*(x \circ q) = f \circ x \circ q,$$

$$(\psi_{EY} \circ f_*)(x)(e) = \psi_{EX}(f \circ x)(e) = f \circ x \circ q.$$

□

Naturality in E. We want to prove that ψ_{EX} are the components of a natural transformation $[A/-, X] \Rightarrow \mathbf{GSet}^t(-, [A, X]_G)$, but before drawing the naturality square as above we shall note that this case is slightly complicated by the fact that it is not immediately clear how the functor $[A/-, X]$ acts on arrows. Indeed let $f \colon E \cong G/H \to F \cong G/H'$ be an arrow of $\mathbf{GSet}^t$ with $H = \mathrm{Fix}(x_0), H' = \mathrm{Fix}(f(x_0))$ and $x_0 \in E$; these choices are justified by the fact that both E and F are transitive G-sets. Now notice that, for $h \in H$, $f(h \cdot x_0) = f(x_0)$ by definition of H and $f(h \cdot x_0) = h \cdot f(x_0)$ by G-invariance of f; thus $h \in H'$ and thus $H \subset H'$. This means that if we let $q \colon A \to A/H, q' \colon A \to A/H'$ be the quotients in $\mathscr{C}$ then there is a unique arrow $\widetilde{f} \colon A/H \to A/H'$ such that $\widetilde{f} \circ q = q'$. The functor $[A/-, X]$ then sends the arrow $f \colon E \to F$ to $\widetilde{f}^* \colon [A/H', X] \to [A/H, X]$.

Now we want to prove the naturality of the following square.

$$\begin{tikzcd}
{[A/H, X]} \arrow[r, "\psi_{EX}"] & \mathbf{GSet}^t(E, [A, X]_G) \\
{[A/H', X]} \arrow[r, "\psi_{FX}"] \arrow[u, "\widetilde{f}^*"] & \mathbf{GSet}(F, [A, X]_G) \arrow[u, "f^*"']
\end{tikzcd}$$

As before we chase a $x \in [A/H', X]$ down the two possible paths to get two arrows of type $E \to [A, X]_G$ and prove that they are the same by evaluating them on $e \in E$:

$$(\psi_{EX} \circ \widetilde{f}^*)(x)(e) = \psi_{EX}(x \circ \widetilde{f})(e) = x \circ \widetilde{f} \circ q = x \circ q',$$

$$(f^* \circ \psi_{FX})(x)(e) = (\psi_{FX}(x) \circ f)(e) = \psi_{FX}(x)(f(e)) = \psi_{FX}(x)(e) = x \circ q'.$$

Note that we used that $f(e) = e$, this holds because the action of G on $G/H \cong E$ and $G/H' \cong F$ is just left multiplication and f is G-invariant. $\square$

3.2 η and ε are the unit and counit of $A \times_G - \dashv [A, -]_G$

η is the unit. Our adjunction is given by the bijection

$$\psi\colon \mathscr{C}(A \times_G E, X) \cong \mathbf{GSet}^t(E, [A, X]_G)$$

natural in $X \in \mathscr{C}$ and $E \in \mathbf{GSet}^t$. To obtain the unit we set $X = A \times_G E$:

$$\mathscr{C}(A \times_G E, A \times_G E) \cong \mathbf{GSet}^t(E, [A, A \times_G E]_G)$$

and so

$$\mathscr{C}(A/H, A/H) \cong \mathbf{GSet}^t(E, [A, A/H]_G)$$

where $H = \mathrm{Fix}(x) \leq G$ ($x \in E$) by definition of $A \times_G E$. Now (the component at E of) the unit is given by the image of $1_{A/H}$ under this bijection. By the discussion in Section 1 if $q\colon A \to A/H$ is the quotient arrow in $\mathscr{C}$ then $\psi(1_{A/H})$ is the map $E \cong [A, A]_G/H \to [A, A/H]_G$ of $\mathbf{GSet}^t$ that factors the map $G \to E$ that sends e to q. But this last map is q^* as in Figure 2.5. $\square$

ε is the counit. Keeping the proof above in mind we set $E = [A, X]$ and obtain

$$\psi\colon \mathscr{C}(A \times_G [A, X], X) \cong \mathbf{GSet}^t([A, X]_G, [A, X]_G)$$

that becomes

$$\mathscr{C}(A/H, X) \cong \mathbf{GSet}^t([A, X]_G, [A, X]_G)$$

with $H = \mathrm{Fix}(x)$ for some $x \in [A, X]$. Notice that since $[A, X] \cong G/H$ by Proposition 2.7 we can consider $1_{[A,X]}$ as a map of type $G/H \to [A, X]$ such that $1_{[A,X]}(e) = x$ (this can be obtained by chasing 1_A around Figure 2.7 keeping in mind that ε_* there is iso). Now we have that $\psi^{-1}(1_{[A,X]})$ is the arrow $A/H \to X$ of $\mathscr{C}$ that factorizes x i.e. $\varphi^{-1}(1_{[A,X]}) = \varepsilon$ as in Figure 2.6. $\square$