Esercizio 1. Let M be an L-structure and let $\psi(x), \varphi(x,y) \in L$. For each of the following conditions, write a sentence true in M exactly when

a. $\psi(M) \in \{\varphi(a,M) : a \in M\}$;
b. $\{\varphi(a,M) : a \in M\}$ contains at least two sets;
c. $\{\varphi(a,M) : a \in M\}$ contains only sets that are pairwise disjoint.

Soluzione 1.

a. $\exists a \forall b(\psi(b) \to \varphi(a,b))$;
b. $\exists a,b,c(\psi(a,b) \oplus \psi(b,c))$;
c. $\forall a,b,c(\psi(a,c) \wedge \psi(b,c) \to \perp)$.

Esercizio 2. Let M be a structure in a signature that contains a symbol r for a binary relation. Write a sentence φ such that

a. $M \models \varphi$ if and only if there is an $A \subseteq M$ such that $r^M \subseteq A \times \neg A$.

Soluzione 2. Let φ be the formula $\forall a \forall b(arb \to \forall c(b \not r c))$.

If $M \models \varphi$ then set $A = \{a \in M \colon \text{there is a } b \in M \text{ such that } ar^M b\}$. Now if $(a,b) \in r^M$ then $a \in A$ by definition of A and there is no $c \in M$ such that $br^M c$ (because $M \models \varphi$) so $b \notin A$. This proves that $r^M \subseteq A \times \neg A$.

Conversely suppose that $r^M \subseteq A \times \neg A$ for some $A \subseteq M$. If $ar^M b$ we immediately have $b \in \neg A$. Now for the sake of contradiction let there be $c \in M$ such that $br^M c$; but this immediately implies $b \in A$ that is absurd. We are forced to conclude that $M \models \varphi$.

Esercizio 3. Let $M \preceq N$ and let $\varphi(x) \in L(M)$. Prove that $\varphi(M)$ is finite if and only if $\varphi(N)$ is finite and in this case $\varphi(N) = \varphi(M)$.

Soluzione 3. We recall that $M \preceq N$ means that M is a $L(M)$-substructure of N such that $N \models \psi$ *if and only if* $M \models \psi$ for all sentences $\psi \in L(M)$. We thus trivially have $\varphi(M) \subseteq \varphi(N)$ so $\varphi(N)$ finite implies $\varphi(M)$ finite.

Now suppose $\varphi(M) = \{m_1, \ldots, m_k\}$ and $\varphi(M) \subset \varphi(N)$. We thus have that $N \models \psi$ where ψ is the formula

$$\exists x (x \neq m_1 \wedge \ldots \wedge x \neq m_k \wedge \varphi(x)).$$

But ψ is a $L(M)$-sentence and thus $M \models \psi$. This is clearly impossible because there sould be an element $m \in M$ such that $M \models \varphi(m)$ but $m \notin \varphi(M) = \{m_1, \ldots, m_k\}$. By contradiction we have $\varphi(N) \subseteq \varphi(M)$.

We conclude that $\varphi(M)$ finite implies $\varphi(M) = \varphi(N)$ and thus $\varphi(N)$ finite as well.

Esercizio 4. Let $M \preceq N$ and let $\varphi(x,z) \in L$. Suppose there are finitely many sets of the form $\varphi(a,N)$ for some $a \in N^{|x|}$. Prove that all these sets are definable over M.

Soluzione 4. We present a solution for the case $|x| = 1$ that we believe should be adjustable to the case $|x| > 1$. We know that $A = \{\varphi(a,N) : a \in N\}$ is finite. For the sake of the argument we define the following equivalence relation on the elements of N

$$a \sim b \quad \Leftrightarrow \quad \varphi(a,N) = \varphi(b,N).$$

One immediately has that if $a \sim m$ for some $m \in M$ then $\varphi(a,N)$ is definable over M (just consider the formula $\varphi(m,x) \in L(M)$). We will show that for all $a \in N$ there is some $m \in M$ such that $a \sim m$ and thus that all sets in A are definable over M.

Choose $m_1, \dots, m_k$ in M such that if $m \in M$ then $m \sim m_i$ for some $1 \le i \le k$; we can always chose a finite number of such elements since A finite implies that $\sim$ has a finite number of equivalence classes. Now assume that there is $n \in N$ such that $n \not\sim m_i$ for all $1 \le i \le k$. This condition is equivalent to requiring that N satisfies the following $L(M)$-sentence

$$\psi \equiv \exists n \big[\exists a \big(\varphi(n,a) \oplus \varphi(m_1,a)\big) \wedge \ldots \wedge \exists a \big(\varphi(n,a) \oplus \varphi(m_k,a)\big)\big].$$

But now since $M \preceq N$ we have $M \models \psi$ which is absurd by the choice of $m_1, \dots, m_k$. By contradiction we conclude that for all $a \in N$ there is some $m \in M$ such that $a \sim m$ and thus $\varphi(a,N)$ is definable over M.