**Esercizio 1.** Let $M$ be an $L$-structure and let $\psi(x), \varphi(x, y) \in L$. For each of the following conditions, write a sentence true in $M$ exactly when

a. $\psi(M) \in \{\varphi(a, M) : a \in M\}$;

b. $\{\varphi(a, M) : a \in M\}$ contains at least two sets;

c. $\{\varphi(a, M) : a \in M\}$ contains only sets that are pairwise disjoint.

**Soluzione 1.**

a. $\exists x \forall y (\psi(y) \leftrightarrow \varphi(x, y))$;

b. $\exists x, y, z(\varphi(x, z) \not\leftrightarrow \varphi(y, z))$;

c. $\forall x, y(x \neq y \rightarrow \exists z(\varphi(x, z) \not\leftrightarrow \varphi(x, z)))$

**Esercizio 2.** Let $M$ be a structure in a signature that contains a symbol $r$ for a binary relation. Write a sentence $\varphi$ such that

a. $M \models \varphi$ if and only if there is an $A \subseteq M$ such that $r^M \subseteq A \times \neg A$.

**Soluzione 2.** Let $\varphi$ be the formula $\forall x \forall y (xry \to \forall z \neg(yrz))$.

If $M \models \varphi$ then set $A = \{a \in M\colon \text{there is a } b \in M \text{ such that } ar^M b\}$. Now if $(a,b) \in r^M$ then $a \in A$ by definition of $A$ and there is no $c \in M$ such that $br^M c$ (because $M \models \varphi$) so $b \notin A$. This proves that $r^M \subseteq A \times \neg A$.

Conversely suppose that $r^M \subseteq A \times \neg A$ for some $A \subseteq M$. If $ar^M b$ we immediately have $b \in \neg A$. Now for the sake of contradiction let there be $c \in M$ such that $br^M c$; but this immediately implies $b \in A$ that is absurd. We are forced to conclude that $M \models \varphi$.

**Esercizio 3.** Let $M \preceq N$ and let $\varphi(x) \in L(M)$. Prove that $\varphi(M)$ is finite if and only if $\varphi(N)$ is finite and in this case $\varphi(N) = \varphi(M)$.

**Soluzione 3.** We recall that $M \preceq N$ means that $M$ is a $L(M)$-substructure of $N$ such that $N \models \psi$ *if and only if* $M \models \psi$ for all sentences $\psi \in L(M)$. We thus trivially have $\varphi(M) \subseteq \varphi(N)$ so $\varphi(N)$ finite implies $\varphi(M)$ finite.

Now suppose $\varphi(M) = \{m_1, \ldots, m_k\}$ and $\varphi(M) \subset \varphi(N)$. We thus have that $N \models \psi$ where $\psi$ is the formula

$$\exists x (x \neq m_1 \wedge \ldots \wedge x \neq m_k \wedge \varphi(x)).$$

But $\psi$ is a $L(M)$-sentence and thus $M \models \psi$. This is clearly impossible because there sould be an element $m \in M$ such that $M \models \varphi(m)$ but $m \notin \varphi(M) = \{m_1, \ldots, m_k\}$. By contradiction we have $\varphi(N) \subseteq \varphi(M)$.

We conclude that $\varphi(M)$ finite implies $\varphi(M) = \varphi(N)$ and thus $\varphi(N)$ finite as well.

**Esercizio 4.** Let $M \preceq N$ and let $\varphi(x,z) \in L$. Suppose there are finitely many sets of the form $\varphi(a,N)$ for some $a \in N^{|x|}$. Prove that all these sets are definable over $M$.

**Soluzione 4.** We present a solution for the case $|x| = 1$ that we believe should be adjustable to the case $|x| > 1$. We know that $A = \{\varphi(a,N)\colon a \in N\}$ is finite. For the sake of the argument we define the following equivalence relation on the elements of $N$

$$a \sim b \quad \Leftrightarrow \quad \varphi(a,N) = \varphi(b,N).$$

One immediately has that if $a \sim m$ for some $m \in M$ then $\varphi(a,N)$ is definable over $M$ (just consider the formula $\varphi(m,x) \in L(M)$). We will show that for all $a \in N$ there is some $m \in M$ such that $a \sim m$ and thus that all sets in $A$ are definable over $M$.

Choose $m_1, \ldots, m_k$ in $M$ such that if $m \in M$ then $m \sim m_i$ for some $1 \le i \le k$; we can always chose a finite number of such elements since $A$ finite implies that $\sim$ has a finite number of equivalence classes. Now assume that there is $n \in N$ such that $n \not\sim m_i$ for all $1 \le i \le k$. This condition is equivalent to requiring that $N$ satisfies the $L(M)$-sentence

$$\psi \equiv \exists x \bigwedge_{i=1}^{k} \neg(x \sim m_i).$$ [1]

But now since $M \preceq N$ we have $M \models \psi$ which is absurd by the choice of $m_1, \ldots, m_k$. By contradiction we conclude that for all $a \in N$ there is some $m \in M$ such that $a \sim m$ and thus $\varphi(a,N)$ is definable over $M$.

[1] We use $x \sim y$ as an abbreviation for $\forall z(\varphi(x,z) \leftrightarrow \varphi(y,z))$.