

Esercizio 1. Let M be an L -structure and let $\psi(x), \varphi(x, y) \in L$. For each of the following conditions, write a sentence true in M exactly when

- a. $\psi(M) \in \{\varphi(a, M) : a \in M\}$;
- b. $\{\varphi(a, M) : a \in M\}$ contains at least two sets;
- c. $\{\varphi(a, M) : a \in M\}$ contains only sets that are pairwise disjoint.

Soluzione 1.

- a. $\exists a \forall b (\psi(b) \rightarrow \varphi(a, b))$;
- b. $\exists a, b, c (\psi(a, b) \oplus \psi(b, c))$;
- c. $\forall a, b, c (\psi(a, c) \wedge \psi(b, c) \rightarrow \perp)$.

Esercizio 2. Let M be a structure in a signature that contains a symbol r for a binary relation. Write a sentence φ such that

- a. $M \models \varphi$ if and only if there is an $A \subseteq M$ such that $r^M \subseteq A \times \neg A$.

Soluzione 2. Let φ be the formula $\forall a \forall b (arb \rightarrow \forall c (b \neq c))$.

If $M \models \varphi$ then set $A = \{a \in M : \text{there is a } b \in M \text{ such that } ar^M b\}$. Now if $(a, b) \in r^M$ then $a \in A$ by definition of A and there is no $c \in M$ such that $br^M c$ (because $M \models \varphi$) so $b \notin A$. This proves that $r^M \subseteq A \times \neg A$.

Conversely suppose that $r^M \subseteq A \times \neg A$ for some $A \subseteq M$. If $ar^M b$ we immediately have $b \in \neg A$. Now for the sake of contradiction let there be $c \in M$ such that $br^M c$; but this immediately implies $b \in A$ that is absurd. We are forced to conclude that $M \models \varphi$.

Esercizio 3. Let $M \leq N$ and let $\varphi(x) \in L(M)$. Prove that $\varphi(M)$ is finite if and only if $\varphi(N)$ is finite and in this case $\varphi(N) = \varphi(M)$.

Soluzione 3. We recall that $M \leq N$ means that M is a $L(M)$ -substructure of N such that $N \models \psi$ if and only if $M \models \psi$ for all sentences $\psi \in L(M)$. We thus trivially have $\varphi(M) \subseteq \varphi(N)$ so $\varphi(N)$ finite implies $\varphi(M)$ finite.

Now suppose $\varphi(M) = \{m_1, \dots, m_k\}$ and $\varphi(M) \subset \varphi(N)$. We thus have that $N \models \psi$ where ψ is the formula

$$\exists x(x \neq m_1 \wedge \dots \wedge x \neq m_k \wedge \varphi(x)).$$

But ψ is a $L(M)$ -sentence and thus $M \models \psi$. This is clearly impossible because there should be an element $m \in M$ such that $M \models \varphi(m)$ but $m \notin \varphi(M) = \{m_1, \dots, m_k\}$. By contradiction we have $\varphi(N) \subseteq \varphi(M)$.

We conclude that $\varphi(M)$ finite implies $\varphi(M) = \varphi(N)$ and thus $\varphi(N)$ finite as well.

Esercizio 4. Let $M \preceq N$ and let $\varphi(x, z) \in L$. Suppose there are finitely many sets of the form $\varphi(a, N)$ for some $a \in N^{|x|}$. Prove that all these sets are definable over M .