Esercizio 1. Let $L = \{<\}$ and let N be a ω_1-saturated extension of $\mathbb{Q}$. Prove that there is an embedding $f : \mathbb{R} \to N$. Is it elementary? Can it be an isomorphism?

Soluzione 1. From Theorem 9.6 N ω_1-saturated implies that N is ω_1-rich in the category $\mathcal{M}$ with elementary maps as arrows. As N is an extension of $\mathbb{Q}$ there is an elementary embedding $\mathbb{Q} \hookrightarrow N$; this can be regarded as an elementary map $\mathbb{R} \to N$ of cardinality $< \omega_1$. Now using that N is ω_1 rich together with the finite character of morphisms (c2 from Definition 7.1) we can extend that map to obtain an embedding f of $\mathbb{R}$ in N that is elementary.

Finally suppose that f is an isomorphism and consider the type $p(x) = \{n < x : n \in \mathbb{N}\}$ with parameters in $\mathbb{N}$ of cardinality $< \omega_1$. This type is realized in N by ω_1-saturation. Now, since f is an isomorphism, there must be a $x^* \in \mathbb{R}$ that realizes the type in $\mathbb{R}$ but this is absurd.

Esercizio 2. Let M and N be elementarily homogeneous structures of the same cardinality λ. Suppose that $M \models \exists x\, p(x) \Leftrightarrow N \models \exists x\, p(x)$ for every $p(x) \subseteq L$ such that $|x| < \lambda$. Prove that the two structures are isomorphic. (Hint: see Theorem 7.8)

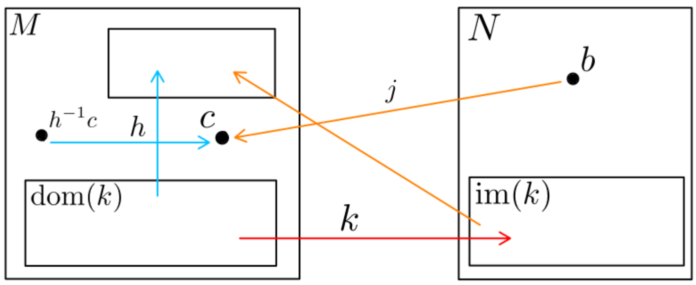
Soluzione 2. We would like to do a back-and-forth construction but in order to do so we need a "morphism extension lemma".

Let $k\colon M \to N$ be an elementary map such that $|\mathrm{dom}(k)| < \lambda$. Now pick $b \in N$ (that we can assume to be outside $\mathrm{im}(k)$) and consider the type

$$p(x) = \mathrm{tp}_N\big(\mathrm{im}k, b\big)$$

where we consider $\mathrm{im}(k)$ as a tuple. Since $|\mathrm{im}(k)| < \lambda$ by assumption $p(x)$ is realized by some $a \in M^{|\mathrm{im}(k)|}$ and $c \in M$. Now let $j\colon N \to M$ be the map with $\mathrm{dom}(j) = \mathrm{im}(k) \cup \{b\}$ that sends the tuple $\mathrm{rm}(k)$ to the tuple a and the element b to the element c. This map is elementary because given $\varphi(x) \in L$ and $a \in N^{|x|}$ such that $N \models \varphi(a)$ we have $\varphi(x) \in p(x)$ and thus $M \models \varphi(ja)$. The composition of elementary maps is elementary so $j \circ k\colon M \to M$ is elementary and, by λ-homogeneity, extends to an automorphism $h\colon M \to M$.

Now consider the map $k'\colon M \to N$ with $\mathrm{dom}(k') = \mathrm{dom}(k) \cup \big\{h^{-1}c\big\}$ defined as $k' = j^{-1} \circ h$. This is an elmentary map and an extension of k such that $b \in \mathrm{im}(k')$.



We can now obtain an isomorphism by a back-and-forth construction using the above "morphism extension lemma" and familiar induction techniques.

Esercizio 3. Let $A \subseteq N \models T_{\mathrm{acf}}$ what is the cardinality of $S_x(A)$, where $|x| = 1$? Recall that $S_x(A)$ is the set of complete types $p(x) \subseteq L(A)$, finitely consistent in N.

Answer the same question for $A \subseteq N \models T_{\mathrm{rg}}$.

Soluzione 3.

Algebraically closed fields. Consider $\overline{A}$ made of all $a \in \mathcal{U}$ (with $\mathcal{U}$ some monster model) that are non-zero and such that there is a polinomial equation $\varphi_a(x) \in L(A)$ such that $\varphi_a(a)$ is true. Since T_{acf} has quantifier elimination we can assume all formulas to be polynomial (un)equations eventually combined with connectives.

Consider $p(x) \in S_x(A)$ and let a be a realization of p. By completeness unless p is the type of all polynomial unequations there is some polynomial equation $\varphi_a(x) \in p$ such that $\varphi_a(a)$ is true and so $a \in \overline{A}$. Now if $b \in \overline{A}$ is another realization of p then $\varphi_a(b)$ must be true as well. This tells us that the type p can only have a finite number of realizations because if this were not the case then the polynomial given by the equation $\varphi_a(x)$ would have arbitrarily large degree. This tells us that $|S_x(A)| \leq |\overline{A}|$.

Now consider $a \in \overline{A}$ and let $p(x) \in S_x(A)$ be the unique (by completeness) type with realization a. By remembering that $p(x)$ has only a finite number of realizations we have $|\overline{A}| \leq |S_x(A)|$. So, in the end, $|S_x(A)| = |\overline{A}|$.

Finally if A is infinite then $|A| = |\overline{A}|$ and if A is finite then $|\overline{A}| = \aleph_0$.

Random graphs. We know that T_{rg} has quantifier elimination so we can assume that all formulas in our types are of the kind $r(x, a)$ or $\neg r(x, a)$ for some $a \in A$, eventually combined with the binary connectives.

By completeness every type $p(x) \in S_x(A)$ is completely determined by a binary choice for every $a \in A$ and thus $S_x(A)$ is in bijection with the set 2^A. So we conclude $|S_x(A)| = |2^A|$.