Esercizio 1. Assume L is countable and let $M \preceq N$ have arbitrary (large) cardinality. Let $A \subseteq N$ be countable. Prove there is a countable model K such that $A \subseteq K \preceq N$ and $K \cap M \preceq N$ (in particular, $K \cap M$ is a model). Hint: adapt the *construction* used to prove the downward Löwenheim-Skolem Theorem.

Soluzione 1. We have $|L(A)| = \aleph_0$. We shall construct a chain $\left(A_i\right)_{i<\omega}$ of subsets of N such that $K = \bigcup_{i<\omega} A_i$ will be the wanted model.

We begin by setting $A_0 = A$ then we have $|A_0| = \aleph_0$ so $|L(A_0)| = \aleph_0$. Now let $\left(\varphi_k(x) \colon k < \aleph_0\right)$ be an enumeration of all the formulas of $L(A_0)$ that are consistent in N. By construction there is a sequence $\left(a_k \in N \colon k < \aleph_0\right)$ such that $N \models \varphi_k(a_k)$ for all $k < \aleph_0$; moreover if $\varphi_k(x) \in L(M)$ we can chose a_k in M because of the Vaught-Tarski test. Finally we set $A_1 = A_0 \cup \left\{a_k \in N \colon k < \aleph_0\right\}$; note that $|A_1| = \aleph_0$. By iteration of this process we obtain the chain $\left(A_i\right)_{i<\omega}$.

As in the proof of the Downward Löwenheim-Skolem by construction we have that $|K| = \aleph_0$ and the Vaught-Tarski test can be used to show that $K \preceq N$.

Similarly if $\varphi(x) \in L(K \cap M)$ then $\varphi(x) \in L(M)$ and so K will contain a solution that is also in M thus $K \cap M \preceq N$ by the Vaught-Tarski test (and particularly $K \cap M$ is a model).

Esercizio 2. Prove either that 1a⇔2a, or that 1b⇔2b. (Choose whichever you prefer.)

1a. $X \subseteq \mathbb{R}$ is open;

2a. $b \approx a \in X \Rightarrow b \in {}^*X$ for every $b \in {}^*\mathbb{R}$.

1b. $X \subseteq \mathbb{R}$ is closed;

2b. $a \in {}^*X \Rightarrow \operatorname{st} a \in X$ for every finite $a \in {}^*\mathbb{R}$.

Open and closed are understood w.r.t. the usual tolopology on $\mathbb{R}$.

Soluzione 2. We prove that 1a⇔2a.

Assume X open and let a be one of its points; then we know that there is some real $r > 0$ such that $\mathbb{R} \models \forall x\big(|a-x| < r \rightarrow x \in X\big)$. Since it is an elementary extension we have that this sentence is also true in ${}^*\mathbb{R}$. Now let $b \in {}^*\mathbb{R}$ be such that $b \approx a$; then $|b-a|$ is an infinitesimal and as such $|b-a| < r$ so $b \in {}^*X$.

Now fix $a \in X$ and assume 2a. Note that if $\delta > 0$ is an infinitesimal then $|a-x| < \delta$ implies $a \approx x$ and thus $x \in {}^*X$. Since non-zero infinitesimals exist (there are actually infinitely many) ${}^*\mathbb{R} \models \exists\delta\big(\delta > 0 \wedge \forall x\big(|a-x| \leq \delta \rightarrow x \in {}^*X\big)\big)$. But now $\mathbb{R}$ is an elementary substructure of ${}^*\mathbb{R}$ so $\mathbb{R} \models \exists\delta\big(\delta > 0 \wedge \forall x\big(|a-x| < \delta \rightarrow x \in X\big)\big)$. Finally, since the choice of a is arbitrary, X is an open set.

Esercizio 3. **(Bonus question)** For which sets $X \subseteq \mathbb{R}$ does the following hold?

2. $b \approx a \in {}^*X \Rightarrow b \in {}^*X$ for every $a, b \in {}^*\mathbb{R}$.