18: THE MATHEMATICS OF OSCILLATORY MOTION

Ut tensio, sic vis.

—ROBERT HOOKE

THE motions of projectiles, of planets, and of light have an obvious importance or attraction, and one would therefore hardly question why men should study these motions. On the other hand, the motion of an object suspended from a spring and bobbing up and down repeatedly seems to offer no more attraction than to while away an idle hour. Of course an idle mind readily becomes the devil's playground, but we shall tempt the devil.

Let us be clear first as to what is different about the motion of objects on springs as compared with the motions we have previously examined. A projectile shot from a cannon traverses its course to the target and stops there. It does not then reverse its motion and return to the cannon. In fact it is apparently so pleased to have completed its motion that it announces its arrival at its destination with a bang. Some motions are never completed but continue indefinitely in one direction. Thus the earth, after completing one revolution around the sun, does not stop and then move in the opposite direction. Projectiles and planets move, so to speak, along one-way streets. On the other hand, the object bobbing up and down in response to the force exerted by the spring travels back and forth endlessly. The object goes nowhere in this oscillatory motion; yet the study of such motions has carried man far in his scientific travels.

Historically, the investigation of oscillatory motions was motivated by the desire to improve methods of telling time. The primary standard of time is, of course, the motion of the earth around the sun, but this natural clock is not particularly useful in the daily affairs of man. In the seventeenth century the need to measure small periods of time accurately for the purpose of telling longitude at sea caused scientists to search for increasingly accurate clocks. The search resulted in some major successes that were at least as valuable for the advancement of mathematics and the study of other phenomena of nature, such as light and sound, as they were for the specific problem of measuring time.

Scientists naturally concentrated on any physical phenomena that seemed to be periodic or repetitive and might therefore be related to the periodic motion of the planets. Two phenomena recommended themselves for closer investigation, the motion of an object or bob, as it is called, on a spring, and the motion of a pendulum. The first of these attracted the attention of Robert Hooke (1635–1703), a contemporary of Newton, a professor of mathematics and mechanics at Gresham College in England, and a noted experimentalist.

Let us consider with Hooke the motion of a bob on a spring. The bob is attached to the lower end of the spring (fig. 93) whose upper end is attached to a fixed support. Gravity will pull the bob

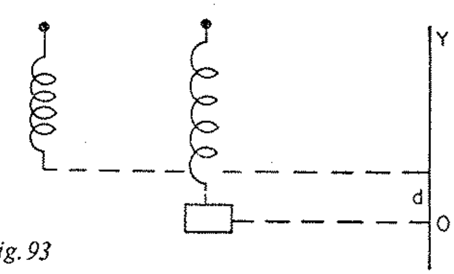


Fig. 93

down until the tension in the spring offsets the force of gravity. Unless disturbed the bob will then remain in a fixed position, which is called its rest or equilibrium position. Since we intend to disturb the bob and set it in motion let us agree to introduce a variable y that will represent its displacement from the rest position. When the displacement is upward we shall consider y to be positive and when downward, negative. Also, we shall ignore the resistance of air. While this resistance does dampen the motion its effect is secondary.

Hooke made a capital discovery about the motion of bobs on springs, but before we describe and utilize it fully let us see what mere observation of such motions can tell us. Let us suppose that the bob is pulled down below its rest position and then released.

We know from experience that the bob will move up past the rest position to some highest point or maximum displacement, then move downward to some lowest point or minimum displacement, and repeat this behavior endlessly, if the damping effect of air and (strictly) any internal energy losses in the spring are negligible. The motion is best exhibited by a graph (fig. 94) that shows how the displacement varies with time from the instant the bob passes the rest position on the way up. Of course the bob moves up and down along a vertical path whereas the graph is spread out horizontally because time values are graphed along the horizontal axis. The graph shows, of course, that the motion of the bob is periodic and that the behavior from O to P is repeated. The interval OP is therefore called the *period,* which is, then, the time required by the bob to go through one complete oscillation.

To undertake a mathematical study of the motion of the bob the first fact one would want is the formula that represents the graph, that is, the formula that relates the displacement and time of motion of the bob. But no one of the functions we have had occasion to use seems to fit this periodic behavior. We are therefore faced with the unfortunate necessity of extending our mathematical tools. Let us examine the graph more closely to see just what is involved.

According to the graph the bob executes one complete oscillation—that is, it moves from the rest position upward to its highest position, back to rest position, down to the lowest position, and then back up to the rest position—in one period. Let us suppose that the period is one second and that the maximum and minimum displacements are 1 and -1, respectively. We note also that during one period the graph naturally divides itself into four quarters, namely, the interval from 0 to ¼ second during which the bob rises to its maximum displacement, the interval from ¼ to ½ second during which the bob goes through the same sequence of displacements but in *reverse* order, and the two quarters from ½ to 1 second during which the bob repeats the motion of the first half except that the displacements are now downward or negative. Every period thereafter the graph repeats exactly the pattern of the first period.

These observations show that the shape of the entire graph is

determined by what happens in the first quarter of a period, since the displacements after that merely repeat in different order or with different sign what happens in this quarter. Hence let us concentrate on the shape of the graph during the first quarter. We notice from the graph that we are dealing with a function, that is, a relation between displacement and time, wherein the displacement starts with the value 0 when $t = 0$, rises rapidly for small values of t, and then rises more slowly to the value 1 when $t = \frac{1}{4}$. To the mathematician these observations on the behavior of the displacement suggest the behavior of $\sin A$ when A varies from 0° to 90°. Indeed, if one were to plot the values of angle A as abscissas and the values of $\sin A$ as ordinates the graph would then have the same shape as the graph in figure 94 as t varies from 0 to ¼.

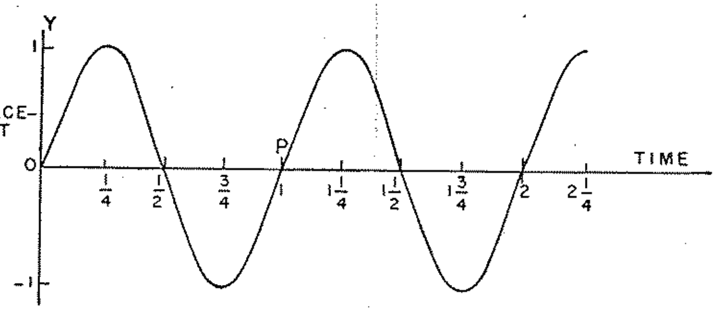
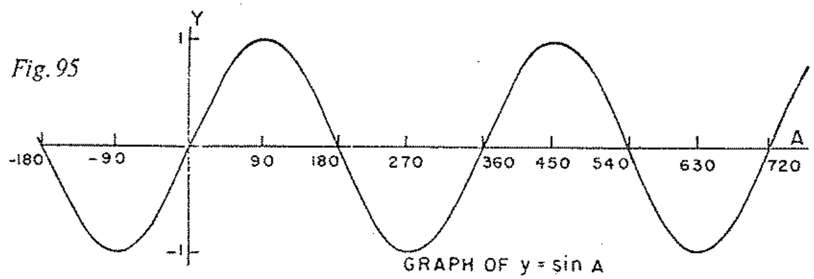


Fig. 94

There are, however, several differences. In the first place the value of $\sin A$ ends abruptly with the value 1 when A reaches 90°. There is no functional behavior corresponding to the second, third, and fourth quarters of the displacement-time relationship. This defect is easily remedied. We shall create a new function that will extend the meaning of $\sin A$. Let us agree to have $\sin A$ vary from 1 to 0 as A increases from 90° to 180°, and, moreover, to have this function decrease in the reverse order from that in which it increases as A varies from 0° to 90°. Thus we require that $\sin 100° = \sin 80°$, $\sin 110° = \sin 70°$, and so forth, until A reaches the value of 180°. In the interval from 180° to 360° we shall agree to have $\sin A$ take on the same values as it does from 0° to 180° except that we shall now require that $\sin A$ be negative. Thus we require that $\sin 190° = -\sin 10°$, $\sin 200° = -\sin 20°$, and so forth.

We can consider angles larger than 360° merely by regarding such angles as generated by a rotation. Thus the minute hand of a clock rotates through 360° in one hour. As it continues to rotate we may regard the additional angle through which it rotates as added to 360°. In an hour and a half the minute hand sweeps through 540°, in two hours through 720°, and so forth. We shall now require that in each 360° interval, beyond the value of 360° for A, the function sin A is to repeat its behavior in the interval from 0° to 360°. Thus, for example, sin 370° = sin 10°, sin 380° = sin 20°, and sin 540° = sin 180° The function $y = \sin A$, which we have just created, is pictured in figure 95.

Fig. 95



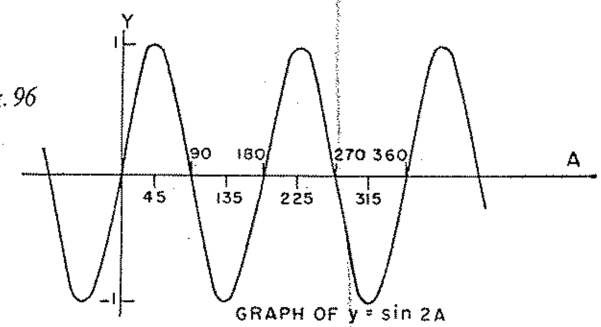
There are occasions when one wishes to use negative angles as well as positive angles. Of course, a negative angle means no more than a rotation of some number of degrees in the direction opposite to whatever direction is called positive. It is conventional to call the direction in which the minute hand of a clock rotates negative; accordingly, the angle formed by the initial and final positions of the minute hand is called negative. Correspondingly, the angle generated by a counterclockwise rotation is called positive. Though minute hands of clocks do not usually rotate in what we have called the positive direction, other devices such as wheels or gears on machines do. Of course the convention that the negative direction be clockwise and the positive be counterclockwise need not be adhered to any more than that the upward direction must always be called positive and the downward be called negative.

Since negative angles are used and since there will be occasion to utilize sin A when A is negative, the function sin A is defined for negative angles as shown in figure 95 so as to fit the agreement

that the function repeats its behavior in each 360° interval. Thus sin (—90°) = sin 270°.

There is, however, a difficulty in attempting to employ the function $y = \sin A$ to describe the motion of the bob on the spring. The bob repeats its behavior each second, while the function $y = \sin A$ repeats its behavior every 360°. This difficulty is readily circumvented. To "speed up" the behavior of $y = \sin A$ we have but to make the values of A change more rapidly while the values of sin A continue their usual course. A simple maneuver will do the trick. Let us consider $y = \sin 2A$. As A varies from 0° to 180°, $2A$ varies from 0° to 360° and sin $2A$ must go through the full range or cycle of values from 0 to 1, to 0, to —1, and then to 0. Thus the function $y = \sin 2A$, as shown in figure 96, repeats its behavior every 180°. If we use the term *frequency* to mean the number of times that the function $y = \sin 2A$ repeats its behavior in 360°, we may say that the frequency of this function is 2.

Fig. 96



We now have a mathematical device that we can carry further. The function $y = \sin 3A$ will have a frequency of 3, or repeat its behavior every 120°. Obviously we can produce a function that repeats its behavior as often as we please merely by using the proper coefficient of the angle A. This coefficient will be the frequency with which the sine function repeats its behavior in each 360° period.

While we are on the subject of creating new trigonometric functions we shall anticipate another need. In discussing the motion of the bob on the spring we supposed that its maximum displacement, or the *amplitude* of the motion, was 1. Let us suppose that each displacement were one-half of what is shown in figure 94. Could a trigonometric function still represent this motion? This question is readily answered. The function $y = (1/2) \sin A$ will have exactly one-half the y-values of the function $y = \sin A$ for the same values of A, and the function $y = (1/3) \sin A$ will have exactly one-third the y-values of $y = \sin A$ for the same values of A. Evidently we can modify the displacement values by any factor we please merely by putting that factor in front of $\sin A$.

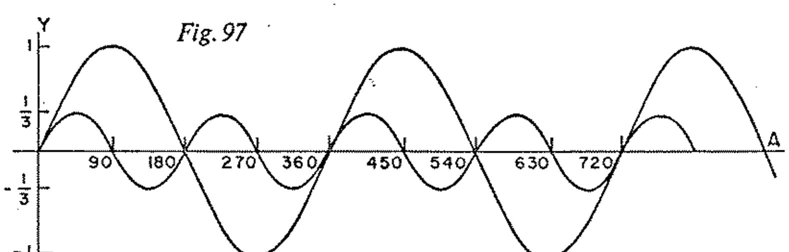
By the same argument we can decrease or increase the displacement values of $y = \sin 2A$ by any constant factor merely by placing that factor in front of $\sin 2A$. Thus $y = (1/3) \sin 2A$ will rise and fall one-third as much as $y = \sin 2A$ in each 180° period (fig. 97).

We have therefore arrived at the following stage. The function

(1) $$y = D \sin FA$$

will have a frequency F in 360°, that is, it will repeat its behavior F times in each 360° interval, and it will have an amplitude or maximum displacement D. It seems clear, then, that by choosing F and D properly we can produce a formula that will describe a sinusoidal, that is, sine-like, function with any frequency and amplitude.

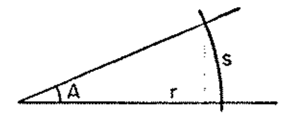
Fig. 97



COMPARISON OF $y = \sin A$ AND $y = \frac{1}{3} \sin 2A$

There is one more detail to be covered before we can describe mathematically the motion of a bob on a spring. We had occasion to point out in the preceding chapter that the size of an angle can be described in terms of radians instead of degrees. An angle can always be thought of as a central angle in a circle, and the number of radians in the angle is the subtended arc divided by the radius,

Fig. 98



that is, s divided by r (fig. 98). In this *radian* system of measuring angles, the sizes of angles usually denoted as varying from 0° to 360° can vary from 0 to 2π because the subtended arc can vary from 0 to 2π times the radius. Thus a 90° angle, which is one-quarter of a complete rotation, would have the size $\pi/2$ in the radian system of measuring angles. The use of radians instead of degrees involves no more difficult an idea than the use of yards instead of feet to measure length. Whereas in the latter case we remember the conversion relationship, that 1 yard is equivalent to 3 feet, in the case of radian measure we need only remember that an angle of 2π radians is equal in size to one of 360°.

Though we could continue to work with angles measured in degrees we shall adopt the radian measure because it will ultimately be more convenient. At the moment the adoption of radian measure will serve no more purpose than conformity to scientific usage. We shall therefore restate the conclusion arrived at in connection with formula (1) as follows. The function

(2) $$y = D \sin FA$$

has a frequency F in 2π radians and an amplitude D.

The function (2) relates two variables, the size of angle A and a variable y that has a specific numerical value for each value of A. But this statement is not exactly right. The fact that the numbers chosen for A were originally suggested by the sizes of angles, and the fact that the y-values were originally suggested by ratios associated with angles, are *not* contained in formula (2). The formula merely relates two variables, A and y. Given a numerical value of A we can find the numerical value of y. The situation here is not different from that involving a simple formula such as $d = 16t^2$. We often say that this formula relates the distance in feet that a body falls to the number of seconds it has been falling. Actually the formula itself does no such thing. As a mathematical formula it relates two variables, d and t. That d can represent the distance a body falls when t represents the number of seconds the body falls is a physical interpretation and application of the mathematical formula. There is, however, no reason why $d = 16t^2$ cannot represent the number of apples that t pigs eat each month.

We are therefore free to use formula (2) to represent some other physical phenomena having little or nothing to do with angles. In particular, we can let A stand for the time that the bob on the spring is in motion and let y stand for the displacement of the bob, and it may very well be that for the proper choice of F and D, the formula can represent the motion of the bob. To emphasize that the values of A in formula (2) may, in some applications at least, represent time values, we shall write

$$y = D \sin Ft \tag{3}$$

and now say that this function has a frequency F in an interval of 2π units of t and an amplitude D.

Up to this point we have been constructing the mathematical equipment needed to study the motion of bobs on springs. To recapitulate, it seemed from mere observation of the motion that some sort of periodic function would be needed to represent it, and we have therefore built up such functions by extending some concepts of trigonometry. Now let us see what all this mathematics will do for us in studying the motion of bobs on springs.

Hooke's great discovery in connection with the action of springs may be explained as follows. We all know from experience that if we stretch or compress a spring the spring exerts a force that tends to restore the normal length. Suppose d is the increase or decrease in length of the spring resulting from extension or contraction. Hooke found that the restoring force the spring exerts is proportional to d; that is, the force is a constant, k, say, times d. This is the meaning of the quotation from Hooke that heads this chapter. The value of the constant k depends upon the elasticity of the spring. Let us apply Hooke's law to the motion of a bob on a spring.

Suppose that the bob is attached to the lower end of the vertical spring shown in figure 93. Since gravity pulls the bob downward the spring will be extended by some amount, say d. The force that gravity exerts on the bob is of course the weight of the bob, and this is $32m$, where m is the mass of the bob. The restoring force, according to Hooke's law, is kd. The bob will settle in what we have called the rest position, which is fixed by the condition that the restoring force will just offset the pull of gravity. Hence

$$32m = kd. \tag{4}$$

But now suppose the bob is pulled further downward so that the new position of the bob is y units below the rest position (fig. 99).

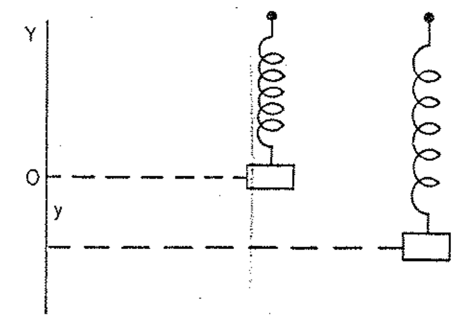


Fig. 99

Then the total extension of the spring is now $d - y$, the minus entering because y-values below the rest position are negative. The restoring force exerted by the spring is now $k(d - y)$. Since the weight of the mass acts downward, the net upward force is

$$kd - ky - 32m. \tag{5}$$

In view of equation (4) the net upward force acting on the mass m is $-ky$. When the bob is released it will begin to move under the action of the net force, $-ky$, exerted by the spring. Now Newton's second law of motion says that the force equals the mass times the acceleration created by the force. Hence

$$ma = -ky, \tag{6}$$

or

$$a = -\frac{ky}{m}. \tag{7}$$

We now have a mathematical law governing the motion of the bob. Let us therefore utilize the law to see what we can establish about this motion. When the motion begins, the bob is at some point below the rest position to which it has been pulled. At this position the velocity is zero and y is negative. Hence, according to equation (7), the acceleration is positive and the bob will begin to acquire positive velocity as it moves upward. This velocity will continue to increase because the acceleration remains positive until $y = 0$ when the bob is at the rest position. The acceleration is now zero but this does not mean that the velocity is zero. In fact the velocity, as we have just noted, has been increasing up to this point and hence the bob is moving very rapidly as it passes the rest position. Since the velocity there is not zero, the bob will continue to move upward past the rest position. But y is now positive and a is accordingly negative. The bob will lose velocity and in fact the larger y becomes the more negative is the acceleration. Hence the velocity will ultimately become zero and the bob will stop moving upward.

Will the motion then cease? No, because acceleration is still present and moreover negative. Hence the velocity will decrease below the zero value and become more and more negative. Physically, the negative velocity means that the bob will reverse direction and move downward. The y-values will now decrease from that of the highest position and the acceleration, though still negative, will likewise decrease in numerical value. Nevertheless, since acceleration is present and negative, the velocity will continue to become more and more negative. When the bob reaches the rest position on the way downward the velocity is highly negative and hence the bob is moving rapidly downward. But as y becomes negative, that is, as the bob falls below the rest position, the acceleration becomes positive. This means that positive increases are being made to a negative velocity and so the velocity becomes less and less negative and ultimately reaches the value zero. At this point the bob will stop moving downward. However, the bob will not remain motionless because the acceleration, now positive and large, will add positive increments to the velocity and so the velocity will increase from the zero value to positive values and the bob will move upward. The behavior of the bob from this position onward repeats the original behavior.

This analysis of formula (7) reveals several features of the motion. The motion is slowest, because the velocity is close to zero, when the bob is near its lowest or highest positions. The motion is fastest when the bob passes through the rest position on the way

upward and on the way downward. Moreover, the entire downward motion has all the features of the upward motion except, of course, for the direction. Finally, the entire upward and downward motion repeats itself because when the bob reaches the lowest position after the first oscillation, the acceleration, velocity, and position are precisely what they were to start with.

This tedious analysis serves one purpose at the moment. Near the beginning of this chapter we graphed the relationship between displacement and time of the bob moving up and down at the lower end of a spring, and we surmised from the appearance of the graph that the motion might be represented by a sinusoidal function. The more careful analysis based on formula (7) shows that the rate at which the displacement of the bob changes does indeed follow the rate at which the y-values of a function such as $y = D \sin Ft$ change. Of course familiarity with such sinusoidal functions is presupposed; this familiarity naturally comes through experience with such functions. The identification of the two functional relationships—the relationship between the displacement of the bob and time of motion on the one hand and $y = D \sin Ft$ on the other—is not mathematically demonstrated by our argument. In a later chapter we shall show how this identification can be established rigorously, but let us now rely upon the above evidence, which in fact is pretty much what Hooke used, to conclude that the bob does pursue a sinusoidal behavior. However, we still do not know what D and F of formula (3) should be to make the formula fit the motion of the bob. As to D, this is the maximum displacement, and its value is determined by how far below the rest position the bob is pulled before it is released. Hence we know D.

What will the frequency F of the motion be? Let us see what determines this frequency. One would expect that the elasticity or stiffness of the spring would be involved. Some springs stretch readily, others have to be pulled hard to stretch them. This property of the spring is represented in the constant k. If the spring is very stiff then k is large. According to formula (7) the larger k is, the greater the acceleration of the mass. Greater acceleration means that the bob will go through the cycles of changes in velocity and therefore in displacement more rapidly; that is, there will be more oscillations per second or the frequency will be larger. On the other hand, if a bob of larger mass is attached to a given spring, then formula (7) says that the acceleration will be less. Hence, for a given spring stiffness, the larger the mass the less the frequency. These considerations suggest that the frequency F of the oscillating mass is determined by k and m, and, moreover, that F must be large when k is large and that F must be small when m is large. To the mathematician these considerations suggest that a possible formula for F is k/m.

However, there is no reason why the formula could not as well be k^2/m^2 or k^3/m^3 or some other one that fits the foregoing considerations. With the mathematics presently at our disposal we could not determine the exact formula for F. An experimental physicist could, however, proceed as follows. He would first determine the value of k. That is, using a known mass he would observe how much the spring is stretched when the mass is attached. This increase in length is d. Then, knowing m and d, he would use formula (4) to calculate k. He would now set the mass in motion and observe what F is for the known k and m. He would find that

$$F = \sqrt{\frac{k}{m}}, \tag{8}$$

where F is the frequency or number of oscillations in 2π seconds. To verify experimentally that (8) is the correct formula he could try several masses on the same spring and also try several different springs.

In a later chapter we shall say more about the determination of F. Let us now, however, accept formula (8) on the basis of experimentation and recognize that we now have determined the D and F which we need to apply formula (3) to the motion of the bob on the spring. The formula that relates the displacement and time of the bob's motion is then

$$y = D \sin \sqrt{\frac{k}{m}}\, t, \tag{9}$$

if the time t is measured from the instant the bob moves upward from the rest position.

Since $\sqrt{k/m}$ is the frequency in 2π seconds, the frequency f in one second is of course

$$f = \frac{1}{2\pi}\sqrt{\frac{k}{m}}. \tag{10}$$

One more quantity is generally desired in connection with the motion of bobs on springs, namely, the period, or the time required to make one complete oscillation. If a man runs 10 feet in 1 second he requires $\frac{1}{10}$ of a second to run 1 foot. Likewise, if a bob makes f oscillations in one second it requires $1/f$ seconds to make one oscillation. Hence the period T is given by the formula

$$T = \frac{1}{f} = \frac{2\pi}{\sqrt{\frac{k}{m}}} = 2\pi\sqrt{\frac{m}{k}}. \tag{11}$$

The remarkable fact, which Hooke observed and emphasized, about formula (11) is that the period is independent of the amplitude. In other words, whether one pulls the bob down a small or large distance from its rest position and then releases it, the period of motion is the same. This result is not in accord with most people's intuitions. When the bob is pulled down a great distance D, the restoring force, according to formula (6), is $-kD$, wherefore the bob will shoot up rapidly and indeed move rapidly thereafter. But this does not mean that the period will be shorter. Actually, since the amplitude D of the motion will be larger, the bob will have to cover more distance in each oscillation and, though it moves rapidly, its period of oscillation will be the same as when the amplitude is small.

We have devoted much space to the derivation of formulas (9), (10), and (11), and it would now seem appropriate that we utilize these formulas to derive some new and useful or at least interesting information about the motion of bobs on springs. Indeed we could do so, but we shall refrain from capitalizing on these accomplishments until we have introduced a closely related phenomenon, the motion of a pendulum. Let us regard the work done thus far as an investment. We have learned some facts about a new type of mathematical function, the sinusoidal function, and we have seen that this function can represent the motion of bobs on springs. Perhaps this information can be turned to good advantage later. As a matter of fact our position is not greatly different from Hooke's. He learned, largely by experimentation, a great deal about the action of springs and sought to apply this knowledge to the construction of an accurate clock. Although he did not succeed in producing the first successful spring-regulated clock, his contributions did ultimately make this device possible.

While Hooke was studying the action of springs, Galileo and Huygens were exploring another device that had attracted Galileo's attention when he was a youth. Galileo had observed the swinging of lamps suspended from long cords in churches. By using his pulse beat to time the swings he found that the time required for a complete back and forth motion, that is, a complete oscillation, was the same whether the lamps swung through wide arcs or narrow ones. This observation excited Galileo's interest in the pendulum, though at the time he did not envisage its use in a clock. What did interest him was that the pendulum seemed to have some special properties. Moreover, because its motion is not hindered as much by air resistance as is the motion of objects sliding or rolling down planes, observations of pendulum motion might agree far better with theoretical studies of idealized situations. Hence Galileo proceeded to study the pendulum.

Galileo's contemporaries thought that the pendulum was kept in its regular motion by the action of the air. But Galileo dismissed this idea because, he said, "in that case, the air must needs have considerable judgment and little else to do but kill time by pushing to and fro a pendent weight with perfect regularity." Just as he ignored the presence of air in his study of straight-line and projectile motion, Galileo decided to study the action of pendulums in a vacuum.

Following Galileo, we shall suppose that a bob is hung at one end of a length of string, and the other end is held fixed. The bob is set in motion by pulling it to one side and then releasing it. What does mathematics tell us about the subsequent motion?

We shall denote by A (fig. 100) the angle the string makes with the vertical when the bob is in some arbitrary position during its motion. If the bob is to the right of the vertical then A will be

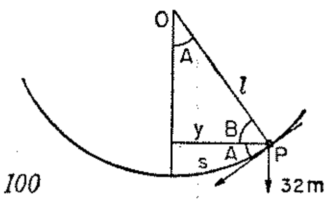


Fig. 100

understood to have positive values, and if to the left, negative values. The bob moves along the arc of a circle, and the direction of its motion at any position P is along the tangent at that position. The tangent at P is perpendicular to the string OP because OP is a radius of the circle. Hence the two angles marked A in figure 100 are equal because both are complements of angle B.

What force acts on the bob? If m is the mass of the bob, the force of gravity pulls the bob straight down with a force of $32m$, which is, of course the weight of the bob. But the bob is not free to fall straight down because the string resists somewhat the pull of gravity. The bob can, however, move in the direction of the tangent, which, as noted, makes an angle A with the horizontal. We may then regard the bob as moving along an inclined plane that makes an angle A with the horizontal. In our study of the motion of an object sliding down an inclined plane we learned that the effectiveness of the force of gravity along the direction of the plane is not $32m$ but $32m \sin A$. Hence this latter quantity is the force acting on the bob and causing it to move along the circle.

There is, however, a matter of sign. When the bob is to the right of its rest position, the force $32m \sin A$ acts to the left; when the bob is to the left of the rest position, the force causes it to move to the right. Thus the force is opposite to the displacement. Hence we must represent the force as $-32m \sin A$. Since this force is, according to Newton's second law, the mass times the acceleration of the bob, we have

$$ma = -32m \sin A$$

or

$$a = -32 \sin A.$$

If we refer to figure 100 we see that

$$\sin A = \frac{y}{l},$$

where y is the *horizontal* displacement of the bob. Hence

$$a = \frac{-32y}{l}. \tag{12}$$

Equation (12), though correct, is not quite what we should like. It says that the acceleration of the bob is a constant, $32/l$, times the horizontal displacement of the bob. But the bob does not move horizontally. Rather, it moves along the arc of a circle and the distance it actually moves along this arc is the length s shown in figure 100. To relate the acceleration to the actual distance moved we shall consciously make an approximation of the type we made in our study of light and, at the moment, hope for the best insofar as the usefulness of our result is concerned. The quantity y is half the chord of the arc $2s$. When the arc is small the chord is a good approximation to the arc. Hence if we restrict the theory to oscillations of small amplitude, we can with some reasonableness replace y by s in (12) and obtain

$$a = -\frac{32}{l}s. \tag{13}$$

This formula gives the acceleration acting on the bob in terms of the *actual* displacement of the bob from the rest position, that is, from the vertical.

We are now going to take advantage, in a big way, of the abstractness of mathematics. We have seen, for example, that the same geometric theorem can apply to many quite different physical situations. The same is true of formulas. What acceleration dictated the motion of the bob on the spring? Formula (7) gives the answer. Now formula (13) governs the motion of the bob on the pendulum, and the two formulas are identical except for notation. Just as k and m were constants in the case of the motion of the first bob, so 32 and l are constants in the motion of the second one. Moreover, just as y in formula (7) is the actual displacement of the bob on the spring, so s in formula (13) is the actual displacement of the second bob. Since the same mathematical formula governs the acceleration in the two cases, and since the acceleration determines the motion, we may conclude that the mathematical deductions from that formula apply to both cases. Hence, in view of

formula (9), the relationship between the displacement and time in the case of the motion of the bob on the pendulum is

$$s = D \sin\sqrt{\frac{32}{l}}\, t, \tag{14}$$

where D is now the maximum displacement of the bob, that is, the largest value of s, and t is time measured from the instant the bob starts to move to the right from the lowest position. It follows from formula (10) that the frequency of the bob, that is, the number of oscillations or complete swings it makes in one second is

$$f = \frac{1}{2\pi}\sqrt{\frac{32}{l}}, \tag{15}$$

and from formula (11) we see that the period of the pendulum is

$$T = 2\pi\sqrt{\frac{l}{32}}. \tag{16}$$

Before we discuss formulas (14) to (16) let us emphasize an aspect of the reasoning that led to them. Because the same formula for acceleration acts in two different situations, and the acceleration determines the final formula for the motion, we conclude that the same final formula must apply in the two different situations. The abstractness of the mathematical formulas pays unexpected dividends, no doubt as compensation for our willingness to think abstractly.

Formula (16) shows exactly what Galileo had observed, namely, that the period of the pendulum is independent of the amplitude or maximum displacement of the bob. Thus, whether the bob is pulled aside to make an angle of 30° with the vertical or so as to make an angle of 5° with the vertical and then released, the period of the subsequent swings is the same. We must remember, however, and we shall return to this point later, that, in the case of the pendulum, formulas (14), (15), and (16) are only approximately correct because we replaced a chord of a circle by an arc, and this approximation is reasonably accurate only for small amplitudes of oscillation. However, if the pendulum can be kept in motion at about the same amplitude, as is done in pendulum clocks by making a weight that falls gradually under the action of gravity apply a force to a mainspring, then the period of the pendulum will be practically constant, though not given exactly by formula (16).

Another deduction from (16), which Galileo also made, is that the period is independent of the mass of the pendulum. Whether the bob is made of lead or cork the period will be the same.

Formula (16) reveals the immense usefulness of the pendulum. By knowing the acceleration due to gravity, the quantity 32 in the formula, and the length l of the string to which the bob is attached, one can predict the period of the pendulum and hence measure time. Moreover, by fixing l, one can make the period whatever he wishes. Many years after Galileo studied the pendulum to learn some facts about its motion he designed a practical pendulum clock and had his students construct one. About fifteen years later, Huygens, working independently, constructed one that gained wide acceptance. Of course pendulum clocks do not work well on rolling ships and so the search for a reliable spring clock continued long after Galileo's and Huygens' time.

In our treatment of the pendulum, as well as in that of the motion of the bob on a spring, we used the fact that the force of gravity acting on a mass m is $32m$. The quantity 32 is of course the acceleration due to gravity at points near the surface of the earth. We know, however, that the acceleration of masses under the pull of gravitation is not always 32. In fact the acceleration is, more generally, the quantity GM/r^2—formula (2) of chapter 16—where G is the gravitational constant, M is the mass of the earth, and r is the distance of the mass being accelerated from the center of the earth. Then the acceleration varies with distance from the center of the earth. Hence, if a pendulum clock is adjusted so that it beats seconds at one location, it will not be accurate at another location if the acceleration due to gravity is different there. The pendulum clock may have to be readjusted if it is moved from one location to another.

It is an ill wind, however, that blows no good. The fact that the period of the pendulum depends upon the acceleration due to gravity was turned to excellent advantage by seventeenth- and eighteenth-century mathematicians and scientists. To follow their work let us first rewrite formula (16). In our derivation of this

result we used 32 for the acceleration due to gravity. Let us now reconsider the derivation with the quantity 32 replaced by the symbol g, which is the customary symbol for the acceleration due to the direct action of gravity, as opposed to the symbol a, which is used for any acceleration, whether due to gravity or some other force. Then the derivation would continue to hold as long as g is constant, which it is in any one location, and formula (16) would read

$$T = 2\pi\sqrt{\frac{l}{g}}. \tag{17}$$

Now suppose that by adjusting l, the period T of a pendulum is made to be one second at a location where g is, say, 32. If the pendulum is moved to another location and if the period is now measured, then formula (17) may be used to calculate g. Hence the acceleration due to gravity can be measured at a variety of locations. This method is very accurate and is used today for precise determinations of g.

As the pendulum is moved from either pole toward the equator it is found that g decreases from the value of 32.257 ft/sec^2 at the poles to 32.089 ft/sec^2 at the equator. This difference, small as it may seem, must be explained.

Newton had considered the shape of the earth. On the assumption that it was homogeneous throughout, that is, it possessed the same density all the way down to the center, and that it was subject to the attraction of the sun and to rotation, he proved that the earth's shape is not a sphere but an ellipsoid of revolution, roughly a sphere flattened somewhat at the top and bottom. Hence an object on the surface of the earth is farthest from the center at the equator and closest to the center at the poles. Since r, the distance from the center of the earth, decreases the closer one is to the poles, the greater g must be as one moves from the equator to either pole. Thus far at least the argument agrees with the measured values of g. However, according to the calculations of Newton and Huygens and the more accurate later calculations of the mathematician Clairaut, the flattening of the earth at the poles is not sufficient to account for the observed variation in g; that is, the distance from the center of the earth to either pole is not sufficiently less than the distance to the equator to account for the larger value of g at the poles.

The full explanation of the observed variation in g was given by Huygens. An object whirled at the end of a string tends to fly off into space and follow the straight-line path that the first law of motion says all objects seek to follow. That the object continues to move in a circle is due to the fact that the hand keeps pulling the object into the center. The force the hand exerts is called a centripetal (center-seeking) force. Objects on the surface of the earth are whirled around by the rotation of the earth and hence are subject to a centripetal force. Where does this force come from? When we examined the motion of the moon around the earth we pointed out that the earth pulls the moon toward the earth and thus prevents the moon from moving along a straight-line path. In other words, the gravitational attraction of the earth supplies the centripetal force. Let us now consider an object on the earth. If the earth were not rotating, it would still of course, exert its gravitational force on the object; and this force would be the weight of the object. But since the earth rotates and keeps objects rotating along with it, the gravitational force must be used partly to supply the centripetal force. Hence the weight of the object is reduced. Since the weight of an object near the surface is gm, a reduction in weight must mean a reduction in g compared to the value on a stationary earth.

Why, then, does the measured g vary from the equator to the poles? An object on the equator is whirled on a circle whose radius is the radius of the earth. As one goes north or south from the equator the circle on which an object is whirled by the rotation of the earth shrinks in size; this circle is in fact the circle of latitude at the object's position. At the poles the circle shrinks to zero. Let us take into account now that an object traveling with velocity v on a circle of radius r is subject to the centripetal acceleration of v^2/r—see chapter 16, formulas (15) and (22). The velocity of an object rotating with the earth is the circumference of the circle on which it travels divided by the time, that is $2\pi r/T$. Hence

$$a = \frac{v^2}{r} = \frac{4\pi^2 r^2}{T^2 r} = \frac{4\pi^2 r}{T^2}. \tag{18}$$

The time of revolution is 24 hours, or 86,400 seconds, no matter where the object is. But the radius of the path depends upon the latitude of the object. Thus $r = 4000$ miles at the equator and $r = 0$ at the poles. Evidently the centripetal acceleration is greatest at the equator and decreases to zero at the poles. The centripetal force exerted by the earth on a mass m is just m times the centripetal acceleration. Since the centripetal force must be subtracted from the total gravitational force obviously the net gravitational force, that is, mg, should be less at the equator than at the poles. Thus the values of g calculated by means of the formula for the period of a pendulum confirm the rotation of the earth, or may be regarded as independent evidence for the rotation.

The value of g varies from place to place on the earth for still another reason. The earth's mass would attract any object as if the mass were concentrated at the center, but only if the earth's mass were uniformly distributed. But large deposits in one area of iron ore, oil, or other highly dense substances disturb the uniform distributions of the earth's mass and cause the value of g to vary. Such variations in g can be calculated through the action of a pendulum, as noted above, and departures from the expected values at a given locale indicate to geologists information as to the local matter in the earth. Detection of such variations in g is one of the methods of geophysical prospecting.

There is a postscript to the story of pendulum motion, written by Huygens. We pointed out above that the period of a pendulum is not strictly independent of the amplitude of swing. Huygens thereupon set for himself the problem of determining the curve along which a bob might swing, such that the period would be exactly the same whether the bob traversed a long or a short arc. It so happens that the curve Huygens found to be the solution of this problem was already being intensively studied by many mathematicians, largely as an intellectual challenge.

The curve itself is actually before our eyes daily, though we do not take the trouble to see it, much less investigate its mathematical properties. Suppose one fixes his attention on a point on the circumference of an automobile tire. The path *of that point* (fig. 101, p. 296) as the tire rolls along the ground is known as a cycloid. The

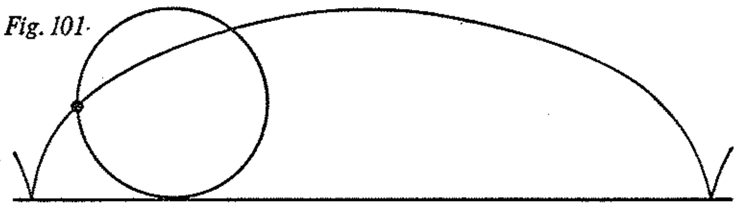


Fig. 101

figure shows what is called one arch of the cycloid; the full curve consists of repeated arches.

Let us visualize this curve as inverted (fig. 102), and let an object slide along this curve under the action of gravity. Huygens proved that no matter at what point A the mass starts it will reach the lowest point O in the same time and will go on to reach the symmetrically located point A′ in that same time.

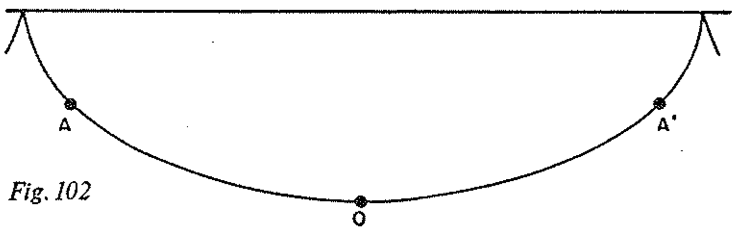


Fig. 102

This mathematical property of the cycloid is of considerable interest in itself, but Huygens, concerned with designing a clock, was not interested in sliding objects. If he could get the bob of a pendulum to follow such a curve under the action of gravity, then he would have the essential device for the truly accurate clock he was seeking. Huygens found the answer to this question through

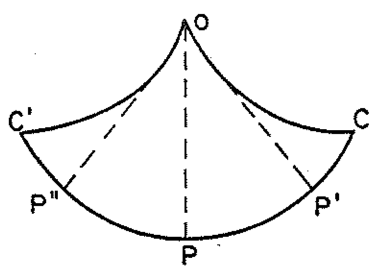


Fig. 103

further mathematical work on the cycloid. Let OC and OC' be identical half cycloidal arches (fig. 103), and OP be a string with a bob at P. As the bob is swung to the right the string will wind itself partly around the cycloidal arc OC and the rest of the string will be pulled taut by the pull of gravity on the bob. Huygens proved that if the length of OP is twice the height of either cycloid OC or OC' then the locus PP' of the bob will also be an arc of a cycloid. Likewise, as the bob swings to the left the path PP'' will be an arc of the same cycloid with P as the lowest point. Mathematically speaking, the arc OC is called the evolute of the cycloidal arc $PP'C$ and OC' is the evolute of the cycloidal arc $PP''C'$.

Hence Huygens had found the way to make the bob swing along the arc of a cycloid and had designed the perfect pendulum. The cycloid is often referred to as the isochronous curve, or the tautochronous curve, because the period of swing of a pendulum following a cycloidal path is truly independent of the amplitude of the swing.