2.1.

$\dot{x} = \sin x$ (1)

$\frac{dx}{dt} = \sin x.$

$dt = \frac{dx}{\sin x}.$

$\int_0^t dt' = \int_0^x \frac{du}{\sin u}$

$t = \int_0^x \frac{\sin u}{\sin^2 u}\,du$

$= \int_0^x \frac{\sin u}{1-\cos^2 u}\,du$

$\cos u = w$ とおく.

$-\sin u\,du = dw.$

u	0	$\to x$
w	1	$\to \cos x$

$\Leftrightarrow t = -\int_1^{\cos x} \frac{dw}{1-w^2}$

$= -\frac{1}{2}\left[\int_1^{\cos x} \frac{dw}{1+w} + \int_1^{\cos x} \frac{dw}{1-w}\right]$

$\left(\frac{1}{(1+w)(1-w)} = \frac{a}{1+w} + \frac{b}{1-w} \right.$

$1 = a(1-w) + b(1+w).$

$(w \to 1)\ 1 = 0 + b \cdot 2$

$\therefore b = \frac{1}{2}$

$(w \to -1)\ 1 = a(2) + 0.$

$\left. \therefore a = \frac{1}{2} \right)$

(w→1)

$t = -\frac{1}{2}\left(\left[\log|1+w|\right]_1^{\cos x} - \left[\log(1-w)\right]_1^{\cos x}\right)$

$= -\frac{1}{2}\Big(\log|1+\cos x| - \log|1-\cos x|$

$- \log 2 + \log 0 \Big).$

$= -\frac{1}{2}\left(\log\left|\frac{1+\cos x}{1-\cos x}\right|\right) + \underbrace{\frac{1}{2}\log\frac{0}{2}}_{\text{const}}$

const より Cとおく.

$\therefore t = \frac{1}{2}\log\left|\frac{1-\cos x}{1+\cos x}\right| + C.$

$(C: \text{const.})$

The initial condition exists

$x = x_0$ where $t = 0.$

Then

$0 = \frac{1}{2}\log\left|\frac{1-\cos x_0}{1+\cos x_0}\right| + C.$

$C = -\frac{1}{2}\log\left|\frac{1-\cos x_0}{1+\cos x_0}\right|$

The solution of the equation (1) is

$t = \frac{1}{2}\log\left|\frac{1-\cos x}{1+\cos x}\right| - \frac{1}{2}\log\left|\frac{1-\cos x_0}{1+\cos x_0}\right|$ (2)

Now,

(1) $x_0 = \frac{\pi}{4}$, for any $t > 0$.

What is charactor of (2), especialy $t \to \infty$.

$t = \frac{1}{2}\log\left|\frac{1+\cos x}{1-\cos x}\right| - \frac{1}{2}\log\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)$

?

(2) for any initial conditions, x_0,

$t \to \infty$. then

How move

$t = x(t).$

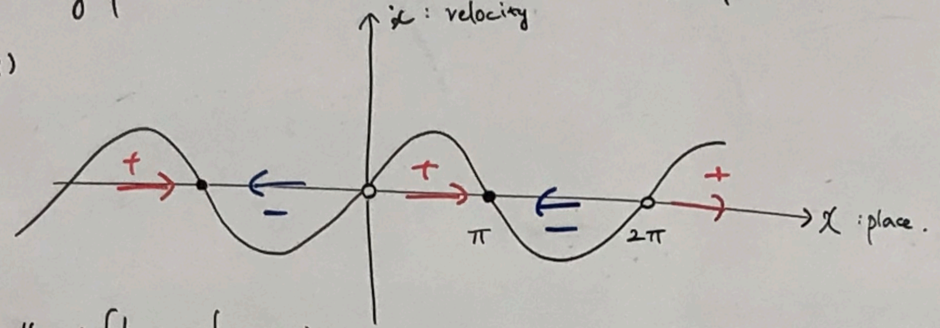
?

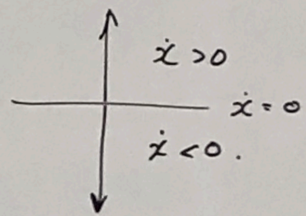
⟶ These are too difficult...

ふごう
まちがい
あり

In contrast, a graphical analysis of (1) is clear and simple as shown in Figure followed:

$\dot{x} = \sin x$. (1)





an allow : flow , $\begin{cases} \rightarrow \dot{x} > 0. \\ \leftarrow \dot{x} < 0 \end{cases}$

簡単なイメージ

FIXED POINT

- ● solid black point : stable fixed points , attractors or repellers → end points of flows
- ○ open circle : unstable fixed points , sinks or sources (湧点) ($\because$ There is no flow x, where $\dot{x} = 0$) → start points of flows

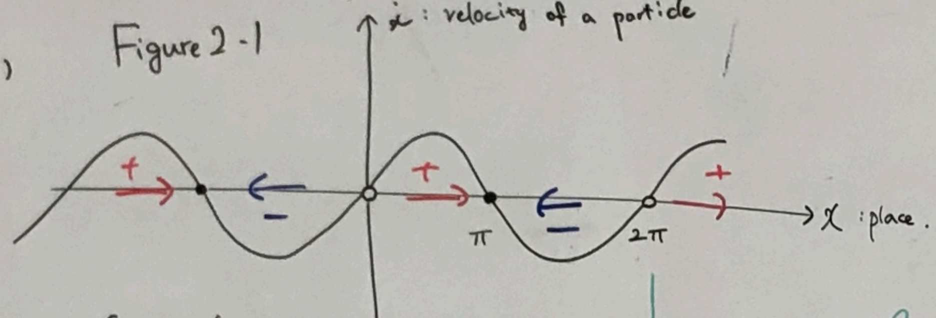
With this Figure,

We can now easily understand the solutions to the differential equation $\dot{x} = \sin x$

We just start our imaginary particle at x_0 and watch how it is carried along by the flow.

$\dot{x} = \sin x$. (1)

Figure 2-1



an allow : flow , $\begin{cases} \rightarrow \ \dot{x} > 0, \\ \leftarrow \ \dot{x} < 0. \end{cases}$

This approach allows us to answer the questions above as follows :

(1) From Figure 2-2 helps to understand a trajectory of the particle.

and Figure 2-3 shows one.

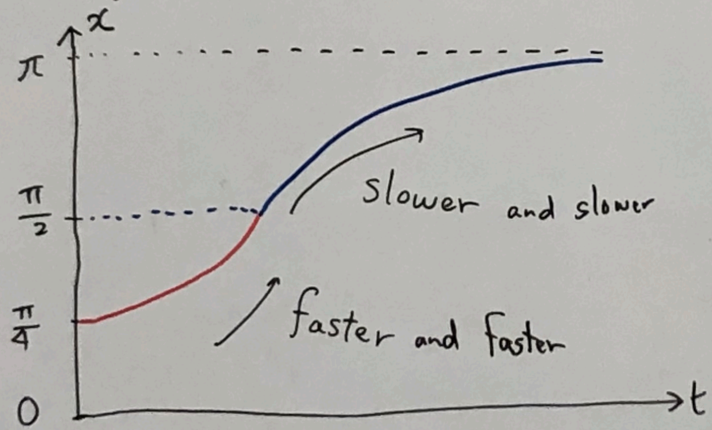
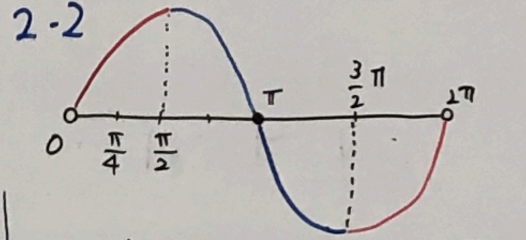


Figure 2-3.

$t \to \infty$, the particle moves and stable $x = \pi$

Figure 2-2 flow → ←



$\frac{d\dot{x}}{dt} = \ddot{x}$: Accelelation ⊕ ⊖

if $\ddot{x} > 0$ (⊕).

a particle moves faster and faster until $\frac{2n+1}{2}\pi$ $(n = 0, 1, 2, \cdots)$.

if $\ddot{x} < 0$ (⊖).

a particle moves slower and slower until $(2n+1)\pi$ $(n = 0, 1, 2, \cdots)$

(2) For any initial condition, x_0. By same way (1), figure 2-3, we can understand the particles trajectories.

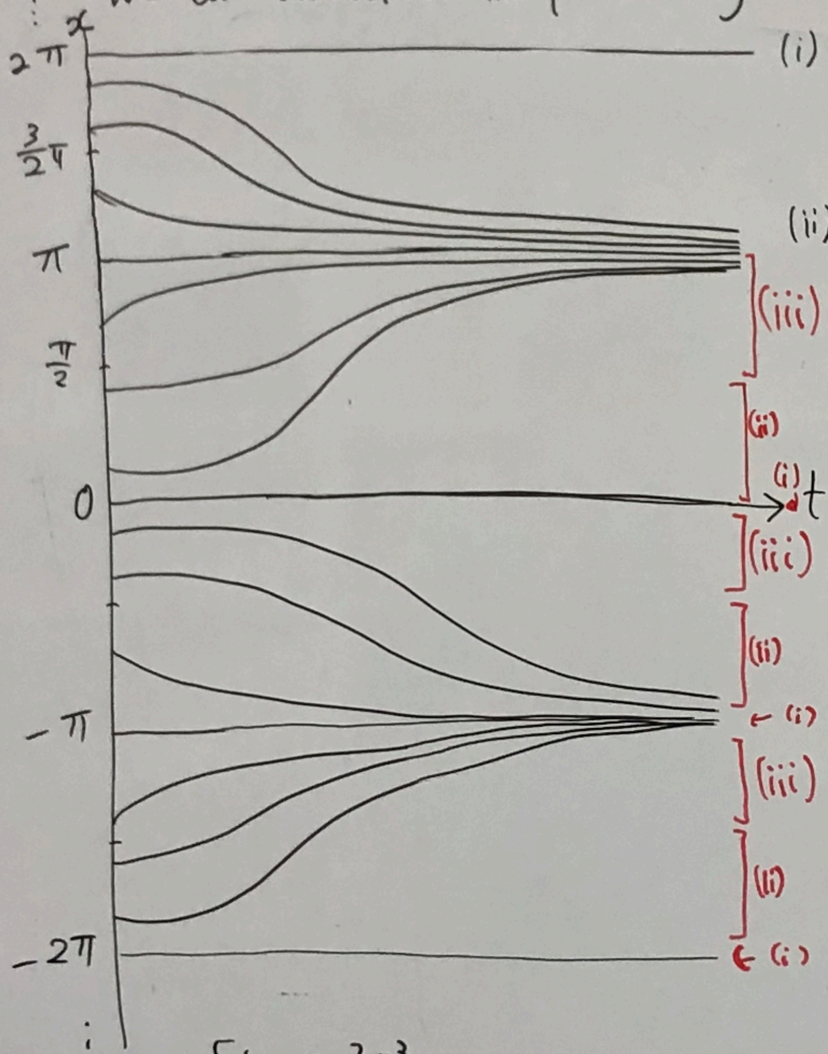


Figure 2-3

(i) $x_0 = 2n\pi$ $(n=0,1,2,\cdots)$, e.g. $x_0 = 0$, $x_0 = \pi$

the particles do not move.

(ii) $2n\pi < x_0 < \frac{2n+1}{2}\pi$ $(n=0,1,2,\cdots)$. e.g. $x_0 = \frac{\pi}{4}$

First, the particles moves to $x = 2(n+1)\pi$ faster and faster until $x = \frac{2n+1}{2}\pi$

Next, from $x = \frac{2n+1}{2}\pi$, these velocities slow down until $x = 2(n+1)\pi$

Finally, $t \to \infty$, they will be stable at $x = 2(n+1)\pi$.

(iii) $\frac{2n+1}{2}\pi < x_0 < 2n\pi$ $(n=0,1,2,\cdots)$ e.g. $x_0 = \frac{3}{2}\pi$

The particles moves to $x = 2n\pi$ and slower and slower.

Afterwards, $t \to \infty$, they also become stable at $x = 2n\pi$.

EXERCISES FOR CHAPTER 2-1.

(みなす. 解釈する)

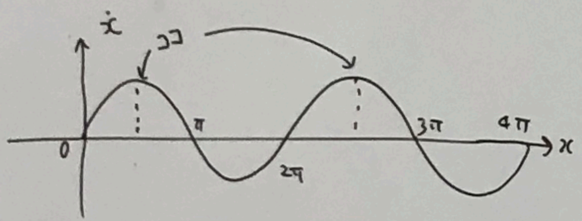
In the next three exercises, interpret $\dot{x} = \sin x$ as a flow on the line.

2.1.1 Find all the fixed points of the flow.

No flows are $\dot{x} = 0$. So we can find x, where $\sin x = 0$.

$$\therefore \quad x = n\pi \quad (n = 0, 1, 2, \cdots).$$

2.1.2. At which points x does the flow have greatest velocity to the right?



$\ddot{x} = 0$ is greatest velocity. Now, a direction of particles is right,

Then the range is $[2n\pi, (2n+1)\pi]$ $(n = 0, 1, 2, \cdots)$. $(\because \dot{x} > 0)$

The maximum velocitys are $x = \frac{5n+1}{2}\pi$ $(n = 0, 1, 2, \cdots)$

EXERCISES FOR CHAPTER 2-1.

2.1.3.

a) Find the flow's acceleration $\ddot{x}$ as a function of x.

$$\ddot{x} = \frac{d\dot{x}}{dt} = \frac{d\dot{x}}{dx} \cdot \frac{dx}{dt} = \frac{d}{dx}(\sin x) \cdot \sin x = \cos x \sin x = \frac{1}{2} \sin 2x$$

b) Find the points x where the flow has maximum positive acceleration.

Same solution at 2.1.2.

EXERCISES FOR CHAPTER 2-1.

$$t = \frac{1}{2}\log\left|\frac{1-\cos x}{1+\cos x}\right| - \frac{1}{2}\log\left|\frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}\right|$$

$$\frac{\sqrt{2}-1}{\sqrt{2}+1}$$

$$\frac{1-\cos x}{1+\cos x} = \tan^2\left(\frac{x}{2}\right)$$

$$= \frac{1}{2}\log\left|\frac{1-\cos x}{1+\cos x}\right| + \frac{1}{2}\log\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right|$$

rationalizing the denominator

$$3+2\sqrt{2}$$

and

$$\sqrt{3+2\sqrt{2}} = 1+\sqrt{2}$$

$$= \log\left(\tan\frac{x}{2}\right) + \log\left(1+\sqrt{2}\right).$$

$$e^t = \left(\tan\frac{x}{2}\right)\left(1+\sqrt{2}\right).$$

$$\frac{e^t}{1+\sqrt{2}} = \tan\frac{x}{2}$$

$$\therefore\ x(t) = 2\,\mathrm{Arctan}\left(\frac{e^t}{1+\sqrt{2}}\right)$$

EXERCISES FOR CHAPTER 2-1.

2.1.4

(a)

$$x(t) = 2\tan^{-1}\left(\frac{e^t}{1+\sqrt{2}}\right) \qquad (\tan^{-1}\alpha = \mathrm{Arctan}\,\alpha)$$

$(t \to \infty)$

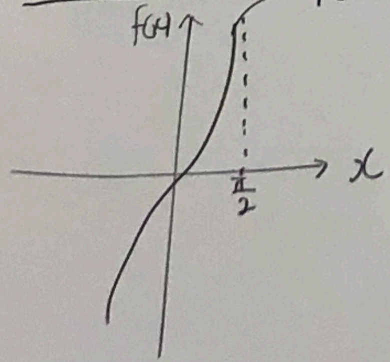
$$\tan\left(\frac{x(t)}{2}\right) = \frac{e^t}{1+\sqrt{2}}.$$

$t \to \infty$ のとき.

$$(右辺) = \infty.$$

$$\Longleftrightarrow \tan\left(\frac{x(t)}{2}\right) \to \infty$$

$\therefore x = \pi$ $f(x) = \tan x.$



EXERCISES FOR CHAPTER 2-1.

2.1.4

(b)

$$t = \frac{1}{2}\log\left|\frac{1-\cos x}{1+\cos x}\right| - \frac{1}{2}\log\left|\frac{1-\cos x_0}{1+\cos x_0}\right|$$

$$= \frac{1}{2}\log\left(\tan^2\frac{x}{2}\right) - \frac{1}{2}\log\left(\tan^2\frac{x_0}{2}\right).$$

$$= \log\left(\frac{\tan\frac{x}{2}}{\tan\frac{x_0}{2}}\right).$$

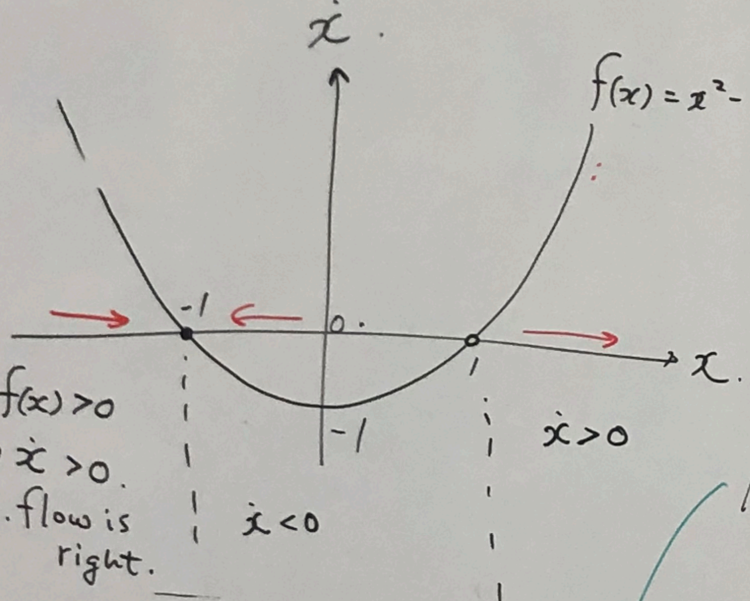
$$e^t\tan\frac{x_0}{2} = \tan\frac{x}{2}.$$

$$x(t) = 2\tan^{-1}\left(e^t\tan\frac{x_0}{2}\right).$$

EXAMPLE 2.2.1.

Find all fixed points for $\dot{x} = x^2 - 1$, and classify their stability.

$$\begin{cases} \dot{x} = f(x) \\ f(x) = x^2 - 1 \end{cases}$$



$f(x) > 0$

$\Leftrightarrow \dot{x} > 0$.

$\therefore$ flow is right.

the fixed points (equilibrium solutions), x^*.

$x^* = -1, 1$.

$\rightarrow \begin{cases} x^* = -1 & \text{is stable,} \\ x^* = 1 & \text{is unstable.} \end{cases}$

From 2.1.

$\dot{x} = \sin x$

extend to any 1-dimensional system

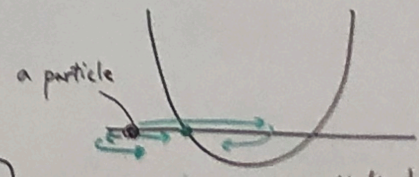
$\dot{x} = f(x)$.

NOTE:

The definition of stable equibrium is based on small disturbances; certain large disturbance may fail to decay (減衰).

Intuition

It does not move to stable fixed point.



Stable becaus a particle have only small disturbances.

2.7.2

$$\dot{x} = x - x^3 \qquad \frac{dV}{dx} = -x + x^3 = 0$$

$$f(x) = -\frac{dV}{dx} = x(1+x)(1-x) \qquad x(x^2-1) = 0$$

$$-\int dV = \int (x - x^3)\,dx$$

$$-V = \frac{x^2}{2} - \frac{x^4}{4} - C$$

$$V = -\frac{x^2}{2} + \frac{x^4}{4} + C$$

Set $C = 0$

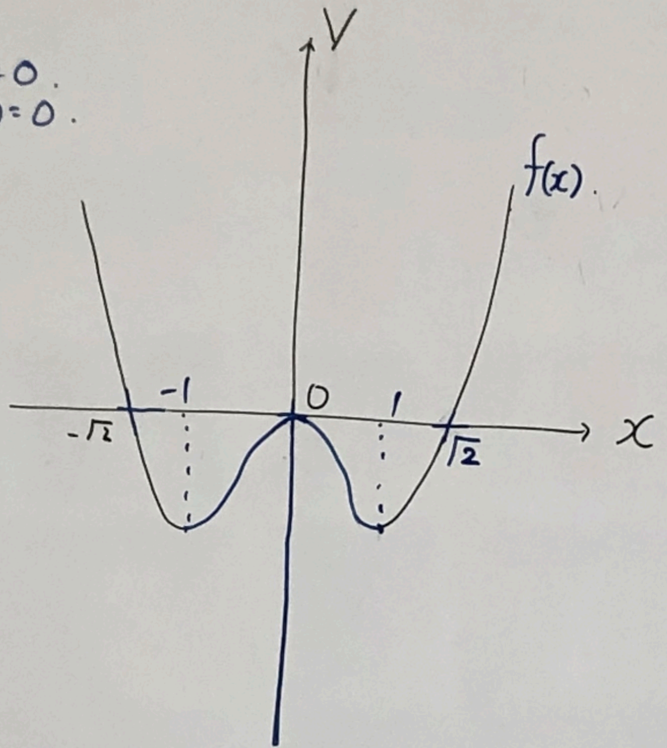
$$V = \frac{1}{2}\left(-1 + \frac{x^2}{2}\right)x^2$$

$$= \frac{1}{4}(x^2 - 2)x^2$$

$$= \frac{x^2}{4}(x+\sqrt{2})(x-\sqrt{2})$$

M or W (trajectory)

$$\boxed{\begin{aligned} &\text{Set } x = 3\sqrt{2} \quad V = \frac{(3\sqrt{2})^2}{4}(3\sqrt{2}+\sqrt{2})(3\sqrt{2}-\sqrt{2}) \\ &= \frac{9 \cdot 2}{4}(3+1)\sqrt{2} \cdot (3-1)\sqrt{2} \\ &= \frac{9}{2} \cdot 2\,(4)(2) \\ &= 72 > 0 \end{aligned}}$$



This potential is called a double-well potential. (2重井戸ポテンシャル, ワインボトル型ポテンシャル)

The system is said to be bistable, since it has two stable equilibria.

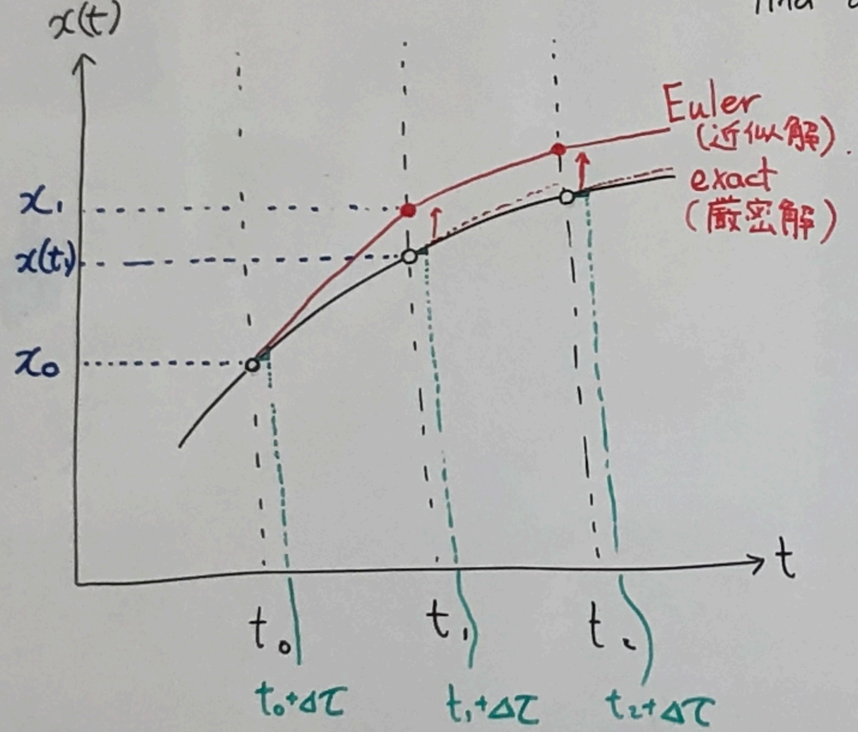
The local minima at $x = \pm 1$ correspond to stable equilibria.

The local maximum at $x = 0$ correspond to unstable equilibrium.

2.8. Solving Equations on the Computer

Euler's Method

The problem can be posed this way : given the differential equation $\dot{x} = f(x)$,
subject to the condition $x = x_0$ at $t = t_0$,
find a systematic way to approximate the solution $x(t)$.



$t_i - t_{i-1} = \Delta t$

Suppose we use the vector field interpretation of $\dot{x} = f(x) \Leftrightarrow f(x)$: velocity at location x.

The rule of Euler's Method.

$$x(t_0 + \Delta t) \approx x_1 = x_0 + f(x_0)\Delta t$$

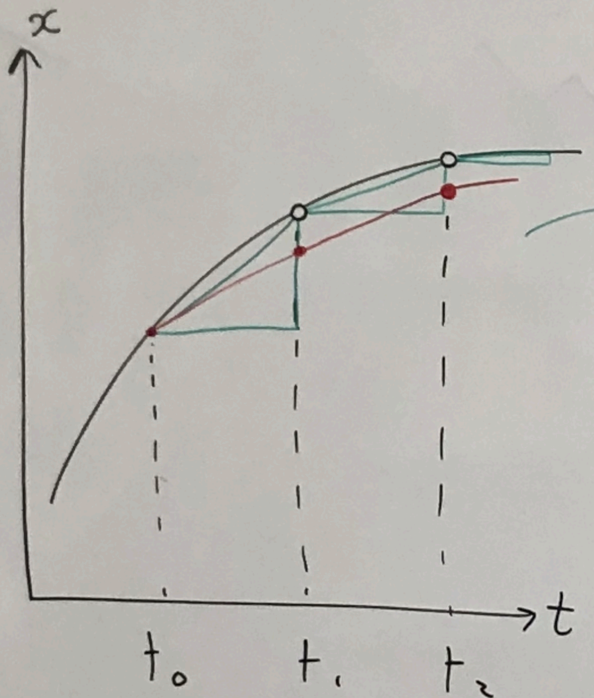
$$x_{n+1} = x_n + f(x_n)\Delta t$$

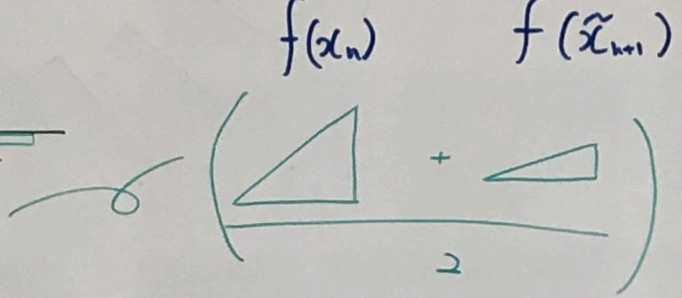
The approximation gets bad in a hurry unless Δt is extremely small.

Next, it contains the conceptual essence of more accurate methods to be discussed.

2.8. Solving Equations on the Computer

Euler's Method : Refinements





$\Rightarrow$ new Velocity

$g(x_n)$

Improved Eular's Method

$$\tilde{x}_{n+1} = x_n + f(x_n)\Delta t$$

$$x_{n+1} = x_n + g(x_n)\Delta t$$

$$g(x_n) = \frac{1}{2}\left[f(x_n) + f(\tilde{x}_{n+1}) \right]$$

2.8. Solving Equations on the Computer

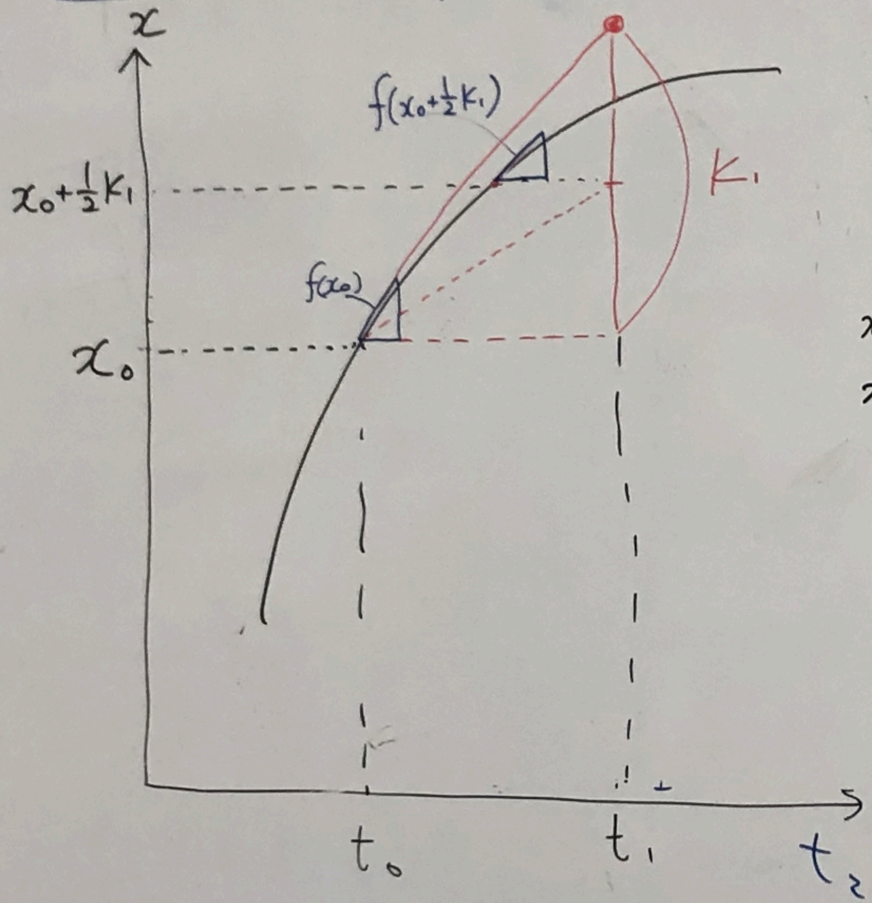
Euler's Method: 1st-order

Improved Euler's Method: 2nd-order

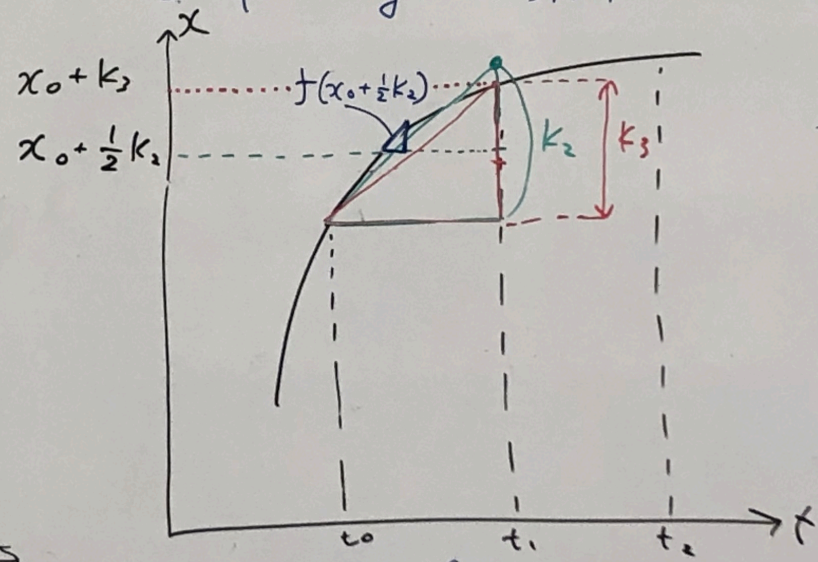
Runge-Kutta method (4th-order)

Intuition Step 1: get k_1, k_2

$$\begin{cases} k_1 = f(x_n)\,\Delta t \\ k_2 = f(x_n + \frac{1}{2}k_1)\,\Delta t \\ k_3 = f(x_n + \frac{1}{2}k_2)\,\Delta t \\ k_4 = f(x_n + k_3)\,\Delta t \end{cases}$$



Step 2: get k_3, k_4



Step 3: Get x_{n+1}

$$\boxed{x_{n+1} = x_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)}$$