State Space Control

The primary disadvantage of the classical approach (using Frecuency domain representation taking the system's [[Transfer Function]] with Laplace) is its limited applicability: It can be applied only to linear, time-invariant systems or systems that can be approximated as such.

The state-space approach can be used to represent nonlinear systems that have backlash, saturation, and dead zone.

Also multiple systems have multiple inputs and outputs that can be compactly represented in state-space-

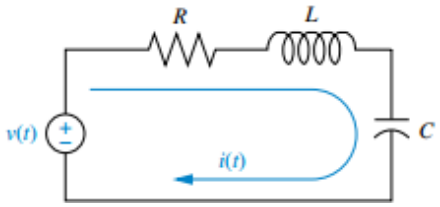
The time-domain approach can also be used for the same class of systems modeled by the classical approach.

Aproach

1. Select a particular subset of all possible system variables and call them the *state* variables.
2. for a *n*th order system we write n simultaneos, first order diferential equations in terms of state variables. we call this system of simultaneus diferential equations **State Equations**
3. if we know the inital condition of all the states variables at t_0 as well as the system input for $t \geq 0$, we can solve the simultaneous DEs for the states variables for $t \geq 0$
4. We algebraically combine the state variables with the system's input and find other system variables for $t \geq t_0$, we call this equation the **Output Equation**
5. We consider the state equations and the output equations a viable representation of the system. we call this representation of the system a **State-Space Representation**

Example:

Let's say we have the following circuit:



We then have:

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = v(t)$$

we can convert this to charge:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = v(t)$$

Then we turn this into a system of equations:

$$\frac{d\mathbf{q}}{dt} = \begin{pmatrix} \dot{q} \\ \ddot{q} \end{pmatrix} = \begin{pmatrix} i \\ \frac{di}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{1}{LC} & \frac{-R}{L} \end{pmatrix} \begin{pmatrix} q \\ i \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{L} \end{pmatrix} v(t)$$

General Representation

The **State Equations** are a set of n simultaneous first oredr ODEs with n variables, where the variables to be solved for are the state variables.

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

where:

- $\mathbf{x}$ = state vector
- $\dot{\mathbf{x}}$ = Derivative respect time of state vector
- $\mathbf{u}$ = input or control vector
- $\mathbf{A}$ = system matrix
- $\mathbf{B}$ = Input Matrix

And the **output equation** it's the algebraic equation that expreses the output of the system as a linear combinations fo the state variables and the inputs.

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}$$

where:

- $\mathbf{y}$ = Output vector
- $\mathbf{C}$ = Output Matrix
- $\mathbf{D}$ = Feedfoward matrix

Applying State-Space Representation.

the first step it's tow select the state vector with the following considerations:

1. A minimum number of state variables must be selected as components of the state vector. and this number must be sufficient to describe completely the state of the system.
2. the components of the state vector must be [[Linear Independence | linearly independent]]. (Variables and their successive derivatives are linearly independent)

The minimun number of variables required equals the order of the differential equation.

Often state variables are chosen to be physical variables of a system, such as position and velocity in a mechanical system. Cases arise where these variables, although linearly independent, are also **decoupled** .

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title: System of Decoupled equations

Some linearly independent variables are not needed to solve for any
of the other linearly independent variables.
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Another case that increases the size of the state vector rises when the added variable is not linearly independent of the other members of the state vector. This usually occurs when a variable is selected as a state variable but its dependence on the other state variables is not immediately apparent.

Linear Systems

A linear system is a system of the form:

$$\begin{aligned} \dot{x} &= ax + by \\ \dot{y} &= cx + dy \end{aligned}$$

where a, b, c, d are parameters, the system can also be written in matrix form:

$\dot{\mathbf{x}} = A\mathbf{x}$

where:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$$

The solutions of $\dot{\mathbf{x}} = A\mathbf{x}$ can be visualized as trajectories moving on the (x, y) phase plane.

given the system $\dot{\mathbf{x}} = A\mathbf{x}$, it becomes obvious that if $\dot{\mathbf{x}} = 0 = A\mathbf{x}$ can be arrived at if $\mathbf{x} = 0$.

if A is non-singular the only equilibrium is the origin, what's standard to do to find solutions is to try an exponential solution: $\mathbf{x} = \mathbf{v}e^{\lambda t}$, and if we plug this in we end up with an Eigenvalue problem:

$(A - \lambda I)\mathbf{v} = 0$

and solving we end up with a solution of the form:

$$\mathbf{x} = c_1\mathbf{v}_1 e^{\lambda_1 t} + c_2\mathbf{v}_2 e^{\lambda_2 t}$$

And depending on the eigenvalues we'll have different behaviors around the fixed points, as we can see in Classification of Linear Systems.

Uncoupling Linear Dynamics

First let's start considering just $\dot{\mathbf{x}} = A\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^n$, and remembering that the basic solution of this is:

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0$$

Where:

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

This becomes easier to understand using eigenvectors and eigenvalues:

$$A\vec{\xi} = \lambda\vec{\xi}$$

Where our eigenvectors are $\vec{\xi}$ and if we write them in in a matrix we'll get the matrix:

$$T = [\xi_1, \xi_2, \ldots, \xi_n]$$

and we write a diagonal matrix of eigenvalues:

$$D = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

Then we write the eigenvalue problem as:

$$AT = TD$$

> *Side Note: This is not true if we have generalized eigenvectors or if you have two eigenvectors two close one to another.*

We represent the system in this form because now instead of using the $\mathbf{x}$ coordinates we want to map $\mathbf{x}$ into a coordinate system where things become simpler, we do this by writing our dynamics in $\mathbf{z}$ coordinates.

$$\mathbf{x} = T\mathbf{z}$$

from $AT = TD$ we get $D = T^{-1}AT$, and rewriting our equations we get:

$$\begin{aligned} \dot{\mathbf{x}} = T\dot{\mathbf{z}} &= A\mathbf{x} \\ T\dot{\mathbf{x}} &= AT\mathbf{z} \\ \dot{\mathbf{z}} &= T^{-1}AT\mathbf{z} = D\mathbf{z} \end{aligned}$$

now our dynamics are uncoupled and everyone is happy:

$$\frac{d}{dt}\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}$$

and we solve this as we would any system of uncoupled linear differential equations:

$$\begin{aligned} \mathbf{z}(t) &= e^{Dt}\mathbf{z}(0) \\ &= \begin{pmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{pmatrix} \mathbf{z}(0) \end{aligned}$$

and if we want to map this back to $\mathbf{x}$, we need to just do the inverse transformation:

$$\mathbf{x}(t) = Te^{Dt}T^{-1}\mathbf{x}(0)$$

The way to think this is the following:

- $\mathbf{x}(0)$ are our initial conditions
- $T^{-1}\mathbf{x}(0) = \mathbf{z}(0)$ are our initial conditions in eigenvector coordinates
- $e^{Dt}\mathbf{z}(0) = \mathbf{z}(t)$ are our system dynamics advanced in time in eigenvector coordinates.
- and finally $T\mathbf{z}(t) = \mathbf{x}(t)$ are our system dynamics advanced in time transformed back into $\mathbf{x}$ coordinates.

Continuous Linear Stability Analysis

Given a Linear System and putting it in it's eigenvector coordinates and back we got:

$$x(t) = Te^{Dt}T^{-1}x(0)$$

where:

$$e^{Dt} = \begin{pmatrix} e^{\lambda_1 t} & & & 0 \\ & e^{\lambda_2 t} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n t} \end{pmatrix}$$

Then we can see that if any of these exponentials blow up to infinity then $\mathbf{x}$ also is going to blow up to infinity.

Given that λ has real and imaginary components the exponential is:

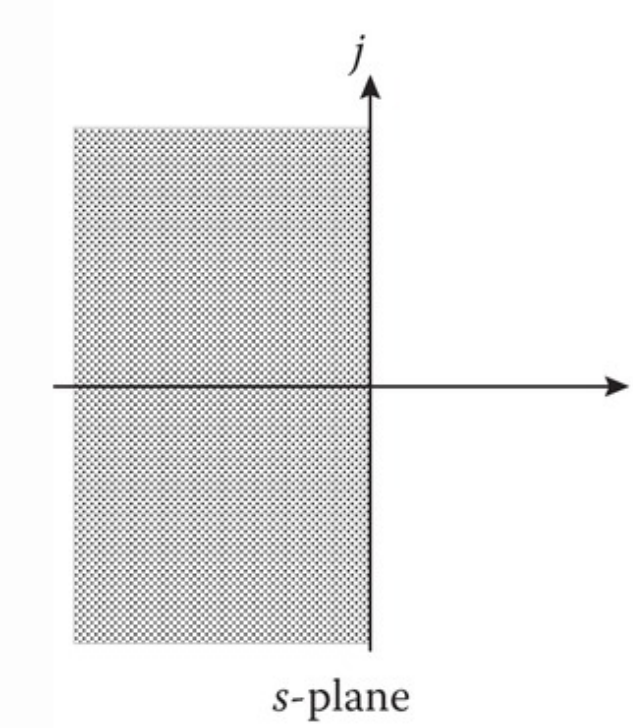
$$\lambda = a + ib$$
$$e^{\lambda t} = e^{at}[\cos(bt) + i\sin(bt)]$$

Where the real part tells us all about the magnitude, and if the exponential is greater than 0, then the system will blow up.

so if $a > 0$ the solution will blow up to infinity, and if $a < 0$ the solution then i have an exponentially decaying response.

Then the system is stable if and only if all off the eigenvalues are negative.

We can see this in the complex plane as:



Where the shadowed part of the graph is the stable space.

Discrete Linear Stability Analysis

In experiments we often don't have a continuous measurement of $\mathbf{x}$ instead, what we have instead are multiple discrete measurements, so what we have in a practice is a system of the form:

$$\mathbf{x}_{k+1} = \tilde{A}\mathbf{x}_k, \qquad \mathbf{x}_k = \mathbf{x}(k\Delta t)$$

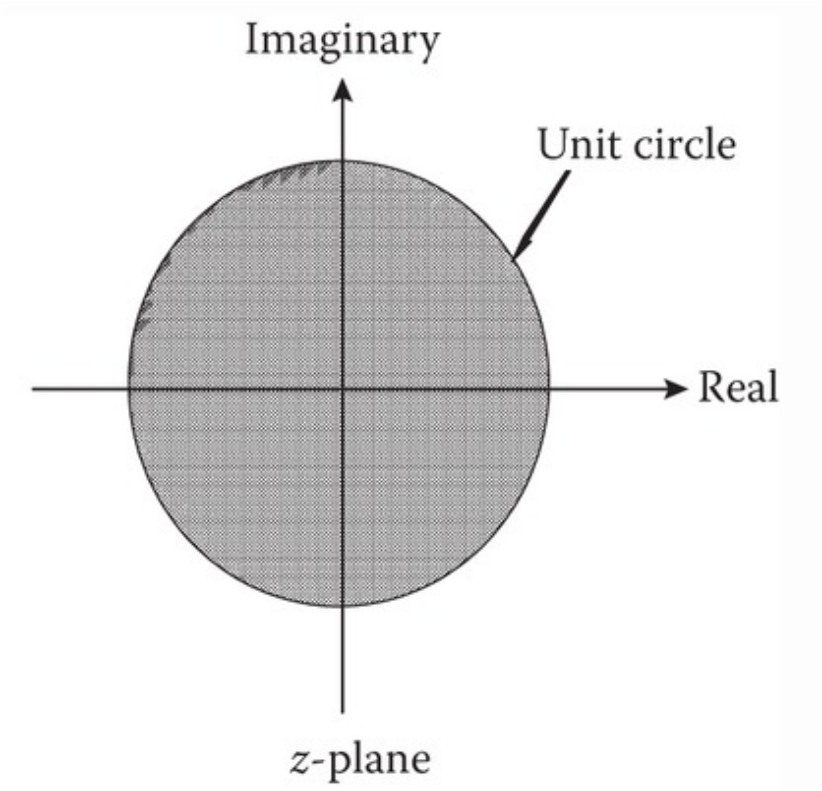
we have measurements at k intervals of time, we also have:

$$\tilde{A} = e^{A\Delta t}$$

now let's say we have an initial condition $\mathbf{x}_0$, given that initial condition we can compute the trajectory by just multiplying by $\tilde{A}$:

$$\begin{aligned}
\mathbf{x}_1 &= \tilde{A}\mathbf{x}_0 \\
\mathbf{x}_2 &= \tilde{A}^2\mathbf{x}_0 \\
\mathbf{x}_1 &= \tilde{A}^3\mathbf{x}_0 \\
&\vdots \\
\mathbf{x}_n &= \tilde{A}^n\mathbf{x}_0
\end{aligned}$$

So the only thing that changes is that the matrix gets exponentiated, and since $\tilde{A} = \tilde{T}\tilde{D}\tilde{T}^{-1}$, then we have powers of the eigenvalues λ in the diagonal matrix, then the radius grows or decays depending on the value of λ, so if the eigenvalues are greater than 1 the system will blow up:



and the stable domain is now the unit circle.

Linearizing at the Fixed Points

Consider the system:

$$\begin{aligned}
\bar{x} &= f(x, y) \\
\bar{y} &= g(x, y)
\end{aligned}$$

and suppose that (x^*, y^*) it's a fixed point, then:

$f(x^*, y^*) = 0$
$g(x^*, y^*) = 0$

Then we apply the following coordinate transformation:

$$u = x - x^* \qquad v = y - y^*$$

denote the components of a small disturbance from the fixed point. To see whether the disturbance grows or decays, we need to derive differential equations for u and v.

since x is constant

$$\dot{u} = \dot{x}$$

by substitution:

$$\dot{u} = f(x^* + u, y^* + v)$$

expanding the [[Taylor series]] and using the chain rule:

$$\dot{u} = f(x^*, y^*) + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + O(u^2, v^2, uv)$$

and since $f(x^*, y^*) = 0$

$$\dot{u} = +u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + O(u^2, v^2, uv)$$

Remember this partial derivatives are evaluated at the fixed point (x^*, y^*), these are numbers not functions.
The quadratic terms are neglegible for small u, v.

Similarly:

$$\dot{v} = +u\frac{\partial g}{\partial x} + v\frac{\partial g}{\partial y} + O(u^2, v^2, uv)$$

Hence the disturbance (u, v) evolves according to:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \text{Quadratic Terms}$$

Where the matrix:
$A = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}$
it's the [[Jacobian Matrix]] at the fixed point (x^*, y^*) and it's the multivariable analog of $f'(x^*)$.

Now if we neglect the quadratic terms we get the **Linearized System**

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Effect of Small Nonlinear Terms

t's okay to neglect the quadratic terms as long as the linearized system is not one of the borderline cases discussed in [[Classification of Linear Systems]], so if it's a saddle, node or a spiral, then the fixed point really it's that.
However in the case of centers, degenerate nodes, stars or non-insolated fixed points, are much more delicate.

Centers with a little nonlinearity can be made into an spiral, similarly, stars and degenerate nodes can be altered by small nonlinearities, but unlike centers, their stability doesn't change. per example a stable star may be changed into an stable spiral, but not into an ustable one

if we are only interested in stability, and not the geometry of the trajectories then fixed points can be classified as:

Robust Cases:

- *Repellers* (also called *Sources*): both eigenvalues have a positive real part
- *Attractors* (also called *Sinks*): Both eigenvalues have negative real part
- *Saddles*: One positive and one negative eigenvalue.

Marginal Cases:

- *Centers*: both eigenvalues are pure imaginary numbers.
- *Higher order and non insolated fixepoints*: at least one eigenvalue is 0

Thus from the point of view of stability, marginal cases are those where at least one eigenvalue satisfies $Re(\lambda) = 0$

Controllability

Controllability of a Continuous Linear System

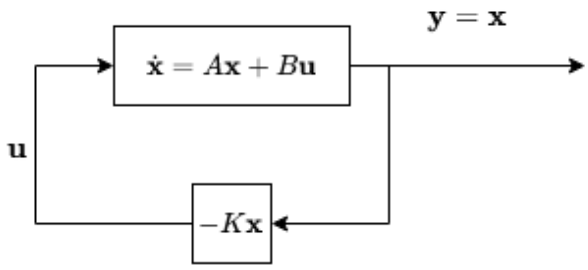
Given a system:

$$\dot{x} = Ax + Bu$$

where:

- $x \in \mathbb{R}^n$
- $u \in \mathbb{R}^q$
- $A \in \mathbb{R}^{n \times n}$
- $B \in \mathbb{R}^{n \times q}$

and we want to design a controller:



we're going to measure the state and if it's deviating from where we want it to be, we control the system to steer it back with the linear control $\mathbf{u} = -K\mathbf{x}$.

if we plug this into the system we get:

$$\dot{\mathbf{x}} = A\mathbf{x} - BK\mathbf{x} = (A - BK)\mathbf{x}$$

So by choosing K we can drive the system where we want it, for many systems.

Our system is **Controllable** if we can choose $\mathbf{u} = K\mathbf{x}$ and place the eigenvalues of the system anywhere we want.

In most control systems we're given A and B, and only have control in $\mathbf{u}$.

There's a special matrix called the **Controllability matrix**:

$$\mathcal{C} = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$$

If this matrix has full-column rank ($\text{rank}(\mathcal{C}) = n$), then the system is controllable, else there's some directions in $\mathbb{R}^n$ that aren't reachable.

And if we look at the SVD of the controllability matrix, if we look at the left singular vectors, they will be ordered from most controllable to least controllable.

The following statements are equivalent:- System is controlable

- I can place the eigenvalues (poles) of the closed loop system anywhere we want.
- full **Reachability** (if the system is controllable i can cook up a control to steer the system anywhere we want), reachable set:
  $R_t = \{\xi \in \mathbb{R}^n | \text{there is an input } \mathbf{u}(t) \text{ so that } \mathbf{x}(t) = \xi\}$
- $R_t = \mathbb{R}^n$

in MatLab it's easy to do this eigenvalue placement with the command:

```
K = place(A,B,eigs)
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In practice there's rich measures of how controllable are certain directions in $\mathbb{R}^n$.

from $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{B}u$ we get a response:

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0)$$

But that's in the absence of control, with only control we would get:

$$\mathbf{x}(t) = \int_0^t e^{A(t-\tau)}B\mathbf{u}(\tau)d\tau$$

which seems a lot like the Convolution of Functions of e^{At} with our input $\mathbf{u}$.

Putting this response together with the initial response without control we would get:

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Reachability

we have **Reachability**, if some vector $\xi \in \mathbb{R}^n$ is reachable, then there's some prescribed control input that will give us $\xi = \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$.

and using Cayley-Hamilton Theorem we can expand this out as:

$$\xi = \int_0^t \Big(\phi_0(t-\tau)u(\tau)IB + \phi_1(t-\tau)u(\tau)AB + \cdots + \phi_{n-1}(t-\tau)u(\tau)A^{n-1}B\Big)d\tau$$

and now we have a finite sum of terms, and each of these is an scalar convolution, so this is:

$$\xi = B\int_0^t \phi_0(t-\tau)u(\tau)d\tau + AB\int_0^t \phi_1(t-\tau)u(\tau)d\tau + \cdots + A^{n-1}B\int_0^t \phi_{n-1}(t-\tau)u(\tau)d$$

so if it's reachable the state ξ can be written as this sum of integrals, and we can write this as:

$$\xi = \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} \begin{bmatrix} \int_0^t \phi_0(t-\tau)u(\tau)d\tau \\ \int_0^t \phi_1(t-\tau)u(\tau)d\tau \\ \vdots \\ \int_0^t \phi_{n-1}(t-\tau)u(\tau)d\tau \end{bmatrix}$$

this is a product of our controllability matrix with the convolutions of our time variant coefficients and our control.

so if the controllability matrix spans all of state space, then we can cook up some u that can get us anywhere in state space. Because the dimension of ξ it's the same as the dimension of what we can control.

if $u \in \mathbb{R}^q$ then there's $n \cdot q$ numbers we need to specify (right of the dot product), to get us to ξ.

And if $q \neq 1$ there are multiple.

Controllability of a Discrete-Time Impulse Response

if we have a discrete time model such as:

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k$$

then an impulse at $k = 0$ is much like an initial condition for $\mathbf{x}_k$ if $\mathbf{x}_k = 0$, so we can think of this as:

$$
\begin{aligned}
\mathbf{u}_0 &= 1 & \mathbf{x}_0 &= 0 \\
\mathbf{u}_1 &= 0 & \mathbf{x}_1 &= B \\
\mathbf{u}_2 &= 0 & \mathbf{x}_2 &= AB \\
\mathbf{u}_3 &= 0 & \mathbf{x}_3 &= A^2B \\
&\vdots & &\vdots \\
\mathbf{u}_m &= 0 & \mathbf{x}_m &= A^{m-1}B
\end{aligned}
$$

This is all to say that if i give a kick to the system, and there's some directions that are not touched by the response, then there's no way i can reach them with control, so this is the space control can reach.

Gremians

We define a gremian at time t as the matrix:

$$W_t = \int_0^t e^{At} BB^T e^{A^T \tau} d\tau$$

This is the time t gramian. and it's an $\mathbb{R}^{n \times n}$ matrix, and because it's real valued and Symmetric we know it's going to have positive real eigenvalues, so we look at the Eigenvalue Decomposition of this:

$$W_t \xi = \lambda \xi$$

Here the eigenvectors corresponding to the biggest eigenvalue are most controllable (i can go farther in these) directions of state-space.

For discrete time systems often we find:

$$W_t \approx \mathcal{C}\mathcal{C}^T$$

But we notice that the eigenvalues and eigenvectors of these, are the singular vectors and singular vectors of $\mathcal{C}$.

So we could look at the SVD decomposition of the controllability matrix $\mathcal{C}$, to look at singular vectors in order to know which are the most controllable directions.

$\mathbf{x}$ might be a really big vector, so it might be too much to ask for all of $\mathbb{R}^n$ to be controllable in some systems, but we don't need full controllability in most cases.

Many directions will be really stable so we won't care about them, hat we care about it's **Stabilizability**, we say that this system is stablilizable if and only if all unstable and lightly damped eigenvectors of A are in controllable subspace.

Popov-Pelevitch-Hautus (PBH) Test

The pair (A, B) is controllable if and only if $\text{rank}[(A - \lambda I)B] = n \forall \lambda \in \mathbb{C}$.

This statement can lead to a few deductions:

- The $\text{rank}(A - \lambda I) = n$ for all values except the eigenvalues λ so we need to test at the eigenvalues of A.
- The only way for $(A - \lambda I)$ to be rank deficient is if it's missing some component in that eigenvalue's direction, so the only way for the rank of $[(A - \lambda I)B]$ to be n is if B has some component in that eigenvalue's direction. So basically B has to have some component in each eigenvalue's direction
- We can then say if B is a random vector then the pair (A, B) will with high probability be controllable.

This test tells us the minimum number of actuators (Columns of B) we need for a given A matrix, and it has to do with the multiplicity of A.

If we have single eigenvalue that's not repeated in A, then it doesn't matter if it's one million-dimensional we can control it with a single column of B, as long as the A matrix has distinct eigenvalues.

If we have a repeated eigenvalue, let's say it's repeated 3 times, then i need three columns of B to fill in that rank, cause $(A - \lambda I)$ will be rank deficient in three dimensions for that eigenvalue.

If we have nearly degenerate eigenvalues we would like to have more columns of B just in case, to boost our controllability.

LQR

Where are the best places to place the poles to stabilize the system? this in an answer we get with the Linear Quadratic Regulator, these work with a cost given by:

$$J = \int_0^\infty (\mathbf{x}^T Q \mathbf{x} + \mathbf{u}^T R \mathbf{u}) dt$$

where Q and R are matrix functions of t, Q it's what adds a cost to the time we spend away from the point we want to stabilize, so it leads to faster solutions, and R it's related the cost of our input as such it regulates the energy spent.

We would build Q to penalize the directions we want to move less.

and there's an optimal control law $\mathbf{u} = -K\mathbf{x}$ that minimizes that cost function, and that is the Linear Quadratic Regulator, this is found by solving the Riccati Equation or the Hamilton Jacobi Bellman Equation (HJB) , in matlab we would find it as:

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K = lqr(A,B,Q,R)
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and we could see where these are placed taking the eigenvalues of K.

This method is expensive ($O(n^3)$) for big systems.

LQR Derivation*

for Linear Systems:

$$\dot{x} = Ax + Bu$$

with cost

$$\ell = x^t Q x + u^t R u$$

with the condition that R is symmetric and positive definite.

$$R = R^T \succ 0$$

and also $Q = Q^T \succeq 0$

Then the cost-to-go function it's also going to be a quadratic of the form:

$$J^*(x) = x^T S x$$

where $S = S^T \succeq 0$

Then we solve the Hamilton Jacobi Bellman (HJB) equation:

$$0 = \min_u \left[\ell(x,u) + \frac{\partial J^*}{\partial x} f(x,u) \right]$$

so we'll minimize:

$$0 = \min_u \left[x^T Q x + u^T R u + \frac{\partial J^*}{\partial x}(Ax + Bu) \right]$$

and we know that the derivative of the cost to go is:

$$\frac{\partial J^*}{\partial x} = 2x^t S$$

then differentiating about u and equating to 0, we'll minimize this.

$$\frac{\partial}{\partial u}\left[\ell(x,u)+\frac{\partial J^*}{\partial x}f(x,u)\right]=2u^TR+2x^TSB=0$$

this yields the optimal policy:

$$u^*=\pi^*(x)=-R^{-1}B^TSx=-Kx$$

going back into the HJB and simplifying:

$$0=x^T[Q-SBR^{-1}B^TS+2SA]x$$

all the terms are symmetric except for the $2SA$, but since $x^TSAx=x^TA^TSx$, we can write:

$$0=x^T[Q-SBR^{-1}B^TS+SA+A^TS]x$$

and since this condition must hold for all x, it's sufficient to consider the matrix equation:

$$0=SA+A^TS-SBR^{-1}B^TS+Q$$

Note that it's quadratic in S making the solution non trivial, but it's well known that the equation has a single positive-definite solution if and only if the system is controllable and there are good numerical methods for finding that solution.