NICHOLS ALGEBRAS OVER FINITE GROUPS: NEW & OLD RESULTS

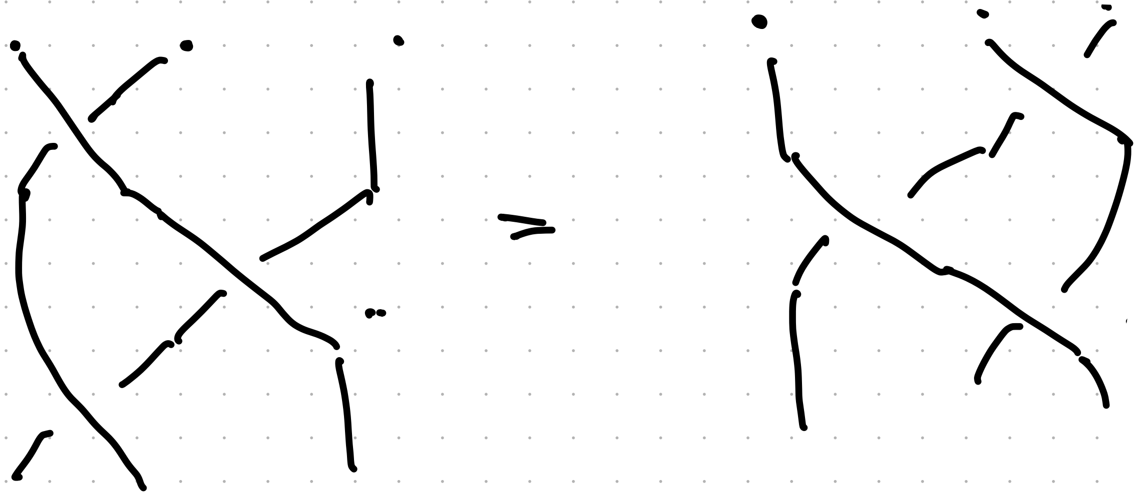
Examples: $S(V)$, ΛV , $U_q(n)$

Ingredient:

Braided vector space (V, c) $\quad$ V vector space $/\Bbbk$

$c \in GL(V^{\otimes 2})$ braiding, i.e.

$$(c\otimes id)(id\otimes c)(c\otimes id) = (id\otimes c)(c\otimes id)(id\otimes c)$$



Braid group

$$B_n = \left\langle \sigma_i ,\ i=1,\dots,n-1 \ \middle|\ \begin{array}{l} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \\ \sigma_i\sigma_j = \sigma_j\sigma_i \quad |i-j|>1 \end{array} \right\rangle$$

c gives rise to a representation $\forall n \geq 2$

$$\rho_n : B_n \longrightarrow GL(V^{\otimes n})$$

$$\sigma_i \longmapsto id^{\otimes i-1} \otimes c \otimes id^{n-i-3}$$

Matsumoto: $\exists \quad M : S_n \longrightarrow B_n$.

The vector space

$$\bigoplus_{n \geq 0} V^{\otimes n} \qquad V^{\otimes 0} = k$$

supports 2 graded algebra structures:

$T_! V \cong k \langle x_1, \dots, x_{\dim V} \rangle$ tensor algebra

$T_* V$ cotensor algebra:

$$(v_1 \otimes \cdots \otimes v_m) \cdot_* (v_{m+1} \otimes \cdots \otimes v_{m+n})$$

$$:= \sum_{\sigma \in \text{Ш}_{m,n}} \rho_{m+n}(M(\sigma))(v_1 \otimes \cdots \otimes v_{m+n})$$

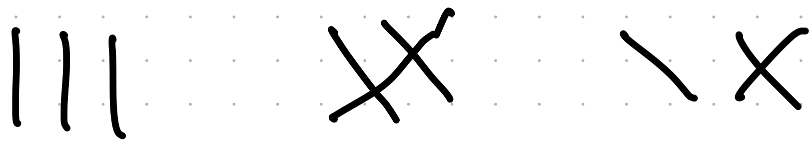
shuffles

$$\text{Ш}_{m,n} = \{ \sigma \in S_{m+n} \text{ st } \sigma(i) < \sigma(j) \text{ if } 1 \leq i < j \leq m \text{ or } m+1 \leq i < j \leq m+n \}$$

$$\begin{tikzcd}
T_! V \arrow[rr, "\exists! \Omega \text{ alg map extending } \mathrm{id}_V"] \arrow[dr, two heads] & & T_* V \\
& T_{!*} V \arrow[ur, hook] &
\end{tikzcd}$$

$T_{!*} V$ = NICHOLS ALGEBRA

eg $m=2$ $n=1$



id (123) (23)

$$(v_1 \otimes v_2) \cdot_* v_3 = v_1 \otimes v_2 \otimes v_3 + M(123) \cdot (v_1 \otimes v_2 \otimes v_3)$$

$$(id \otimes c)(v_1 \otimes v_2 \otimes v_3) =$$

$$= v_1 \otimes v_2 \otimes v_3 + (c \otimes id)(id \otimes c)(v_1 \otimes v_2 \otimes v_3)$$

$$+ (id \otimes c)(v_1 \otimes v_2 \otimes v_3)$$

Why do we care?

CLASSIFICATION OF HOPF ALGEBRAS — Andruskiewitsch, Schneider

MAUE'S CONJECTURE — 2017 Ellenberg-Tran-Westerland

<u>QUESTIONS</u>:

- For which (V,c) is $T_{!*}(V)$ finite dim^e?
- For which (V,c) is $T_{!*}(V)$ finitely presented?

SPECIAL FAMILY OF BVS:

BVS of Group type

Given:
G group

$$\left.\begin{array}{l} V \in kG\text{-mod} \\ V = \bigoplus_{g \in G} V_g \\ h \cdot V_g \subseteq V_{hgh^{-1}} \end{array}\right\} \text{YD module}$$

Then $\exists$

$$c : V \otimes V \longrightarrow V \otimes V$$
$$v \otimes w \longmapsto g \cdot w \otimes v \,, \quad v \in V_g$$

ANSWERS:

G abelian: full answer

Andruskiewitsch, Schneider, Heckenberger, Angiono, ...

G non-abelian: open

Examples of fd^{e} Nichols algebras are rare

<u>CONJECTURE</u>

G simple nonabelian

$\Rightarrow \nexists$ finite dim^{e} Nichols algebras over G

The conjecture is confirmed for:

- ALTERNATING GROUPS
- SPORADIC GROUPS

(up to a few classes in Fi_{22}, B, M)

Andruskiewitsch Fantino, Graña, Vendramin, 2011

Beltrán Cubillos, 2020

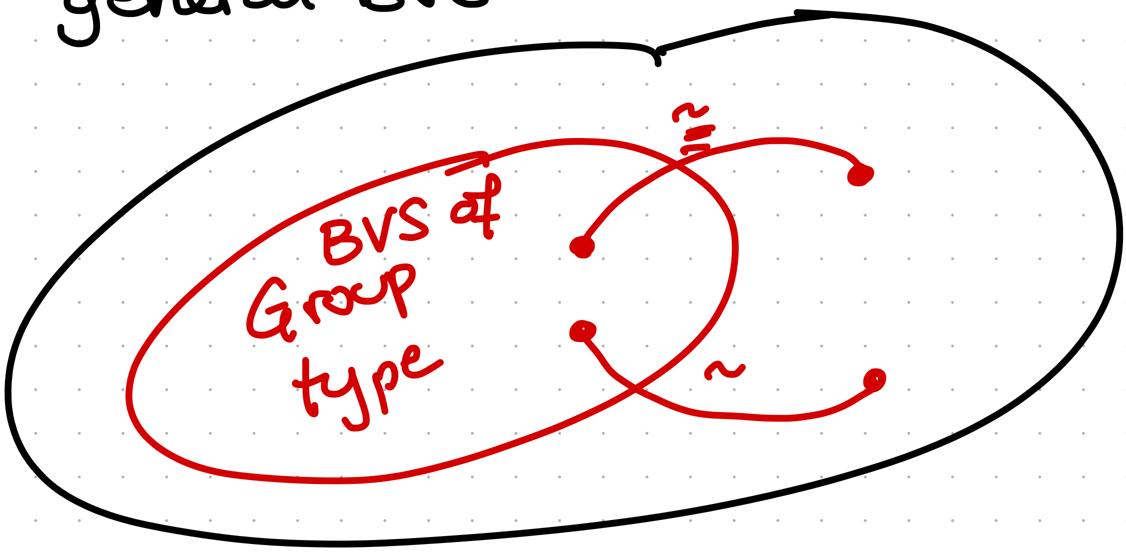
- SIMPLE GROUPS OF LIE TYPE $q = p^m$
  - $PSL_n(q)$: $n > 3$ or $n = 3, q > 2$
  - $PSp_{2n}(q)$: $n > 2$ or $n = 2, q > 3$
  - $P\Omega^+_{4n}(2^m)$
  - $E_7(2^m)$, $E_8(2^m)$, $F_4(2^m)$, $G_2(2^m)$
  - ${}^{(3)}D_4(2^m)$, $P\Omega^-_{4n}(2^m)$

Andruskiewitsch-C-García 2015-2023

- SUZUKI AND REE GROUPS C.-Costantini, 2021

For the remaining groups: if $T_{!*}(V)$ is finite dimensional $\Rightarrow$ support of the grading of V is a conjugacy class of elements of order coprime to p (A-C-G)

It can be useful to consider more general BVS



$\cong$ isomorphic

$\sim$ in Hilbert series

How? (Graña)

RACKS + COCYCLES

RACK

X set, with binary operation $\triangleright : X \times X \xrightarrow{\triangleright} X$.

such that: ① $x \triangleright -$ bijective $\forall x$

② $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z) \quad \forall x, y, z$

EXAMPLE $\quad X \subseteq G \quad$ conj classe

$\triangleright$ = conjugation

$x \triangleright y = x y x^{-1}$ for $x, y \in X$

RACK COCYCLE $\quad q : X \times X \to \mathbb{C}^n$

$$q(x, y \triangleright z)\, q(y, z) = q(x \triangleright y, x \triangleright z)\, q(x, z)$$

A pair (X, q) gives a BVS:

$V = kX \otimes \mathbb{C}^n, \quad c_q \in GL(V^{\otimes 2})$

$$c_q((x \otimes v), (y \otimes w)) = ((x \triangleright y)\, q(x, y) w) \otimes (x \otimes v)$$

<u>EXAMPLE</u>: $G = S_n \qquad n \geq 3$

$X = \mathcal{O} = \{(i\ j),\ i \neq j\}$

$$q^+(\tau, (i\ j)) = \begin{cases} 1 & \text{if } (\tau(i) - \tau(j))(i-j) > 0 \\ -1 & \text{otherwise} \end{cases}$$

- q^+ is of group type (Milinski-Schneider)

$\mathcal{O} = \mathcal{O}_{(12)} \qquad C_{S_n}(12) \cong S_2 \times S_{n-2}$

$\rho = \mathrm{sgn} \boxtimes \mathrm{id}$ on $\mathbb{C}$

$$V = \mathrm{Ind}_{C_{S_n}(12)}^{S_n}(\rho) = \mathbb{C}S_n \otimes_{\mathbb{C}C_{S_n}(12)} \mathbb{C}_\rho$$

grading: $V_{\sigma(12)\sigma^{-1}} = \mathbb{C}\sigma \otimes_{\mathbb{C}C_{S_n}(12)} \mathbb{C}_\rho$

- $q^- \equiv -1$

q^- is <u>NOT</u> of group type

FOMIN-KIRILLOV ALGEBRAS: (MOTIVATION: SCHUBERT CALCULUS)

For an integer $n \geq 3$ denote by $\mathcal{E}_n$ the algebra (of type A_{n-1}) with generators $x_{(ij)}$, where $1 \leq i < j \leq n$, and relations

$$\begin{aligned}
&x_{(ij)}^2 = 0, && \text{for } 1 \leq i < j \leq n,\\
&x_{(ij)}x_{(jk)} = x_{(jk)}x_{(ik)} + x_{(ik)}x_{(ij)}, && \text{for } 1 \leq i < j < k \leq n,\\
&x_{(jk)}x_{(ij)} = x_{(ik)}x_{(jk)} + x_{(ij)}x_{(ik)}, && \text{for } 1 \leq i < j < k \leq n,\\
&x_{(ij)}x_{(kl)} = x_{(kl)}x_{(ij)}, && \text{for any discinct } i, j, k, l.
\end{aligned}$$

These are the degree 2 relations of $T_{!*}(\mathcal{O}, q^+)$!!

- For $n = 3, 4, 5$

$$\mathcal{E}_n \cong T_{!*}(\mathcal{O}, q^+) \text{ \& fd}^{e}$$

Milinski-Schneider, F-K, Garcia-Garcia Iglesias, Graña, Roos

$n \geq 6$??

Vendramin: $T_{!*}(\mathcal{O}, q^+) \sim T_{!*}(\mathcal{O}, -1)$

Bazlov generalized the construction of FK algebras to arbitrary finite Coxeter groups

G. Maret - C. : Vendramin's result holds for all Bazlov's algebras

RK: Hopefully the pair $(\mathcal{O}, -1)$ can be attacked by geometric methods using Kapranov-Schechtman, C-Esposito-Rubio y Degressi