Isomorphisms, automorphisms and torsion units of integral group rings of finite groups – a survey

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Groups and their actions
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$\mathrm{V}(RG)$	group of normalized units of RG, i.e.

$$\mathrm{V}(RG) = \left\{ \sum_{g \in G} u_g g \in \mathrm{U}(RG) : \sum_{g \in G} u_g = 1 \right\}$$

In other words, $\mathrm{V}(RG)$ consists of the units of augmentation 1.

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Moreover G denotes a finite group, if not other stated.

Because G lives naturally in the units of RG, $\mathbb{Z}G$ rsp. it is natural to expect answers to the basic question if the unit group $U(\mathbb{Z}G)$ is considered. There the normalized units $V(\mathbb{Z}G)$, i.e. the units of augmentation 1 contain all informations.

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- If H is a group basis of $\mathbb{Z}G$ and $h \in H$. Then the sum over the conjugates of h in H is called a class sum.
  The class sums of a group basis form a $\mathbb{Z}$ - basis of the centre of $\mathbb{Z}G$.

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  The class sums of a group basis form a $\mathbb{Z}$ - basis of the centre of $\mathbb{Z}G$.
- If H and G are group bases then their class sums coincide. The correspondence is compatible with the power map on the classes.

Most of these fundamental facts are due to G. Higman (1940) and S. D. Berman (1953). The last one is due to G. Glauberman and D.S.Passman. .

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H.Zassenhaus stated three conjectures on torsion subgroups of $\mathbb{Z}G$ in the seventies of the last century.

The second one, denoted by **ZP2**, says that all group bases of $\mathbb{Z}G$ are conjugate within $\mathbb{Q}G$. This provides a strong positive answer to IP.

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- G a p-group , G nilpotent (K.W.Roggenkamp-L.L.Scott 1987, A.Weiss 1987)
- $\mathbb{Z}G$ determines the chief series of G.
  In particular all group bases of $\mathbb{Z}G$ have the same composition factors.
  (Ki., R.Lyons, R.Sandling, D.Teague 1990)
  Among other it follows that IP is valid for simple groups and their automorphism groups.

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1995 M.Mazur discovered for the semidirect product $X = G \cdot C_\infty$ of a finite group G with the infinite cyclic group C_∞ – acting on G via an automorphism τ of G – a connection between the normalizer problem and isomorphisms of group rings. Denote such a semidirect prduct by X_τ then

$$RX_\tau \cong RX_{id}$$

provided no prime divisor of $|G|$ is invertible in the commutative ring R and τ is given by conjugation with a unit normalizing G in RG,

whileas

$$X_\tau \cong X$$

iff τ is an inner automorphism of G.

Counterxeample ctd

K.W.Roggenkamp and A.Zimmermann constructed for a group ring RG with semilocal coefficient ring

$$R = \mathbb{Z}_{\pi(G)} = \bigcap_{p \in \pi(G)} \mathbb{Z}_p, \quad |G| = 2^7 \cdot p^2 \cdot q^2 \text{ with odd primes } p \neq q,$$

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So by Mazur's construction this yields a counterexample to the isomorphism problem for $R(G \cdot C_\infty)$

but this is not a counterexample to NP for $\mathbb{Z}G$ (M.Hertweck 1997).

The Counterexample

The counterexample to IP for $R = \mathbb{Z}$ and a finite group G has been constructed by M.Hertweck in his thesis 1997 (published 2001). The group G has order

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Remark. Hertweck constructs - as first step for his counterexample to IP - a counterexample to NP for integral group rings. By Mazur's result this yields a counterexample to IP for integral groups of infinite groups. For the counterexample for IP for $\mathbb{Z}G$ with finite G Hertweck uses several nontrivial modifications of Mazur's construction.
It is unknown whether NP necessarily plays a role for a counterexample to IP.

Automorphisms, Notations and Definitions

$\mathrm{Aut}\mathbb{Z}\mathrm{G}$ = ring automorphisms of $\mathbb{Z}G$.
$\mathrm{Aut_n}\mathbb{Z}\mathrm{G}$ = ring automorphisms which preserve augmentation, also called normalized automorphisms.

Let X be a group basis of $\mathbb{Z}G$. Then $\tau \in \mathrm{AutX}$ induces uniquely a normalized ring automorphism also denoted by τ

$\tau \in \mathrm{Aut}\mathbb{Z}\mathrm{G}$ is called central, if it fixes the centre of $\mathbb{Z}G$ elementwise.

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If for each group basis X of $\mathbb{Z}G$ each $\sigma \in \mathrm{Aut_n}\mathbb{Z}\mathrm{G}$ is the product of a group automorphism of X and a central automorphism of $\mathrm{Aut}\mathbb{Z}\mathrm{G}$ then we say that

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Note

$$\text{ZP2} \implies \mathrm{AUT}$$

$$\text{ZP2} \iff \mathrm{AUT} + \mathrm{IP}.$$

Results on AUT

ZP2 and therefore AUT holds for the following classes of finite groups.

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- 18 of the 26 sporadic simple groups (F.Bleher, Ki. 2000)
- Finite simple groups of Lie type of small rank and all finite simple groups with abelian Sylow subgroups (F.Bleher 1999)
- Finite Coxeter groups (F.Bleher, M.Geck, Ki. 1997)

Results on AUT ctd

However counterexamples to AUT have been constructed by K.W.Roggenkamp and L.L.Scott (1988), L.Klingler (1991), P.F.Blanchard and M.Hertweck. The smallest ones have order 96 and are due to P.F.Blanchard (1997) and M.Hertweck (2003).

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F* - Theorem, Roggenkamp-Scott 1988

Assume that G has a normal p - subgroup N with $C_G(N) \subset N$ then AUT is valid for $\mathbb{Z}G$.

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A summary is given in
M.Hertweck, Units of p - power order in principal blocks of p - constrained groups, J.of Alg. **464**, (2016) 348-356.

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Proposition (Ki. 1987)

Let $\mathcal{C}$ be a class of finite groups closed under direct products. Assume that AUT holds for $\mathbb{Z}G$ for every $G \in \mathcal{C}$. Then ZP2 holds for every $G \in \mathcal{C}$.

The condition that $F^*(G)$ is a p-group is closed under direct products. That for a group basis X of $\mathbb{Z}G$ the generalized Fitting subgroup $F^*(X)$ is a p-group if, and only if, $F^*(G)$ is a p - group follows from the fact that group bases have the same normal subgroup lattice and corresponding normal subgroups have the same order.

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Theorem

ZP2 holds for $\mathbb{Z}G$ provided $F^*(G)$ is a p - group.

IP is „almost "true

Corollary 1

Let G be an arbitrary finite group then ZP2 (and thus also the isomorphism problem IP) has a positive answer for the semidirect product $\mathbb{Z}(F_p G \cdot G)$.

Here $F_p G$ denotes the additive group of the modular group ring and G acts just by multiplication on it.

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Note that $\mathbb{Z}G$ is a subring and a quotient of $\mathbb{Z}(F_pG \cdot G)$. A group basis H of $\mathbb{Z}G$ sits in a group basis of $\mathbb{Z}(F_pG \cdot G)$ if, and only if, $H \cong G$.

Applications of the F^* - theorem

Theorem (Ki.1991)

Suppose that $G/F(G)$ is abelian. Then IP holds for G and Sylow subgroups of group bases are conjugate within $\mathbb{Q}G$.
In particular IP holds for supersoluble groups.

With similar methods it follows that IP is valid for

- Nilpotent-by- abelian p - group -by- abelian p' - group. (Hertweck 1992)
- Frobenius or 2-Frobenius groups (Ki. 1991).

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Theorem F.Eisele and L-Margolis (2018)

There is a metabelian group G of order $2^7 \cdot 3^2 \cdot 5 \cdot 7^2 \cdot 19^2$ and $u \in V(\mathbb{Z}G)$ such that u is not conjugate within $\mathbb{Q}G$ to an element of G.
The unit has order $7 \cdot 19$.

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$G \cong C \cdot X$ with C and X cyclic of coprime order (A.Valenti 1994, C.Polcino Milies-J.Ritter-S.K.Sehgal 1984).

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ZP1 holds if G is cyclic-by-abelian (M.Caicedo, L.Margolis and A.del Rio 2013), for all groups of order ≤ 143 (A.Bächle, A.Herman, A.Konovalov, L.Margolis, G.Singh (2017)),

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for certain $\mathrm{PSL}(2, q)$, $q \in \{8, 9, 11, 13, 16, 19, 23, 25, 32\}$ or q a Fermat or Mersenne prime

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They are no longer conjectures but still problems for classes of group rings of finite groups.

But **ZP3** and so all three Zassenhaus conjectures hold for nilpotent groups (A.Weiss 1991) or if
$G \cong C \cdot X$ with C and X cyclic of coprime order (A.Valenti 1994, C.Polcino Milies-J.Ritter-S.K.Sehgal 1984).

ZP1 holds if G is cyclic-by-abelian (M.Caicedo, L.Margolis and A.del Rio 2013),
for all groups of order ≤ 143 (A.Bächle, A.Herman, A.Konovalov, L.Margolis, G.Singh (2017)),
for certain $\mathrm{PSL}(2,q),\ q \in \{8, 9, 11, 13, 16, 19, 23, 25, 32\}$ or q a Fermat or Mersenne prime
(Luthar-Passi, Hertweck, Ki.-Konovalov, Bächle-Margolis, Margolis-delRio-Serrano)

Open results, questions

- IP for groups of odd order

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- IP for 3-step abelian groups
- IP , ZP2 rsp. for groups with $O_{p'}(G) = 1$ for some prime p.

  L.L.Scott (1992) calls it plausible that ZP2 might be valid for such groups and points out that these groups deserve special attention since every finite group is a subdirect product of groups of this form.

Are there suitable replacements for ZP 1 and ZP3 ?

Replacements for ZP1

Several questions have been posed which are weaker than ZP1, e.g.
SP Does the order of a torsion element of $V(\mathbb{Z}G)$ coincide with the order of a group element of G (the so-called Spectrum question SP) ?
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- Positive answer to Question OG
- u is conjugate to an element of G in $\mathbb{Q}S_G$.
- (Conjecture of A.A.Bovdi 1987) Let $u = \sum_{g \in G} z_g g \in \mathbb{Z}G$. Then for each $m \in \mathbb{N}$ with $m \neq o(u)$ the coefficients of elements of order m of u sum up to zero , i.e.

$$\sum \quad z_g = 0.$$

p - elements

For elements of prime power order no counterexample to ZP1 is known.

Theorem (F.Eisele-L.Margolis 2022)

ZP1 holds for units of $V(\mathbb{Z}G)$ of prime order p provided a Sylow p - subgroup of G has order p.

Known results on Bovdi's conjecture

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- G arbitrary , u has prime order p.

Further Results

The following two results are joint work with A. Bächle and M. Serrano (2019).

Proposition 1 (Bächle, Ki. - Serrano)

Suppose that G has a nilpotent Hall subgroup N such that G/N is abelian. Then there is a group H containing G as subgroup such that ZP1 holds for $\mathbb{Z}H$.

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The statement of Proposition 1 is slightly stronger than an affirmative answer to Question OG.

Further Results ctd

Proposition 2 (Bächle, Ki., Serrano)

Suppose that G has a normal Sylow p - subgroup P such that Bovdi's conjecture has an affirmative answer for $\mathbb{Z}G/P$. Then it has also an affirmative answer for $\mathbb{Z}G$.

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Corollary

Suppose that G has a supersoluble normal Hall - subgroup H such that Bovdi's conjecture has an affirmative answer for $\mathbb{Z}G/H$. Then it has also an affirmative answer for $\mathbb{Z}G$.

Note. With respect to supersoluble groups ZP1 is still open.

Comparison with the counterexamples to ZP1

The counterexamples G of F. Eisele and L. Margolis to ZP1 are metabelian. They have even an abelian normal Hall subgroup A such that G/A is abelian.

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So the result of M. Dokuchaev and S. K. Sehgal shows that for these groups Bovdi's conjecture holds and Proposition 1 that these groups may be even embedded into larger groups for which ZP1 holds.

Sylow in $V(\mathbb{Z}G)$

A Sylowlike theorem in $V(\mathbb{Z}G)$ may have the following form

Let H be a finite p - subgroup of $V(\mathbb{Z}G)$. Then H is conjugate within $\mathbb{Q}G$ to a subgroup of G.

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Of course one could also try to prove as a first goal weaker statements (so-called weak Sylow like theorems), weak means isomorphism instead of conjugacy

The first case, G a p - group

The results of Roggenkamp-Scott and Weiss on $\mathbb{Z}G$ for G a p - group establish a Sylow like theorem in this case. Conjugacy is in this case even given in $\mathbb{Z}_p(G)$.

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It was a long standing question (the so-called modular isomorphism problem) whether the modular group algebra of a p - group over F_p determines the group up to isomorphism. This has been recently solved.

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Theorem (D.Garcia-Lucas, L.Margolis, A.del Rio 2021)

There are non-isomorphic groups of order 2^9 such that their group algebras over the field of two elements 2 are isomorphic.
In particular there are 2-blocks which do not determine their defect group up to isomorphism.

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and with respect to specific primes, if

- $G/O_{p'}(G)$ has a normal Sylow p - subgroup (A. Weiss 1993)
- $G = PSL(2, r^f)$ if $p \neq r$ or $p = r = 2$ or $f = 1$. (M.Hertweck - C.Höfert - Ki. 2009, L.Margolis 2016)

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A weak Sylowlike theorem holds, if

- G has cyclic Sylow p - subgroups. (Ki. for p=2 2007, Hertweck for p odd 2008)
- 2 - subgroups of $V(\mathbb{Z}G)$ are isomorphic to subgroups of G if Sylow 2 - subgroups of G are abelian, quaternion or dihedral. (Bächle-Ki. 2011, Ki. 2015, Margolis 2017)

Characters and Sylow numbers

For a finite group H denote by $X(H)$ its ordinary character table and by $Spec(H)$ its spectral table, i.e. the character table including the head line.

Question G.Navarro 2003

Let G and U be finite groups with the same ordinary character table, i.e. $X(G) = X(U)$. Do for each prime p the number of Sylow subgroups coincide, i.e. $n_p(G) = n_p(U)$?

Note that $\mathbb{Z}G \cong \mathbb{Z}U \Longrightarrow Spec(G) = Spec(U) \Longrightarrow X(G) = X(H)$.

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Note that $\mathbb{Z}G \cong \mathbb{Z}U \Longrightarrow Spec(G) = Spec(U) \Longrightarrow X(G) = X(H)$.
Thus results concerning character tables yield results for group rings.

Sylow for group bases of $\mathbb{Z}G$

So it is a natural question to consider Sylow like theorems between group bases of integral group rings.

Question Sylow for group bases

Let U be a finite p - subgroup of a group basis X of $V(\mathbb{Z}G)$. Is U conjugate within $\mathbb{Q}G$ to a subgroup of G?
Have G and X the same number of Sylow p - subgroups ?

Properties reflected by character tables

The ordinary character table $X(G)$ determines

- the normal subgroup lattice of G (G.Glauberman)

The ordinary character table does not determine the orders of the representatives of the conjugacy classes but the primes dividing the order of a representative (G.Higman).

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Properties reflected by character tables

The ordinary character table $X(G)$ determines

- the normal subgroup lattice of G (G.Glauberman)
- the chief series of G (Ki. 1989)
- whether G has abelian Sylow subgroups and if so their isomorphism type (Ki. and R.Sandling 1989)

The ordinary character table does not determine the orders of the representatives of the conjugacy classes but the primes dividing the order of a representative (G.Higman).

G.Navarro and N.Rizo 2017

If G is p - soluble then

$$Spec(G) = Spec(H) \Longrightarrow n_p(G) = n_p(H).$$

Ki. and I.Köster 2017

If G is nilpotent-by-nilpotent or if G is a Frobenius group, then

$$X(G) = X(H) \Longrightarrow n_p(G) = n_p(H) \forall p.$$

As one may expect with respect to integral group rings more is known

Theorem

Let G be a p - constrained group and let q be a prime not dividing $O_{p'}(G)$. Let X be a group basis of $\mathbb{Z}G$.

a) A q - subgroup U of X is conjugate within $\mathbb{Q}G$ to a subgroup of G (Ki.-Roggenkamp 1993).

b) $n_p(G) = n_p(X) \forall p$ (Ki.-Köster 2017).

Part a) is an application of the F^* - theorem.

p - soluble groups are p - constrained.

Thus a Sylow like theorem for group bases holds for integral group rings of finite soluble groups.

Thank you for your attention