Representations of G-posets and canonical Brauer induction

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Groups and their actions: algebraic, geometric and combinatorial aspects. Levico Terme, June 3–7, 2024

Overview

- **①** *G*-posets and their representations
- The canonical Brauer induction formula
- Categorification of the canonical Brauer induction formula

1. G-posets and their representations

- G: finite group.
- k: commutative ring.

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Example The set of subgroups of G together with the conjugation action of G.

If X is a G-poset, one can form a category $\mathscr{C}(X)$ as follows:

- Objects: the elements of X.
- $\bullet \operatorname{Hom}_{\mathscr{C}(X)}(x,y) := \{g \in G \mid x \leq gy\}.$
- Composition: $x \xrightarrow{g} y \xrightarrow{h} z = x \xrightarrow{gh} z$ $(x \le gy, y \le hz \Rightarrow x \le gy \le g(hz) = (gh)z).$
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Note that $\operatorname{End}_{\mathscr{C}(X)}(x) = G_x^{\operatorname{op}}$, the stabilizer of x in G, with the opposite multiplication $(g,h) \mapsto hg$). In particular any endomorphism is an isomorphism.

Definition A representation of a G-poset X over k is a functor $F: \mathscr{C}(X)^{\mathrm{op}} \to {}_k \mathrm{mod}$. Representations of X over k form an abelian category $\mathcal{P}_k(X)$. Note that for any $g \in G$ and $x \leq y$ in X one has commutative diagrams



Moreover, F(x) is a kG_x -module.

Example Let X be the set of subgroups of G endowed with G-conjugation and let $V \in {}_{kG}\mathsf{Mod}$. One can form the representation $H \mapsto V^H := \{v \in V \mid hv = v \text{ for all } h \in H\}$. This defines a functor

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The restriction maps are inclusions and the conjugation map $c_{g,H}$ is the application of g on V^H .

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This is also the category algebra $k\mathscr{C}(X)^{\mathrm{op}}$. If X is finite, A(X) has the identity element

$$1_{A(X)} = \sum_{x \in X} e_x,$$

where $e_x = (x, 1, x) = id_x$.

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Consequence: The category $A_k(X)$ mod has many special properties. For instance, every finitely generated $A_k(X)$ -module has a finite projective resolution.

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Proposition (B.-Monteiro 2004) One can explicitly determine the central idempotents of $A_k(X)$ in terms of the central idempotents of the various group algebras kG_x .

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2. The canonical Brauer induction formula

In this section, $k = \mathbb{C}$.

 $R(G) := \text{ring of virtual characters of } G = \text{Grothendieck ring of } \mathbb{C}_G \text{mod.}$

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Theorem (Brauer 1947) For every $\chi \in R(G)$ there exist $H_i \leq G$, $\varphi_i \in \hat{H}_i$, $n_i \in \mathbb{Z}$, i = 1, ..., r, such that

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Consider the set

$$\mathcal{M}_{\mathcal{G}} := \{ (H, \varphi) \mid H \leq G, \varphi \in \hat{H} \}.$$

It is a G-poset via $(K, \psi) \leq (H, \varphi) : \iff K \leq H \text{ and } \psi = \varphi|_K$, together with the G-conjugation action $(g, (H, \varphi)) \mapsto {}^g(H, \varphi) = ({}^gH, {}^g\varphi).$

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Also, the diagram

$$R_{+}(G) \xrightarrow{b_{G}} R(G)$$

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$$a_{G}(\chi) = \sum_{\substack{(H_{0},\varphi_{0}) < \cdots < (H_{n},\varphi_{n}) \\ \text{mod } G}} (-1)^{n} (\chi|_{H_{n}},\varphi_{n}) [H_{0},\varphi_{0}]_{G}.$$

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Thus, if χ is afforded by $V \in \mathbb{C}_G \mod$ then

$$\chi = \sum_{\substack{(H_0, \varphi_0) < \dots < (H_n, \varphi_n) \\ \operatorname{mod} G}} (-1)^n \operatorname{ind}_{H_0}^G [V^{(H_n, \varphi_n)}],$$

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• Symonds 1991: geometric interpretation of this formula.

• Objects: Pairs (M, \mathcal{L}) with $M \in \mathbb{C}_G \mod$ and $\mathcal{L} = \{L_1, \dots, L_n\}$ a set of 1-dimensional \mathbb{C} -subspaces of M that are permuted by G and satisfy $M = L_1 \oplus \cdots \oplus L_n$.

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Every L_i has a stabilizing pair $(H_i, \varphi_i) \in \mathcal{M}_G$. For $(H, \varphi) \in \mathcal{M}_G$ set

$$M((H,\varphi)) := \bigoplus_{\substack{L_i \in \mathscr{L} \\ (H_i,\varphi_i) = (H,\varphi)}} L_i \quad \text{and} \quad M^{((H,\varphi))} := \bigoplus_{\substack{L_i \in \mathscr{L} \\ (H_i,\varphi_i) \geq (H,\varphi)}} L_i .$$

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• $\operatorname{Hom}_{\mathbb{C}_G \operatorname{\mathsf{mon}}}(M,N)$ is the set of $f \in \operatorname{Hom}_{\mathbb{C}_G}(M,N)$ satisfying

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 \mathbb{C}_{G} mon is a \mathbb{C} -linear additive category, but not abelian.

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Proposition (B. 2001) Every indecomposable object in \mathbb{C}_G mon is of the form $\operatorname{Ind}_H^G(\mathbb{C}_\varphi) = \mathbb{C} G \otimes_{\mathbb{C} H} \mathbb{C}_\varphi$ for some $(H, \varphi) \in \mathcal{M}_G$, uniquely determined up to conjugation, and the Grothendieck group of \mathbb{C}_G mon is $R_+(G)$.

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Definition The functors $\mathcal{I} \colon \mathbb{C}_G \mod \to \mathcal{P}(\mathcal{M}_G)$ and $\mathcal{J} \colon \mathbb{C}_G \mod \to \mathcal{P}(\mathcal{M}_G)$ are defined by

$$\mathcal{I}(V) = \left(V^{(H,\varphi)}\right)_{(H,\varphi) \in \mathcal{M}_G} \quad \text{and} \quad \mathcal{J}(M) := \left(M^{((H,\varphi))}\right)_{(H,\varphi) \in \mathcal{M}_G}.$$

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Proposition (B. 2001) \mathcal{I} and \mathcal{J} are fully faithful embeddings of \mathbb{C}_G mod and \mathbb{C}_G mon into the full subcategory $\mathcal{P}'(\mathcal{M}_G)$ of $\mathcal{P}(\mathcal{M}_G)$ consisting of those functors F, such that $h \in H$ acts on $F(H, \varphi)$ via multiplication with $\varphi(h)$.

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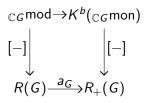
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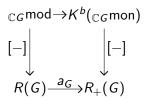
Moreover, if $\mathcal{J}(M_*)$ is this projective resolution of V then $a_G([V]) = \sum_{i \geq 0} (-1)^i [M_i]$ in $R_+(G)$. In particular, one has a commutative diagram



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Remark For given $V \in \mathbb{C}_G$ mod one can find an M_* of length \leq longest strictly ascending chain in the set of subspaces $V^{(H,\varphi)} \neq 0$.

Set

$$\varepsilon := \sum_{(H,\varphi) \in \mathcal{M}} \frac{1}{|H|} \sum_{h \in H} \varphi(h^{-1}) \cdot ((H,\varphi), h, (H,\varphi)) \in A(\mathcal{M}_G).$$

Then ε is an idempotent and $\mathcal{P}'(\mathcal{M}_G)$ corresponds under the equivalence $\mathcal{P}(\mathcal{M}_G)\cong {}_{A(\mathcal{M}_G)}$ mod to the full subcategory of ${}_{A(\mathcal{M}_G)}$ mod consisting of those modules on which ε acts as identity.

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Proposition (B.-Monteiro 2024) One has $A(\mathcal{M}_G)\varepsilon \subseteq \varepsilon A(\mathcal{M}_G)$. In particular, $A(\mathcal{M}_G)\varepsilon = \varepsilon A(\mathcal{M}_G)\varepsilon$ and $a\varepsilon = \varepsilon a\varepsilon$ for all $a\in A(\mathcal{M}_G)$.

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Proposition (B.-Monteiro 2024) One has $A(\mathcal{M}_G)\varepsilon \subseteq \varepsilon A(\mathcal{M}_G)$. In particular, $A(\mathcal{M}_G)\varepsilon = \varepsilon A(\mathcal{M}_G)\varepsilon$ and $a\varepsilon = \varepsilon a\varepsilon$ for all $a\in A(\mathcal{M}_G)$. As a consequence, left $A(\mathcal{M}_G)$ -modules on which ε acts as identity are literally the same thing as left $\varepsilon A(\mathcal{M}_G)\varepsilon$ -modules. Moreover, projective objects of $\mathcal{P}'(\mathcal{M}_G)$ are also projective in $\mathcal{P}(\mathcal{M}_G)$.

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Example For $F \in \mathcal{P}(\mathcal{M}_G)$ and fixed $(H, \varphi) \in \mathcal{M}_G$ consider the functor

$$F\mapsto F(H,\varphi)/\sum_{(H,\varphi)\leq (H',\varphi')}r_{(H,\varphi)}^{(H',\varphi')}(F(H',\varphi'))\in {}_{\mathbb{C}G_{(H,\varphi)}}\mathsf{mod}\,.$$

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