Some problems on modular group algebras

Diego García-Lucas

Universidad de Murcia, Spain

Groups and their actions
Levico Terme, 2024

Tensor factorizations

- k field.
- A k-algebra.

Tensor factorizations

- k field.
- A k-algebra.

Definition

A has the tensor Krull-Schmidt property if whenever

- $A \cong \bigotimes_{i=1}^{n} A_i \cong \bigotimes_{j=1}^{m} B_j$, and
- each of the A_i's and B_j's is tensor indecomposable,

then $n = m$ and $A_i \cong B_i$ after (possibly) rearranging the indices.

Tensor factorizations

- k field.
- A k-algebra.

Definition

A has the tensor Krull-Schmidt property if whenever

- $A \cong \bigotimes_{i=1}^{n} A_i \cong \bigotimes_{j=1}^{m} B_j$, and
- each of the A_i's and B_j's is tensor indecomposable,

then $n = m$ and $A_i \cong B_i$ after (possibly) rearranging the indices.

This is not the case in general:

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong \mathbb{C} \otimes_{\mathbb{R}} M_2(\mathbb{R}),$$

where $\mathbb{H}$ is the ring of real quaternions.

Tensor factorizations of local algebras

There are no examples of local algebras for which the tensor Krull-Schmidt property fails.

Tensor factorizations of local algebras

There are no examples of local algebras for which the tensor Krull-Schmidt property fails.

- Let $\mathbb{L}_k$ be the class of local augmented k-algebras.

Theorem (Horst, 1987)

Suppose that

1. $char(k) = 0$,
2. $N \otimes R \cong N \otimes S$ *in* $\mathbb{L}_k$, *and*
3. R *is noetherian and* N *artinan.*

Then $R \cong S$.

Tensor factorizations of local algebras

Theorem (Horst, 1987)

Suppose that

1. $char(k) = 0$,
2. $A \in \mathbb{L}_k$ *is noetherian.*

Then there is a decomposition

$$A \cong B \otimes A_1 \otimes \cdots \otimes A_n$$

such that

1. *each* A_i *is tensor indecomposable, and*
2. B *has no artinian tensor factors.*

Moreover, this decomposition is unique up to isomorphism and reordering.

Tensor factorizations of local algebras

Theorem (Horst, 1987)

Suppose that

1. $char(k) = 0$,
2. $A \in \mathbb{L}_k$ *is noetherian.*

Then there is a decomposition

$$A \cong B \otimes A_1 \otimes \cdots \otimes A_n$$

such that

1. *each* A_i *is tensor indecomposable, and*
2. B *has no artinian tensor factors.*

Moreover, this decomposition is unique up to isomorphism and reordering.

No analogue is known when $\mathrm{char}(p) > 0$.

Tensor factorizations of local group algebras

In 1995 Carson and Kovacs addressed this problem restricted to group rings.

- k field of characteristic p,
- G finite p-group.

Tensor factorizations of local group algebras

In 1995 Carson and Kovacs addressed this problem restricted to group rings.

- k field of characteristic p,
- G finite p-group.

Observe that

$$\{\text{group rings over } k\} \cap \mathbb{L}_k = p\text{-group rings over } k$$

Tensor factorizations of local group algebras

In 1995 Carson and Kovacs addressed this problem restricted to group rings.

- k field of characteristic p,
- G finite p-group.

Observe that

$$\{\text{group rings over } k\} \cap \mathbb{L}_k = p\text{-group rings over } k$$

Question

Does kG have the Krull-Schmidt property in $\mathbb{L}_k$?

Tensor factorizations of local commutative group algebras

Theorem (Carlson-Kovacs, 1995)

Suppose that

1. *G abelian finite p-group, and*
2. *$kG = A_1 \otimes A_2$.*

Then $G = G_1 \times G_2$ such that $A_i \cong kG_i$ for each i.

Tensor factorizations of local commutative group algebras

Theorem (Carlson-Kovacs, 1995)

Suppose that

1. *G abelian finite p-group, and*
2. $kG = A_1 \otimes A_2$.

Then $G = G_1 \times G_2$ such that $A_i \cong kG_i$ for each i.

This implies that kG, if G is abelian, has the tensor Krull-Schmidt property.

Tensor factorizations of local commutative group algebras

Theorem (Carlson-Kovacs, 1995)

Suppose that

1. *G abelian finite p-group, and*
2. *$kG = A_1 \otimes A_2$.*

Then $G = G_1 \times G_2$ such that $A_i \cong kG_i$ for each i.

This implies that kG, if G is abelian, has the tensor Krull-Schmidt property.

Questions on modular p-group algebras

Let's drop the commutativity assumption.

Question

- *Does every tensor factorization of kG come from a direct decomposition of G?*

Questions on modular p-group algebras

Let's drop the commutativity assumption.

Question

- *Does every tensor factorization of kG come from a direct decomposition of G?*
- *Does every directly indecomposable group have tensor indecomposable group algebra over k?*

Tensor factorizations with a commutative factor

Does every tensor factorization of kG come from a direct decomposition of G?

Theorem (GL-del Río-Sakurai, in progress)

Suppose that

1. $\mathbb{F}_p G = A_1 \otimes A_2$, *and*
2. A_1 *is commutative.*

Then $G = G_1 \times G_2$ *and* $A_i \cong \mathbb{F}_p G_i$ *for each* i.

Tensor factorizations with a commutative factor

Does every tensor factorization of kG come from a direct decomposition of G?

Theorem (GL-del Río-Sakurai, in progress)

Suppose that

1. $\mathbb{F}_p G = A_1 \otimes A_2$, *and*
2. A_1 *is commutative.*

Then $G = G_1 \times G_2$ *and* $A_i \cong \mathbb{F}_p G_i$ *for each* i.

This implies that $\mathbb{F}_p G$ admits a unique decomposition $A_1 \otimes A_2$ with A_1 commutative and A_2 without commutative tensor factors.

Tensor indecomposability

Does every directly indecomposable group have tensor indecomposable group algebra over k?

Tensor indecomposability

Does every directly indecomposable group have tensor indecomposable group algebra over k?

Carlson and Kovacs consider this last question in their paper, providing a positive answer if

- either G is abelian,
- or G has order 8.

Tensor indecomposability

Does every directly indecomposable group have tensor indecomposable group algebra over k?

Carlson and Kovacs consider this last question in their paper, providing a positive answer if

- either G is abelian,
- or G has order 8.

Corollary (GL, del Río, Sakurai)

Suppose that G is indecomposable and

- *either G can be generated by 3 elements, or*
- *G' is cyclic.*

Then $\mathbb{F}_p G$ is indecomposable.

More questions on modular p-group algebras

Question

What information about G can be recovered from kG?

More questions on modular p-group algebras

Question

What information about G can be recovered from kG? And how?

More questions on modular p-group algebras

Question

What information about G can be recovered from kG? And how?

Question (Modular Isomorphism Problem)

Can the isomorphism type of G be recovered from $\mathbb{F}_p G$?

More questions on modular p-group algebras

Question

What information about G can be recovered from kG? And how?

Question (Modular Isomorphism Problem)

Can the isomorphism type of G be recovered from $\mathbb{F}_p G$?

Answer:

- If $p = 2$, it can't (GL-Magolis-del Río, 2022).
- If $p > 2$, we do not know.

The modular isomorphism problem

Question (Modular Isomorphism Problem)

Can the isomorphism type of G be recovered from $\mathbb{F}_p G$?

Yes, provided that one of the following holds:

- G is abelian (Deskins, 1956).
- G is metacyclic (Bagiński 1988, Sandling 1996).
- $\gamma_2(G)^p \gamma_3(G) = 1$ (Sandling, 1989).
- ...

The modular isomorphism problem

Question (Modular Isomorphism Problem)

Can the isomorphism type of G be recovered from $\mathbb{F}_p G$?

Yes, provided that one of the following holds:

- G is abelian (Deskins, 1956).
- G is metacyclic (Bagiński 1988, Sandling 1996).
- $\gamma_2(G)^p \gamma_3(G) = 1$ (Sandling, 1989).
- ...

The modular isomorphism problem

Theorem (GL-Brenner)

Suppose that

- $p > 2$ *and*
- $|G : \mathrm{Z}(G)| \leq p^3$.

Then the isomorphism type of G can be recovered from $\mathbb{F}_p G$.

The modular isomorphism problem

Theorem (GL-Brenner)

Suppose that

- $p > 2$ *and*
- $|G : \mathrm{Z}(G)| \leq p^3$.

Then the isomorphism type of G *can be recovered from* $\mathbb{F}_p G$.

Theorem (GL-del Río)

Suppose that

- $p > 2$,
- G *can be generated by* 2 *elements,*
- G' *is cyclic, and*
- *either* $|G'| \leq p^3$ *or* $|\gamma_3(G)| \leq p$.

Then the isomorphism type of G *is determined by* $\mathbb{F}_p G$.

The modular isomorphism problem

Theorem (GL-Brenner)

Suppose that

- $p > 2$ *and*
- $|G : \mathrm{Z}(G)| \leq p^3$.

Then the isomorphism type of G *can be recovered from* $\mathbb{F}_p G$.

Theorem (GL-del Río)

Suppose that

- $p > 2$,
- G *can be generated by* 2 *elements,*
- G' *is cyclic, and*
- *either* $|G'| \leq p^3$ *or* $|\gamma_3(G)| \leq p$.

Then the isomorphism type of G *is determined by* $\mathbb{F}_p G$.

These results both fail when $p = 2$.

Thanks for your attention.