The classification of the prime graphs of finite solvable cut/rational groups

Sara Cebellán Debón
(joint work with Diego García-Lucas and Ángel del Río)

University of Murcia

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Graphs defined on groups

$$\{Groups\} \longrightarrow \{Graphs\}$$
$$G \longmapsto \Gamma(G)$$

Which properties of the group G can be recovered from the graph-theoretical properties of $\Gamma(G)$?

Examples:

- Commuting graph
- Cyclic graph
- Solvable conjugacy class graph
- Prime graph

Prime graph of a finite group

G a finite group

We construct the graph $\Gamma_{GK}(G) = \{V_{vertices}, E_{edges}\}$ as follows:

- $V = \pi(G) = \{\text{primes dividing } |G|\}$
- $p, q \in V$ different,

$$p - q \in E \iff \exists\, g \in G : \ |g| = pq$$

Definition (Prime graph or GK-graph)

$\Gamma_{GK}(G)$ is the *prime graph* or *Gruenberg-Kegel graph* of G.

Example:

C_6 2 ●—● 3 S_3 2 ● ● 3

Example: A_5 has prime graph 2 ● ● 3
5 ●

Some results about prime graphs:

- Gruenberg-Kegel: G finite solvable group,

  $\Gamma_{GK}(G)$ disconected $\iff$ G Frobenius or 2-Frobenius.

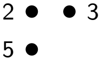
- Lucido: G finite group such that $\Gamma_{GK}(G)$ is a tree $\implies |\pi(G)| \leq 8$.

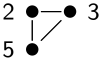
- Gavrilyuk-Khramtsov-Kondrat'ev-Maslova: Characterization of the prime graphs of finite groups with at most 5 vertices.

- Gruber-Keller-Lewis-Naughton-Strasser: Characterization of the prime graphs of finite solvable groups.

Theorem (Gruber-Keller-Lewis-Naughton-Strasser)

A graph Γ is isomorphic to the prime graph of a finite solvable group if and only if its complement $\overline{\Gamma}$ is 3-colorable and triangle-free.

Example: $\Gamma =$ 

$\Longrightarrow \overline{\Gamma} =$ 

, $\Gamma \neq \Gamma_{GK}(G)$ for G finite solvable

- Question 1: What are the prime graphs of finite solvable groups satisfying some property P?
- Question 2 (Realizability): Given a graph Γ, is there a finite solvable group G satisfying some property P such that $\Gamma = \Gamma_{GK}(G)$?

$P =$ rational/cut.

Rational/Cut = Inverse-semirational

G a finite group

$g \in G$ *rational* $\overset{def}{\iff} (\forall h$ generator of $\langle g \rangle, h \sim g)$

Definition (Rational)

A group G is *rational* if every element of G is rational.

Example: S_n is rational for every $n \in \mathbb{N}$.

$g \in G$ *inverse semi-rational* $\overset{def}{\iff} (\forall h$ generator of $\langle g \rangle, h \sim g$ or $h \sim g^{-1})$

Definition (Cut = Inverse-semirational)

A group G is *cut* if every element of G is inverse-semirational.

Example: The monster group M is cut.

$$\{\text{Rational groups}\} << \{\text{Cut groups}\}$$

Remark (Bächle-Caicedo-Jespers-Maheshwary)

Among groups of order at most 1023:

- *78.55% are cut groups*
- *0.52% are rational.*

$g \in G$,

$$B_G(g) = \frac{N_G(\langle g \rangle)}{C_G(g)} \lesssim Aut(\langle g \rangle)$$

φ = Euler's totient function

- g rational $\iff |B_G(g)| = \varphi(|g|)$
- g inverse-semirational $\iff \begin{cases} g \text{ rational or} \\ |B_G(g)| = \varphi(|g|)/2 \text{ and } g \not\sim g^{-1} \end{cases}$

The classification of the prime graphs of finite solvable rational/cut groups

Example: There are no finite cut groups having $(3-7)$ as prime graph.

G cut and $g \in G$ of order 21 $\implies 2 \mid |B_G(g)|$

- Is it possible to classify the prime graphs of finite rational/cut groups?

  Difficult to approach! Each prime p divides the order of S_p.

- What about finite solvable rational/cut groups?

Theorem (Gow)

Let G be a finite solvable rational group. Then $\pi(G) \subseteq \{2,3,5\}$.

Theorem (Bächle)

Let G be a finite solvable cut group. Then $\pi(G) \subseteq \{2,3,5,7\}$.

The classification of Bächle-Kiefer-Maheshwary-del Rio

GK-graphs of solvable finite cut/rational groups:

- 1 vertex:

a) 2 •

C_2

b) 3 •

C_3

- 2 vertices:

c) 2 • • 3

S_3

d) 2 •—• 3

$S_3 \times C_2$

e) 2 •

5 •

$K = C_5^2 \rtimes_{Fr} Q_8$

f) 2 •
|
5 •

$K \times C_2$

g) 3 •

7 •

$C_7 \rtimes_{Fr} C_3$

The classification of Bächle-Kiefer-Maheshwary-del Rio

GK-graphs of solvable finite cut/rational groups:

- 1 vertex:

  a) $2 \bullet$ — C_2

  b) $3 \bullet$ — C_3

- 2 vertices:

  c) $2 \bullet \quad \bullet 3$ — S_3

  d) $2 \bullet\!\!-\!\!\bullet 3$ — $S_3 \times C_2$

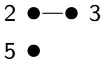
  e) $2 \bullet$, $5 \bullet$ — $K = C_5^2 \rtimes_{Fr} Q_8$

  f) $2 \bullet$ — $5 \bullet$ — $K \times C_2$

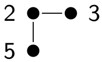
  g) $3 \bullet$, $7 \bullet$ — $C_7 \rtimes_{Fr} C_3$

GK-graphs of solvable finite cut/rational groups:

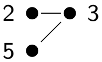
- 3 vertices:

h) 

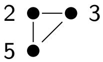
$L = C_5^2 \rtimes_{Fr} Dic_3$

i) 

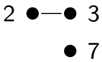
$L \times C_2$

j) 

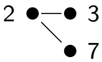
$K \times C_3$

k) 

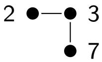
$K \times S_3$

l) 

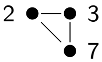
$M = C_7 \rtimes_{Fr} C_6$

m) 

$M \times C_2$

n) 

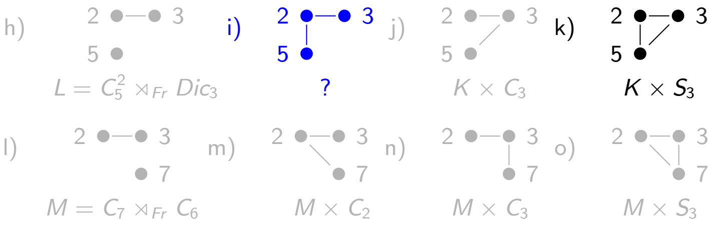
$M \times C_3$

o) 

$M \times S_3$

GK-graphs of solvable finite cut/rational groups:

- 3 vertices:



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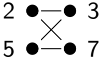
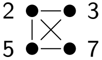
GK-graphs of solvable finite cut/rational groups:

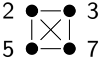
- 4 vertices:

p) 

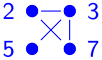
$N = K \times (C_7 \rtimes_{Fr} C_3)$

q) 

$N \times C_2$

r) 

$N \times S_3$

s) 

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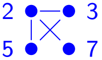
t) 

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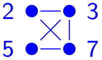
u) 

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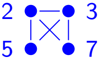
v) 

?

GK-graphs of solvable finite cut/rational groups:

- 4 vertices:

p) 

$N = K \times (C_7 \rtimes_{Fr} C_3)$

q) 

$N \times C_2$

r) 

$N \times S_3$

s) 

?

t) 

?

u) 

?

v) 

?

Summarizing...

To complete the classification we have to answer the following questions.

For rational groups:

Question (Bächle-Kiefer-Maheshwary-del Rio)

Is $(3 - 2 - 5)$ *the GK-graph of a finite solvable rational group?*

For cut groups:

Question (Bächle-Kiefer-Maheshwary-del Rio)

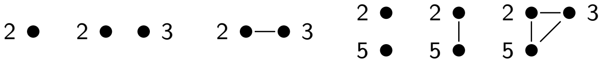
Which of the four graphs $s), t), u)$ *and* $v)$ *are realizable as the GK-graph of some finite solvable cut group?*

For rational groups:

Theorem (CD, Garcia-Lucas, del Río)

$(3 - 2 - 5)$ *is not the GK-graph of a finite solvable rational group.*

The GK-graphs of non-trivial finite solvable rational groups are precisely the following:



Corollary

If G is a finite solvable rational group of order divisible by 15*, then G has elements of order* 6, 10 *and* 15.

For cut groups:

At the moment...we do not know much. But we have the following conjecture.

Conjecture

A finite solvable cut group having GK-graph s), t), u) or v) has Fitting length at least 5*.*

This is true for *s*).

Equivalently,

Conjecture

The GK-graphs of finite solvable cut groups with Fitting length at most 4 are the graphs a) - r).

Thank you!

G a finite solvable rational group and $V \in \mathrm{Syl}_5(G)$

Theorem (Hegedüs)

V is normal and elementary abelian.

Proposition (Hegedüs)

Assume V is minimal normal and let S be a complement of V in G. If $H \leq S$ is minimal such that there exists an $\mathbb{F}_5 H$-submodule $W \leq V$ satisfying some properties then one of the following holds:

- $H/K \cong Q_8$ *and* $W = \mathbb{F}_5^2$,
- $H/K \cong C_3 \rtimes C_4$ *and* $W = \mathbb{F}_5^2$,
- $H/K \cong \mathrm{SL}(2,3)$ *and* $W = \mathbb{F}_5^2$,
- $H/K \cong SG[192, 989]$ *and* $W = \mathbb{F}_5^4$,
- $H/K \cong SG[96, 191]$ *and* $W = \mathbb{F}_5^4$ *or*
- H/K *has order* 144

where $K = C_H(W)$.

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