Rationality of finite groups: Groups with quadratic field of values

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Introduction

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Associating a field to any element $g \in G$ and having some global proprieties of those field can restrict a lot the structure of the group itself.

Notation

- Every group is finite.
- $x \sim y$ denotes the conjugation in the group.
- $|g|$ is the order of the element $g \in G$.
- $\mathbb{Q}_n := \mathbb{Q}(e^{2\pi i/n})$
- $Irr(G)$ denotes the set of irreducible complex characters of the group G.
- Cl_G denotes the set of conjugacy classes of G.

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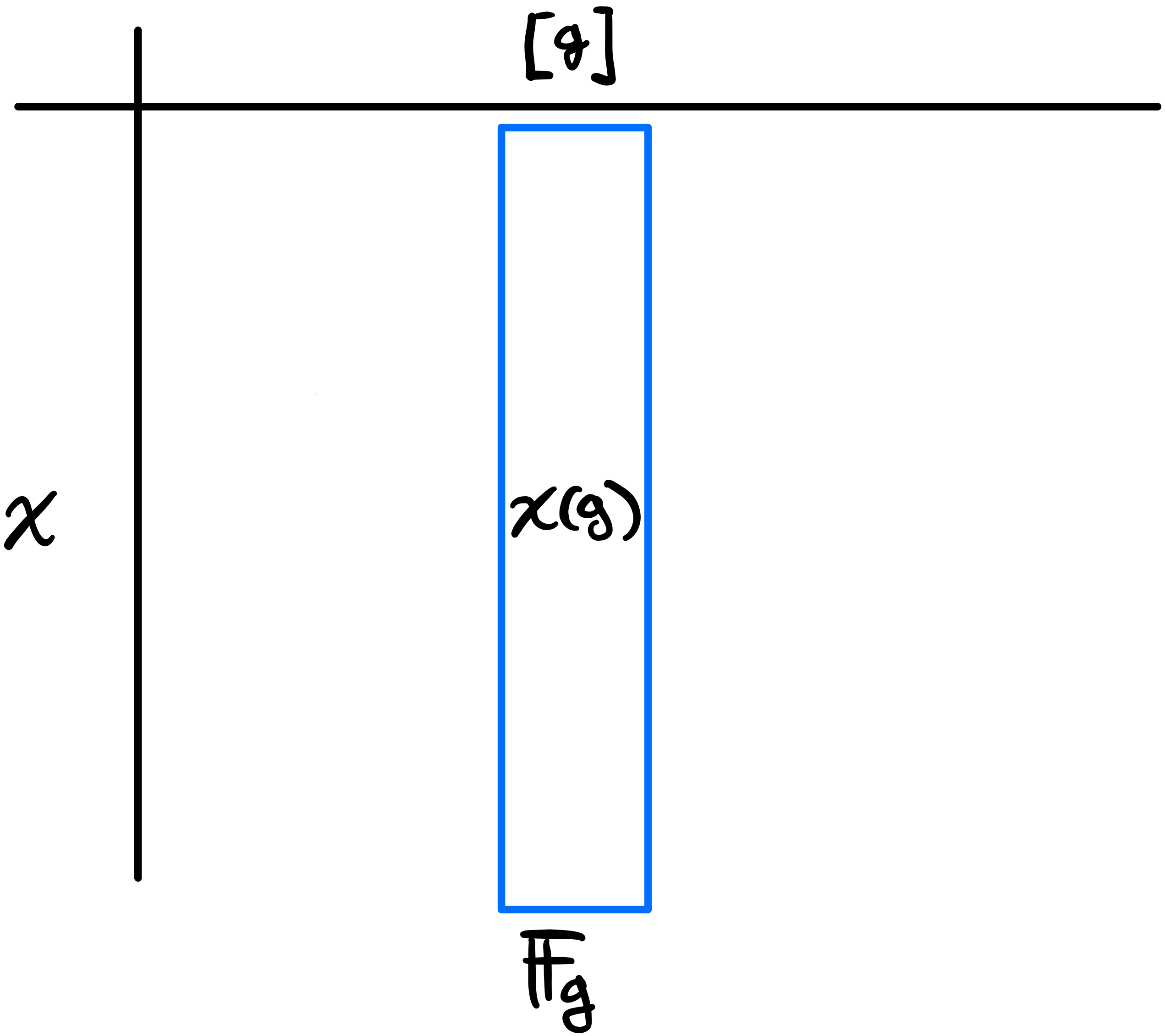
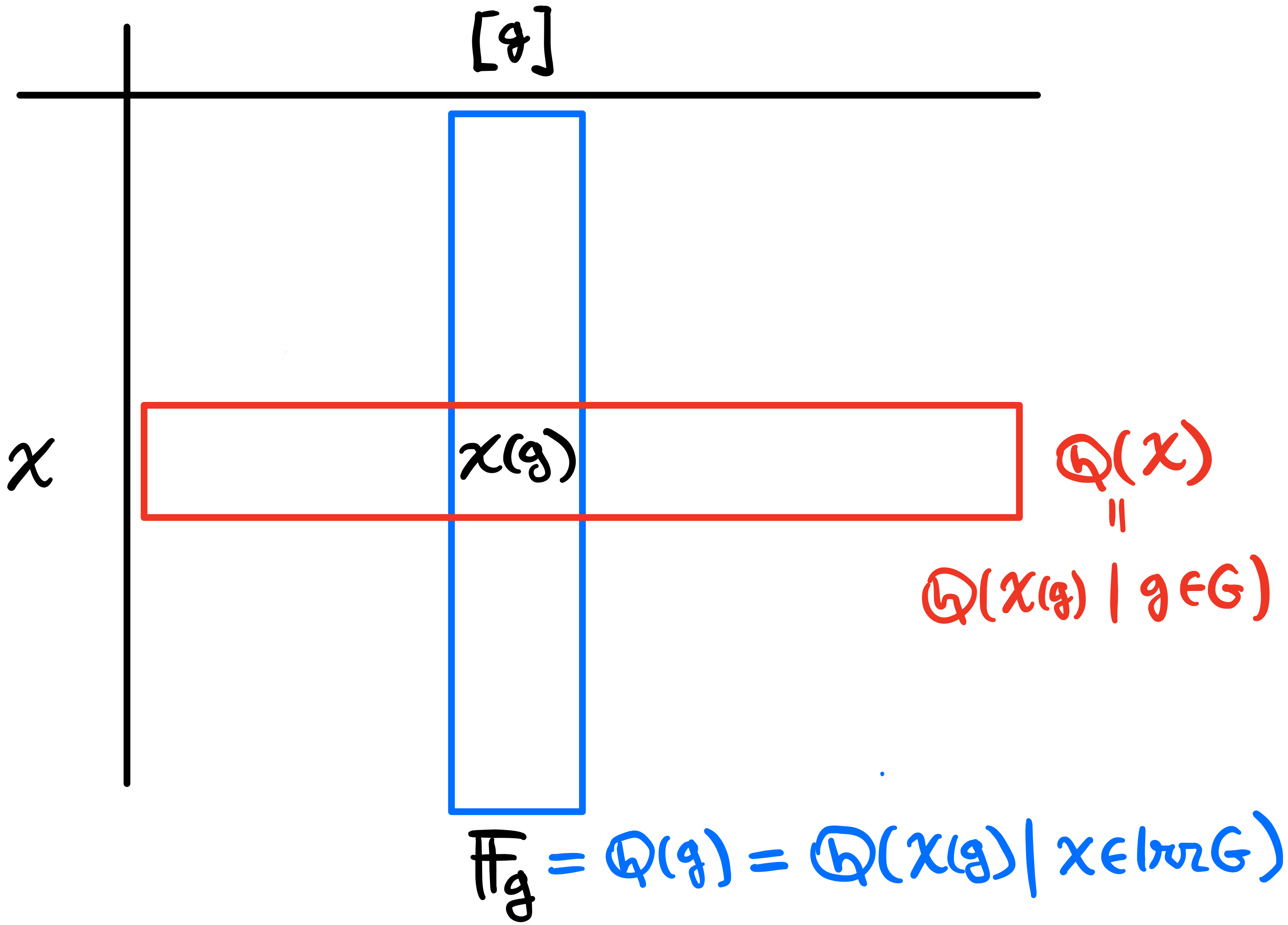
$$\mathrm{Stab}_{\mathrm{Gal}(\mathbb{Q}_n/\mathbb{Q})}[g] = H \qquad \mathbb{F}_g := \mathrm{Fix}(H)$$

$\mathbb{F}_g$ is called field of value of g

	$[g]$
χ	$\chi(g)$


[g]
χ
χ(g)
F_g


$[g]$
χ
$\chi(g)$
$\mathbb{Q}(\chi) = \mathbb{Q}(\chi(g) \mid g \in G)$
$\mathbb{F}_g = \mathbb{Q}(g) = \mathbb{Q}(\chi(g) \mid \chi \in \mathrm{Irr}\,G)$

Some definitions

Definition

A group G is called **quadratic rational** iff $\forall \chi \in \mathrm{Irr}(G)$ then $[\mathbb{Q}(\chi) : \mathbb{Q}] \leq 2$, where $\mathbb{Q}(\chi) = \mathbb{Q}(\chi(g) | g \in G)$.

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hence consider $\sigma \in \mathrm{Gal}(\mathbb{Q}(g)/\mathbb{Q})$

$[g]^\sigma \neq [g]$

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Definition

An element x is called semirational if exists an integer $(m_x, |x|) = 1$ such that for any integer $(j, |x|) = 1$ then

$$x^j \sim x \text{ or } x^j \sim x^{m_x}$$

Some remarks

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A group is called UNIFORMLY SEMIRATIONAL (USR) if there exists an integer r $(r, \exp(G)) = 1$ st: G is r-semirational

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- D_{10} is ± 3-semirational but not inverse semirational.
- `SmallGroup(32, 42)` is quadratic rational but not semirational.
- `SmallGroup(32, 9)` is semirational but not quadratic rational.

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The following are equivalent.

1. G is inverse semirational.
2. If $\mathbb{Q}G \cong \bigoplus_{k=1}^{m} M_{n_k}(D_k)$ is the Wedderburn decomposition where $m, n_k \in \mathbb{Z}_{\geq 1}$ and D_k rational division algebras for each k, then

$$\mathcal{Z}(D_k) \cong \mathbb{Q}(\sqrt{-d})$$

for some $d \in \mathbb{Z}_{\geq 0}$ square free.

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4. G is **cut**.

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In general we have the inclusion

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Definition

A finite group G is called **cut** (central units trivial) iff

$$\mathcal{Z}(\mathcal{U}(\mathbb{Z}G)) = \pm\mathcal{Z}(G)$$

Gruenberg-Kegel Graph (Prime graph)

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Definition

The prime graph (or Gruenberg-Kegel graph) is the undirected loop-free and multiple-free graph whose vertices are the primes in the prime spectra of G, and two vertices p and q are joined by an edge, if and only if G contains an element of order pq.

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$$\pi(G) \subseteq \{2, 3, 5, 7, 13\}$$

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Results on prime graphs of solvable groups

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Let G a finite solvable group, then $\Gamma_{GK}(G)$ has at most 2 connected components, and has exactly 2 components, if and only if G is a Frobenius group or a 2-Frobenius group.

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In particular, since we want to work with quadratic rational groups, I was interested in classifying quadratic rational Frobenius groups.

Classification of quadratic rational Frobenius groups

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Theorem A. *Let H be a Frobenius complement of a quadratic rational group.*

(1) *If $|H|$ is even then*

(a) *If G is rational then G is isomorphic to one of the following:*
- (i) $C_3^n \rtimes_{Fr} C_2$.
- (ii) $C_3^{2n} \rtimes_{Fr} Q_8$.
- (iii) $C_5^2 \rtimes_{Fr} Q_8$.

(b) *If G is cut non-rational then G is isomorphic to one of the following:*
- (i) $C_3^{2n} \rtimes_{Fr} C_4$
- (ii) $C_5^n \rtimes_{Fr} C_4$
- (iii) $C_7^n \rtimes_{Fr} C_6$
- (iv) $C_7^{2n} \rtimes_{Fr} (C_3 \times Q_8)$
- (v) $C_5^2 \rtimes_{Fr} (C_3 \rtimes C_4)$
- (vi) $C_5^2 \rtimes_{Fr} SL(2,3)$
- (vii) $C_7^2 \rtimes_{Fr} SL(2,3)$

(c) *If G is non cut and 2-USR then G is isomorphic to one of the following:*
- (i) $C_3^{4n} \rtimes_{Fr} Q_{16}$
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- (iii) $C_5^{2n} \rtimes_{Fr} (C_3 \rtimes Q_8)$
- (iv) $C_7^2 \rtimes_{Fr} SL(2,3).C_2$

(d) *If G is not 2-USR and 4-USR then G is isomorphic to one of the following:*
- (i) $C_5^{2n} \rtimes_{Fr} Q_8$
- (ii) $C_5^{2n} \rtimes_{Fr} (C_3 \rtimes C_4)$
- (iii) $C_5^{2n} \rtimes_{Fr} SL(2,3)$
- (iv) $C_5^n \rtimes_{Fr} C_2$ *w*
- (v) $C_5^a \times C_5^b \rtimes C_4$
- (vi) $C_5^{2n} \rtimes_{Fr} C_6$
- (vii) $C_{13}^n \rtimes_{Fr} C_6$
- (viii) $C_3^{4n} \rtimes_{Fr} (C_5 \rtimes C_4)$
- (ix) $C_7^{4n} \rtimes (C_3 \times Q_8)$
- (x) $C_{13}^{2n} \rtimes_{Fr} (C_3 \times Q_8)$
- (xi) $C_5^{4n} \rtimes_{Fr} (C_3 \rtimes Q_{16})$
- (xii) $C_5^{4n} \rtimes (SL(2,3).C_2)$
- (xiii) $C_{11}^2 \rtimes_{Fr} SL(2,5)$

In particular $C_{11}^2 \rtimes SL(2,5)$ is the only non solvable Frobenius quadratic rational group.

(2) *If $|H|$ is odd, then $H \cong C_3$ and one of the following holds:*

(a) *K is cut 2-group admitting a fixed-point-free automorphism of order 3. In particular $|K| = 4^a$ for some $a \in 2\mathbb{Z}_{\geq 1}$ and is an extension of an abelian group of exponent 4 by another abelian group of exponent 4.*

(b) *K is a 7-group admitting a fixed-point-free automorphism of order 3 that fixes every cyclic subgroup, of exponent 7 and it's an extension of an elementary abelian 7-group by another elementary abelian 7-group.*

In particular, every quadratic rational Frobenius group is uniformly semirational.

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If $r^2 \equiv 1 \bmod \exp(G) \longleftrightarrow$ cut Groups

Theorem (MV)

Let G be a solvable group. Suppose that G is r-semirational with some r such that $r^2 \equiv 1 \bmod n$. Then:

$$\pi(G) \subseteq \{2, 3, 5, 7\}$$

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Equivalences of r-semirational groups

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Let G be a group with exponent n, $\mathbb{F}$ be a subfield of $\mathbb{Q}_n$ fixed by the cyclic subgroup generated by $\sigma_r \in \mathcal{G}al(\mathbb{Q}_n/\mathbb{Q})$ such that $(r, n) = 1$ and $\sigma_r(\zeta_n) = \zeta_n^r$. Then the following are equivalent:

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1. G is r-semirational.
2. G is quadratic rational with $\forall \chi \in \mathrm{Irr}\, G$ st: $\mathbb{Q}(\chi) \cap \mathbb{F} = \mathbb{Q}$.

Which r makes a group r-semirational

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Definition

Let G be an USR group and $n = exp(G)$ then we call:

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We can observe that, fixed the group G, R_G is the coset of the group:

$$H_G = \{r \in \mathcal{U}(\mathbb{Z}/n\mathbb{Z}) | g^r \sim g \ \forall g \in G\} \cong \mathcal{G}al(\mathbb{Q}_n/\mathbb{Q}(G))$$

. And in particular

$$(\mathcal{U}(\mathbb{Z}/n\mathbb{Z}))^2 \leq H_G$$

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Table: Possible R_G for quasi-rational 2-groups with exponent at least 8

$\{-1,3\}$	$\{-1,-3\}$	$\{3,-3\}$	$\{-1\}$	$\{3\}$	$\{-3\}$
$\langle a\rangle_8 : \langle x\rangle_2$ $a^x = a^{-3}$	$\langle a\rangle_8 : \langle x\rangle_2$ $a^x = a^3$	$\langle a\rangle_8 : \langle x\rangle_2$ $a^x = a^{-1}$	$\langle a\rangle_8 \times \langle b\rangle_4 : \langle x\rangle_2$ $a^x = a^3$ $b^x = a^4b^{-1}$ $a^y = a^5b^2$ $b^y = b^{-1}$	$\langle a\rangle_8 \times \langle b\rangle_4 : \langle x,y\rangle_2$ $a^x = a^{-1}$ $b^x = a^4b^{-1}$ $a^y = a^5b^2$ $b^y = b^{-1}$	$\langle a\rangle_8 \times \langle b\rangle_4 : \langle x,y\rangle_2$ $a^x = a^{-1}$ $b^x = a^4b^{-1}$ $a^y = a^5b^2$ $b^y = a^4b$

$\{2,3\}-$groups

$\{\pm 5, \pm 7\}$	$\{\pm 7, \pm 11\}$	$\{\pm 5, \pm 11\}$
$\langle a \rangle_{24} : \langle x, y \rangle_2$ $a^x = a^{-1}$ $a^y = a^{-11}$ `SmallGroup(96,115)`	$\langle a \rangle_{24} : \langle x, y \rangle_2$ $a^x = a^{-1}$ $a^y = a^{-5}$ `SmallGroup(96,121)`	$\langle a \rangle_{24} : \langle x, y \rangle_2$ $a^x = a^{-1}$ $a^y = a^{7}$ `SmallGroup(96,117)`
$\{-1, -7, 5, 11\}$	$\{-1, -11, 5, 7\}$	$\{-1, -5, 7, 11\}$
$\langle a \rangle_{24} : \langle x, y \rangle_2$ $a^x = a^{-11}$ $a^y = a^{-5}$ `SmallGroup(96,183)`	$\langle a \rangle_{24} : \langle x, y \rangle_2$ $a^x = a^{-5}$ $a^y = a^{11}$ `SmallGroup(96,120)`	$\langle a \rangle_{24} : \langle x, y \rangle_2$ $a^x = a^{5}$ $a^y = a^{-11}$ `SmallGroup(96,113)`
	$\{-1, -11, -5, -7\}$	
	$\langle a \rangle_{24} : \langle x, y \rangle_2$ $a^x = a^{5}$ $a^y = a^{11}$ `SmallGroup(96,118)`	
$\{-1, 11\}$	$\{7, -5\}$	$\{7, 11\}$
`SmallGroup(192,95)`	`SmallGroup(192,305)`	`SmallGroup(192,412)`
$\{5, 7\}$	$\{-1, -7\}$	$\{\pm 7\}$
`SmallGroup(192,414)`	`SmallGroup(192,713)`	`SmallGroup(192,415)`
$\{-1, 7\}$	$\{-7, -5\}$	$\{5, -7\}$
`SmallGroup(192,418)`	`SmallGroup(192,435)`	`SmallGroup(192,623)`
$\{-1, -5\}$	$\{\pm 5\}$	$\{11, -5\}$
`SmallGroup(192,440)`	`SmallGroup(192,949)`	`SmallGroup(192,438)`
$\{-1, 5\}$	$\{5, 11\}$	$\{11, -7\}$
`SmallGroup(192,1396)`	`SmallGroup(192,632)`	`SmallGroup(192,726)`
$\{7\}$	$\{-5\}$	$\{-1\}$
`SmallGroup(192,424)`	`SmallGroup(192,445)`	`SmallGroup(192,634)`
$\{5\}$	$\{11\}$	$\{-11\}$
`SmallGroup(192,595)`	`SmallGroup(192,631)`	?

In the previous situation we have fixed an exponent but we can observe that the situation seems indipendent to the choice of the exponent but more like on the prime spectra of the group. In fact H_G can be viewed as a subgroup of $\mathcal{U}(\mathbb{Z}/n\mathbb{Z})/\mathcal{U}(\mathbb{Z}/n\mathbb{Z})^2$.

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Question: Fix a coset R_G that is realized by some group of exponent n. Suppose to have another m such that

$$\mathcal{U}(\mathbb{Z}/n\mathbb{Z})/\mathcal{U}(\mathbb{Z}/n\mathbb{Z})^2 \cong \mathcal{U}(\mathbb{Z}/m\mathbb{Z})/\mathcal{U}(\mathbb{Z}/m\mathbb{Z})^2$$

Can we find another group H of exponent m such that $R_H = R_G$?

Theorem (delRio, MV)

Let $p \in \{2, 3\}$ and R any coset as before. Then for any n positive integer then exists a group G of exponent p^n that is USR and realizes R, meaning $R_G = R$.

Thank you