Reasoning About Lists

Laws of Concat

Recall the concatenation operation ++ on lists.   concat

We would like to verify that concatenation is associative, and that it admits the empty list Nil as neutral element to the left and to the right:

```
(xs ++ ys) ++ zs  =  xs ++ (ys ++ zs)
       xs ++ Nil  =  xs
       Nil ++ xs  =  xs
```

Q: How can we prove properties like these?

Laws of Concat

Recall the concatenation operation ++ on lists.

We would like to verify that concatenation is associative, and that it admits the empty list `Nil` as neutral element to the left and to the right:

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        Nil ++ xs  =  xs
```

Q: How can we prove properties like these?

A: By *structural induction* on lists.

Reminder: Natural Induction

Recall the principle of proof by *natural induction*:

To show a property $P(n)$ for all the integers $n \geq b$,

- Show that we have $P(b)$ (*base case*),
- for all integers $n \geq b$ show the *induction step*:

  if one has $P(n)$, then one also has $P(n + 1)$.

Example

Given:

```
def factorial(n: Int): Int =
  if (n == 0) 1           // 1st clause
  else n * factorial(n-1) // 2nd clause
```

Show that, for all n >= 4

```
factorial(n) >= power(2, n)
```

$$2^n$$

Base Case

Base case: 4

This case is established by simple calculations:

```
factorial(4) = 24 >= 16 = power(2, 4)
```

Induction step: n+1

We have for n >= 4:

```
factorial(n + 1)
```

$$\text{factorial}(n) \geq 2^n$$

Induction Step

Induction step: n+1

We have for n >= 4:

```
factorial(n + 1)

>=  (n + 1) * factorial(n)   // by 2nd clause in factorial
```

Induction Step

Induction step: n+1

We have for `n >= 4`:

```
factorial(n + 1)

>= (n + 1) * factorial(n)    // by 2nd clause in factorial

>  2 * factorial(n)          // by calculating
```

Induction Step

Induction step: n+1

We have for n >= 4:

```
  factorial(n + 1)

  >=  (n + 1) * factorial(n)   // by 2nd clause in factorial

  >   2 * factorial(n)         // by calculating
             \//
  >=  2 * power(2, n)          // by induction hypothesis
```

Induction Step

Induction step: n+1

We have for `n >= 4`:

```
factorial(n + 1)

>=  (n + 1) * factorial(n)   // by 2nd clause in factorial

>   2 * factorial(n)         // by calculating

>=  2 * power(2, n)          // by induction hypothesis

=   power(2, n + 1)          // by definition of power
```

$$2 \times 2^n = 2^{n+1}$$

Referential Transparency

Note that a proof can freely apply reduction steps as equalities to some part of a term.

That works because pure functional programs don't have side effects; so that a term is equivalent to the term to which it reduces.

This principle is called *referential transparency*.

Structural Induction

The principle of structural induction is analogous to natural induction:

To prove a property `P(xs)` for all lists `xs`,

- show that `P(Nil)` holds (*base case*),
- for a list `xs` and some element `x`, show the *induction step*:
  if `P(xs)` holds, then `P(x :: xs)` also holds.

Example

Let's show that, for lists xs, ys, zs:

```
(xs ++ ys) ++ zs  =  xs ++ (ys ++ zs)
```

To do this, use structural induction on xs. From the previous implementation of concat,

```scala
def concat[T](xs: List[T], ys: List[T]) = xs match {
  case List() => ys
  case x :: xs1 => x :: concat(xs1, ys)
}
```

distill two *defining clauses* of ++:

```
      Nil ++ ys  =  ys                // 1st clause
(x :: xs1) ++ ys  =  x :: (xs1 ++ ys)   // 2nd clause
```

Base Case

Base case: `Nil`

For the left-hand side we have:

```
(Nil ++ ys) ++ zs
```

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For the left-hand side we have:

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=   ys ++ zs        // by 1st clause of ++
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For the right-hand side, we have:

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Nil ++ (ys ++ zs)
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Base Case

Base case: `Nil`

For the left-hand side we have:

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(Nil ++ ys) ++ zs

= ys ++ zs      // by 1st clause of ++
```

For the right-hand side, we have:

```
Nil ++ (ys ++ zs)

= ys ++ zs      // by 1st clause of ++
```

This case is therefore established.

Induction step: `x :: xs`

For the left-hand side, we have:

```
((x :: xs) ++ ys) ++ zs
```

Induction step: `x :: xs`

For the left-hand side, we have:

```
((x :: xs) ++ ys) ++ zs

=   (x :: (xs ++ ys)) ++ zs     // by 2nd clause of ++
```

Induction Step: LHS

Induction step: `x :: xs`

For the left-hand side, we have:

```
  ((x :: xs) ++ ys) ++ zs

=   (x :: (xs ++ ys)) ++ zs     // by 2nd clause of ++

=   x :: ((xs ++ ys) ++ zs)     // by 2nd clause of ++
```

Induction Step: LHS

Induction step: `x :: xs`

For the left-hand side, we have:

```
  ((x :: xs) ++ ys) ++ zs

=  (x :: (xs ++ ys)) ++ zs     // by 2nd clause of ++

=  x :: ((xs ++ ys) ++ zs)     // by 2nd clause of ++

=  x :: (xs ++ (ys ++ zs))     // by induction hypothesis
```

For the right hand side we have:

```
(x :: xs) ++ (ys ++ zs)
```

For the right hand side we have:

```
(x :: xs) ++ (ys ++ zs)

=  x :: (xs ++ (ys ++ zs))    // by 2nd clause of ++
```

So this case (and with it, the property) is established.

Exercise

Show by induction on xs that xs ++ Nil = xs.

How many equations do you need for the inductive step?

- ● 2
- ○ 3
- ○ 4

Base case: $xs = Nil$

Nil ++ Nil

$= Nil$ // by 1st clause

Induction step: $x :: xs$

$(x :: xs)$ ++ Nil $= x :: xs$?

$= x :: (xs$ ++ $Nil)$ // 2nd clause

$= x :: xs$ // eq. i.h.

✓