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Inverses and n-uncial property of Jacobian elliptic functions

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Abstract. The inverses of Jacobi elliptic functions possess an apparently-non-crucial property: they provide almost-everywhere-conformal maps on a hemisphere onto a torus and so, onto a parallelogram. Thus, they produce map projections on the sphere generalizing the famous quincuncial projection of Charles S. Peirce. Besides providing a general practical definition of n-uncial map and proving that all the considered inverse elliptic functions are n-uncial, we give operative handy formulas to calculate these maps. To the best of our knowledge, these useful formulas have not been all together published before, except for Pierce projection. We look forward to their numerical implementation. By the way, we also classify the resulting map projections according the number of singularities.

Keywords: Elliptic functions and integrals (MSC 33E05), conformal mappings of special domains (MSC 30C20), numerical methods in conformal mapping theory (MSC 30C30), compact riemann surfaces and uniformization (MSC 30F10)

1. Introduction

In the theory of functions of a complex variable, a function is called elliptic if it is meromorphic doubly periodic, *cf.* Lang [11]. These fundamental properties, arised from the addition theorems, were absolutely necessary to Abel [1] and Jacobi [9] in their foundational works on the subject. During the nineteenth century, other related properties of elliptic functions were discovered little by little. Here we concentrate our efforts in a property that proves to be a key ingredient to conceive geographical projections with great utility and gracefulness. We refer specifically to a problem addressed in a paper by Richelot [19] and in Durege's well-known textbook [4]. A step forward was given by Peirce in his laconic paper [15], which was unriddled a little bit later by Pierpont [16].

In what follows we generalize Peirce projection through the notion of n-uncial function and prove that all the inverse functions of Jacobi elliptic functions are n-uncial with $n = 1, 3, 5$. The number n denotes the number of singularities of the map. Within this framework, Peirce projection results to be the unique function of this type which has $n = 5$. Certainly, the adjective quincuncial, introduced by Peirce [15], comes from the Latin word *quincuncialis*, relating to the noun *quincux*, an arrangement of five objects in a square or rectangle, one at each corner and one in the middle. In simple words, *quincux* is the usual arrangement of the five-spot face on six-sided dice, playing cards, and dominoes. *Quincux* also denotes a type of church from the second Byzantine Golden Age: a five-domed temple based on the domed

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cross element. The idea is interesting to the geometer since it evokes an almost-everywhere-conformal transformation of a hemisphere onto a cuadrilateral with five singular points (one in the middle and one at each corner), a key feature of Peirce quincuncial projection.

The importance of the n-uncial property is both theoretical and practical. On the one hand, being a characterizing property of the inverses of Jacobi elliptic functions, this property can be used to give an alternative definition of elliptic function, as far as the inverse of a function carries essentially the same information of the original function. This is particularly meaningful for elliptic functions because they are intrinsically meromorphic and the n-uncial property is about the number of singular points. On the other hand, n-uncial projections provide almost-everywhere conformal maps on the sphere onto a plane region and this is helpful to numerous applications. According to Peirce [15], the quincuncial map is useful for "meteorological, magnetological and other purposes". Remarkably enough, he does not mention "cartographic purposes". In general, cartographers indeed find the quincuncial projection interesting (cf. [12]), although completely impractical and this often happens to be the case with map projections based upon complex functions (and the associated computational intricateness). However, Peirce projection has shown to be advantageous because it tessellates the plane, that is, the hemispheres mosaic or tile indefinitely to cover the whole plane in an almost-everywhere-conformal manner. Being so, it faithfully displays nearly every locus on the sphere near to its neighboring loci (not all map projections share this feature). This explains the convenience of the map to earth magnetology and metereology, the primary interest of Peirce in 1879. In 1946, the U.S.A. Coast and Geodetic Survey [23] use the projection to draw a world map of air routes. In astronomy, Taylor and Bell [24] have graphed suitable quincuncial maps of the surface of the earth showing umbral limit lines for eclipses. And just as importantly, a group of biologists [17] have recently (2019) plotted on a Peirce chart the geographic distribution of Eurasian and American samples of genetic human male data in order to formulate hypotheses concerning the history "that shaped the present-day biological and cultural diversity" of the Americas. In three-dimensional computer graphic design, this mapping has also been applied to display spherical panoramas (e.g. [6]). Perhaps most remarkably, the projection serves as well for developing generalized longitudes and latitudes allowing to display the complete surface of a highly irregular body, such a comet. The literature on these appropriate maps for a comet surface is interesting and we refer the interested reader to [7,8].

That an inverse of an elliptic function can supply a plane conformal representation of a sphere might be suggested by the theory of general Schwarz-Christoffel mappings, *cf.* Driscoll and Trefethen [3]. However, here we are interested in maps of Riemann surfaces with boundary [5,10] that are conformal everywhere, except in a finite number of points. By the way, the striking 3D graphs of the complex modulus of elliptic functions exhibit certain *cornettos* about the poles and so, they insinuate us to consider the projections of these graphs onto their domains.

This paper is organized as follows. After reviewing some preparatory facts on Jacobi elliptic functions and setting up the basic notations in Section 2, we describe a general procedure to compute the values of Jacobi elliptic inverse functions in Section 3. Along the way, we introduce innovative convenient notations to write the complex modulus and argument of an arbitrary Jacobi elliptic function and, as a noteworthy outcome, we obtain a complete list of practical formulas for the inverses. In Section 4, we discuss the novel notion of n-uncial map. Among other things, we prove that Peirce projection is just but a particular case of these maps; in fact, it is the only n-uncial map with $n = 5$. Finally, in Section 5, we draw conclusions from the previous results and explain what we should expect from the graphical representations of these general mappings.

2. Jacobi elliptic functions

By their definition, elliptic functions map meromorphically a torus $\mathbb{T}$ to the Riemann sphere $\mathbb{S}$. We furnish $\mathbb{T}$ and $\mathbb{S}$ with their usual conformal structures. That is to say, $\mathbb{S}$ is the one-point compactification of the complex plane $\mathbb{C}$, provided with the atlas containing the stereographic projection; and $\mathbb{T} = \mathbb{C}/\Lambda$, where $\Lambda \subset \mathbb{C}$ is a lattice of rank 2, related to some fundamental parallelograms and to a fundamental pair of periods. Briefly, we endow both $\mathbb{T}$ and $\mathbb{S}$ with their customary structures of Riemann surfaces.

In his *Fundamenta nova* [9], Jacobi himself introduced a handful of elliptic functions as a sort of inverses to incomplete elliptic real integrals of the first kind

$$u = \int_0^{\varphi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}},$$

where $0 < k < 1$ is a parameter called elliptic modulus, *cf.* [13,22]. $u = u(\varphi)$ is odd and increasing and its inverse function $\varphi = \varphi(u)$ is the elliptic amplitude. This amplitude is likewise an odd increasing function, which increases by $\frac{\pi}{2}$ when u increases by

$$K = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}.$$

K is known as the complete elliptic integral of Legendre's first kind with elliptic modulus k. Given an elliptic modulus k, its complementary modulus k' is defined by $k'^2 = 1 - k^2$. The complete elliptic integral associated with the complementary modulus k' of k is written K'.

Jacobi elliptic complex functions are the function composition of the amplitude followed by a trigonometric or a trigonometric-like function. For example, the *sinus amplitudinis*, sine of the amplitude or elliptic sine, is just but the function $\mathrm{sn} u = \sin \varphi$. After the improved notations of Gudermann and Glaisher, the twelve Jacobi elliptic functions are nowadays denoted by $\mathrm{pq} u$, where p and q are any of the letters c (cosine), s (sine), n and d (delta). The symbol $\mathrm{pp} u$ stands for the unity. The basic Jacobi elliptic functions are

$$\mathrm{cn} u = \cos \varphi, \quad \mathrm{sn} u = \sin \varphi \quad \text{and} \quad \mathrm{dn} u = \sqrt{1 - k^2 \sin^2 \varphi}.$$

The remaining nine Jacobi elliptic functions are defined by

$$\mathrm{nc} u = \tfrac{1}{\cos \varphi} = \sec\varphi, \ \mathrm{cs} u = \tfrac{\cos \varphi}{\sin \varphi} = \cot \varphi, \ \mathrm{cd} u = \tfrac{\cos \varphi}{\sqrt{1 - k^2 \sin^2 \varphi}},$$

$$\mathrm{ns} u = \tfrac{1}{\sin \varphi} = \csc \varphi, \ \mathrm{sc} u = \tfrac{\sin \varphi}{\cos \varphi} = \tan \varphi, \ \mathrm{sd} u = \tfrac{\sin \varphi}{\sqrt{1 - k^2 \sin^2 \varphi}},$$

$$\mathrm{nd} u = \tfrac{1}{\sqrt{1 - k^2 \sin^2 \varphi}}, \ \ \mathrm{dc} u = \tfrac{\sqrt{1 - k^2 \sin^2 \varphi}}{\cos \varphi}, \ \ \mathrm{ds} u = \tfrac{\sqrt{1 - k^2 \sin^2 \varphi}}{\sin \varphi}.$$

After several complicate attempts using only the addition formulas, Jacobi [9] discovered that the right setting for the study of these functions should be based on the notion of theta function. In terms of the modern Neville theta functions θ_p and θ_q, Jacobi elliptic functions can be certainly defined as

$$\mathrm{pq} u = \frac{\theta_p u}{\theta_q u}.$$

We refer the interested reader to the details in the classic book of Neville [14].

With the aid of theta functions, it is not hard to see that the periods, zeros and poles of Jacobi elliptic functions are those shown in Table 1. Each pair of integer values n, m determines a unique zero and a unique pole. The periods, zeros and poles for the remaining Jacobi elliptic functions can be easily found from Table 1 and the definition of the functions.

Table 1
Fundamental periods, zeros and poles of Jacobi elliptic functions. $n, m \in \mathbb{Z}$

Function	Fundamental periods	Zeros	Poles
cn	$4K, 2(K+iK')$	$(2m+1)K+2niK'$	$2mK+(2n+1)iK'$
sn	$4K, 2iK'$	$2mK+2niK'$	$2mK+(2n+1)iK'$
dn	$2K, 4iK'$	$(2m+1)K+(2n+1)iK'$	$2mK+(2n+1)iK'$
cs	$2K, 4iK'$	$(2m+1)K+2niK'$	$2mK+2niK'$
cd	$4K, 2iK'$	$(2m+1)K+2niK'$	$(2m+1)K+(2n+1)iK'$
sd	$4K, 2(K+iK')$	$2mK+2niK'$	$(2m+1)K+(2n+1)iK'$

Now, Jacobi elliptic functions defined on a fundamental parallelogram spanned by a pair of periods (or a torus, if we like) on the sphere are not injective. Such fundamental regions contains two zeros and two poles and, more generally, any of these functions has multiplicity two or it is "two-to-one". Nonetheless, each fundamental parallelogram can be split into two smaller parallelograms in such a way that the restriction of the elliptic function to any of them is one-to-one. In the interior of such smaller parallelograms lie exactly one pole and one zero (singular points). For our purposes here, we will even divide each of such smaller parallelograms into two lesser parallelograms, each containing only one singular point (pole or zero). From now on, with no harm to the argumentation, we shall deal exclusively with these injective restrictions of Jacobi elliptic functions. Stated briefly, the domain of such restricted functions is a "small enough" open parallelogram and the range is an open hemisphere of the Riemann sphere. In addition, the only singular point lies always at the center of the parallelogram. Due to the symmetries introduced by the group generated by the fundamental periods, there are many possible choices of such open parallelograms.

In the next section we will make use of many famous identities of the Jacobi elliptic functions, or shortly, elliptic identities. Among them, the most important are perhaps the celebrated addition formulas such as

$$\mathrm{sn}(u+v) = \frac{\mathrm{sn}u\ \mathrm{cn}v\ \mathrm{dn}v + \mathrm{sn}v\ \mathrm{cn}u\ \mathrm{dn}u}{1-k^2\mathrm{sn}^2u\ \mathrm{sn}^2v} \quad \text{and} \quad \mathrm{dn}(u+v) = \frac{\mathrm{dn}u\ \mathrm{dn}v - k^2\mathrm{sn}u\ \mathrm{sn}v\ \mathrm{cn}u\ \mathrm{cn}v}{1-k^2\mathrm{sn}^2u\ \mathrm{sn}^2v}.$$

There are as well essential relations between squares of the functions or Pythagorean-like identities, for example,

$$\mathrm{sn}^2u + \mathrm{cn}^2u = 1 \quad \text{and} \quad k^2\mathrm{sn}^2u + \mathrm{dn}^2u = 1.$$

For the inversion of odd elliptic functions we will resort to some, less-obvious, identities. They comprise the so-called real Jacobi transformations [9,20], for instance,

$$k'\mathrm{sd}u = -\mathrm{cn}(u+K).$$

Also, an important example of an imaginary Jacobi transformation will be given and used in the next section. By the way, the functions cn, dn, cd and their multiplicative inverses (reciprocal functions) are even. The remaining Jacobi elliptic functions are odd.

In the subsequent sections, we will use freely all the identities we could prove, or find in the literature. In particular, we apply those elliptic identities in Chapter 16 of Abramowitz and Stegun [2]. We have also found serviceable the 74 identities in Weisstein [25] and those in Reinhardt and Walker [18].

3. Inverse elliptic functions

Up to this point, we have denoted the elliptic functions by the symbols pq$u$. Since from now it is necessary to specify the elliptic modulus k, we will employ the notation pq(u,k). The idea behind the

Table 2
Even functions: dependence of x, y on the complex moduli of some elliptic functions

Function	Condition on x	Condition on y
cn^{-1}	$\mathrm{cn}(2x,k) = \frac{{}^{k}_{u}\rho^{2}_{cn} - {}^{k}_{u}\rho^{2}_{sn}\,{}^{k}_{u}\rho^{2}_{dn}}{1-k^2\,{}^{k}_{u}\rho^{4}_{sn}}$	$\mathrm{cn}(2y,k') = \frac{1-k^2\,{}^{k}_{u}\rho^{4}_{sn}}{{}^{k}_{u}\rho^{2}_{cn} + {}^{k}_{u}\rho^{2}_{sn}\,{}^{k}_{u}\rho^{2}_{dn}}$
dn^{-1}	$\mathrm{dn}(2x,k) = \frac{{}^{k}_{u}\rho^{2}_{dn} - k^2\,{}^{k}_{u}\rho^{2}_{sn}\,{}^{k}_{u}\rho^{2}_{cn}}{1-k^2\,{}^{k}_{u}\rho^{4}_{sn}}$	$\mathrm{dc}(2y,k') = \frac{{}^{k}_{u}\rho^{2}_{dn} + k^2\,{}^{k}_{u}\rho^{2}_{sn}\,{}^{k}_{u}\rho^{2}_{cn}}{1-k^2\,{}^{k}_{u}\rho^{4}_{sn}}$
cd^{-1}	$\mathrm{cd}(2x,k) = \frac{{}^{k}_{u}\rho^{2}_{cd} - {}^{k}_{u}\rho^{2}_{sn}}{1-k^2\,{}^{k}_{u}\rho^{2}_{sn}\,{}^{k}_{u}\rho^{2}_{cd}}$	$\mathrm{dn}(2y,k') = \frac{{}^{k}_{u}\rho^{2}_{cd} + {}^{k}_{u}\rho^{2}_{sn}}{1+k^2\,{}^{k}_{u}\rho^{2}_{sn}\,{}^{k}_{u}\rho^{2}_{cd}}$

following procedure is to generalize the methods of Pierpont [16] and Solanilla, Oostra and Yañez [21], which elucidate the Peirce's laconic paper [15]. Simply put, we extend a well-known formula for the function $\mathrm{cn}(u, 1/\sqrt{2})$ to an arbitrary Jacobi elliptic function $\mathrm{pq}(u,k)$.

In the first place, we need to introduce a new notation for the complex modulus and the principal argument of these functions. The new symbols consist of a principal (Greek) letter and four attributes ascribing qualities of the principal letter. Such principal letter are, as it is usual, ρ (rho) and θ (theta), which is distinct from a theta function. The attributes are written counterclockwise beginning from the bottom right corner of the principal letter. They specify a Jacobi elliptic function, an (integer) exponent or power, an elliptic modulus and the point at which the function is evaluated. With reference to the common notations of Complex Analysis, they translate as

$$ {}^{k}_{u}\rho^{n}_{pq} := |\mathrm{pq}(u,k)|^n \quad \text{and} \quad {}^{k}_{iu}\theta_{pq} := \mathrm{Arg}(\mathrm{pq}(iu,k)). $$

The exponent of the principal argument Arg is omitted because it will always be the unity.

3.1. Inverses of the even elliptic functions

By generalizing the method in Solanilla, Oostra and Yañez [21], we notice first that any inverse function can be expressed as a function of the complex moduli of some Jacobi elliptic functions.

Proposition 1. Let $\mathrm{pq}(u,k) = {}^{k}_{u}\rho^{1}_{pq}\exp({}^{k}_{u}\theta_{pq})$ be an even Jacobi elliptic function and write its inverse function as $u = x + iy$. Then, the real and imaginary parts x, y of u satisfy the conditions in Table 2, according to the respective inverse elliptic function.

Proof. The function dn^{-1} serves as example of the general procedure. By virtue of the addition formulas,

$$ \mathrm{dn}(2x,k) = \mathrm{dn}(u+\bar{u},k) = \frac{\mathrm{dn}(u,k)\mathrm{dn}(\bar{u},k) - k^2\mathrm{sn}(u,k)\mathrm{sn}(\bar{u},k)\mathrm{cn}(u,k)\mathrm{cn}(\bar{u},k)}{1-k^2\mathrm{sn}^2(u,k)\mathrm{sn}^2(\bar{u},k)}, $$

$$ \mathrm{dn}(2yi,k) = \mathrm{dn}(u-\bar{u},k) = \frac{\mathrm{dn}(u,k)\mathrm{dn}(\bar{u},k) + k^2\mathrm{sn}(u,k)\mathrm{sn}(\bar{u},k)\mathrm{cn}(u,k)\mathrm{cn}(\bar{u},k)}{1-k^2\mathrm{sn}^2(u,k)\mathrm{sn}^2(\bar{u},k)}. $$

Now, for the involved functions, $\mathrm{pq}(\bar{u},k) = \overline{\mathrm{pq}(u,k)}$. Finally, again by an addition theorem,

$$ \begin{aligned} \mathrm{dn}(0+iv,k) &= \frac{\mathrm{dn}(0,k)\mathrm{cn}(v,k')\mathrm{dn}(v,k')}{1-\mathrm{dn}^2(0,k)\mathrm{sn}^2(v,k')} - i\frac{k^2\mathrm{sn}(0,k)\mathrm{cn}(0,k)\mathrm{sn}(v,k')}{1-\mathrm{dn}^2(0,k)\mathrm{sn}^2(v,k')} \\ &= \frac{\mathrm{cn}(v,k')\mathrm{dn}(v,k')}{1-\mathrm{sn}^2(v,k')} = \frac{\mathrm{cn}(v,k')\mathrm{dn}(v,k')}{\mathrm{cn}^2(v,k')} = \mathrm{dc}(v,k'). \end{aligned} $$

□

Table 3
Even functions: the complex moduli in Table 2 are properly related to the elliptic function

Function	Required moduli
cn^{-1}	${}^{k}_{u}\rho^4_{sn} = 1 - 2\,{}^{k}_{u}\rho^2_{cn}\cos\left(2\,{}^{k}_{u}\theta_{cn}\right) + {}^{k}_{u}\rho^4_{cn}$
	${}^{k}_{u}\rho^4_{dn} = k^4\,{}^{k}_{u}\rho^4_{cn} + 2k^2k'^2\,{}^{k}_{u}\rho^2_{cn}\cos\left(2\,{}^{k}_{u}\theta_{cn}\right) + k'^4$
dn^{-1}	${}^{k}_{u}\rho^4_{sn} = k^{-4}\left(1 - 2\,{}^{k}_{u}\rho^2_{dn}\cos\left(2\,{}^{k}_{u}\theta_{dn}\right) + {}^{k}_{u}\rho^4_{dn}\right)$
	${}^{k}_{u}\rho^4_{cn} = k^{-4}\left({}^{k}_{u}\rho^4_{dn} - 2k'^2\,{}^{k}_{u}\rho^2_{dn}\cos\left(2\,{}^{k}_{u}\theta_{dn}\right) + k'^4\right)$
cd^{-1}	${}^{k}_{u}\rho^4_{sn} = \dfrac{1 - 2\,{}^{k}_{u}\rho^2_{cd}\cos\left(2\,{}^{k}_{u}\theta_{cd}\right) + {}^{k}_{u}\rho^4_{cd}}{1 - 2k^2\,{}^{k}_{u}\rho^2_{cd}\cos\left(2\,{}^{k}_{u}\theta_{cd}\right) + k^4\,{}^{k}_{u}\rho^4_{cd}}$

Remark 1. By these means, the problem of finding the complex inverses is reduced to the restrictions of the elliptic functions to the real axis. Accompanying relations hold for the other even inverse elliptic functions. The real inverses in the formulas for the imaginary part 'y' of 'u' have elliptic modulus k', instead of k.

In a second step, for each function pq^{-1} in Table 2, the complex moduli necessary to compute x and y depend exclusively on the complex modulus ${}^{k}_{u}\rho^1_{pq}$ and argument ${}^{k}_{u}\theta_{pq}$ of $\mathrm{pq}(u,k)$. This makes the formulas in Table 2 useful.

Proposition 2. The complex moduli required to compute the inverse functions $u = x + iy$ of each even Jacobi elliptic function can be expressed in terms of the complex modulus and argument of its corresponding Jacobi elliptic function. The expressions accomplishing this goal are given in Table 3.

Proof. Again for dn^{-1}, we start with $k^2sn^2(u,k)+dn^2(u,k) = 1$, *i.e.*, $k^2\,{}^{k}_{u}\rho^2_{sn}e^{2i\,{}^{k}_{u}\theta_{sn}} = 1 - {}^{k}_{u}\rho^2_{dn}e^{2i\,{}^{k}_{u}\theta_{dn}}$. By multiplying this equation by its complex conjugate,

$$k^4\,{}^{k}_{u}\rho^4_{sn} = 1 - {}^{k}_{u}\rho^2_{dn}\left(e^{2i\,{}^{k}_{u}\theta_{dn}} + e^{-2i\,{}^{k}_{u}\theta_{dn}}\right) + {}^{k}_{u}\rho^4_{dn} = 1 - 2\,{}^{k}_{u}\rho^2_{dn}\cos\left(2\,{}^{k}_{u}\theta_{dn}\right) + {}^{k}_{u}\rho^4_{dn}.$$

In like manner, $k^2cn^2(u,k) + k'^2 = dn^2(u,k)$ or $k^2\,{}^{k}_{u}\rho^2_{cn}e^{2i\,{}^{k}_{u}\theta_{cn}} = {}^{k}_{u}\rho^2_{dn}e^{2i\,{}^{k}_{u}\theta_{dn}} - k'^2$. Therefore,

$$k^4\,{}^{k}_{u}\rho^4_{cn} = {}^{k}_{u}\rho^4_{dn} - 2k'^2\,{}^{k}_{u}\rho^2_{dn}\cos\left(2\,{}^{k}_{u}\theta_{dn}\right) + k'^4.$$

□

To sum up, we arrange together the previous results to obtain a set of appropriate formulas.

Theorem 1. The inverses $u = x + iy$ of the even Jacobi elliptic functions can be calculated by the real-variable formulas in Table 4.

3.2. Inverses of the odd elliptic functions

In order to find the inverses of these functions we rely on some real and imaginary Jacobi transformations [9,20] and so, some extra work must be made to do so. This method constitute one of the central contributions of this paper. We certainly simplify the early efforts of Richelot [19] and Durège [4]. As before, we first reduce the calculation of $z = x + iy$ to some expressions involving real Jacobi elliptic functions and certain complex moduli.

Proposition 3. Let $\mathrm{pq}(u,k) = {}^{k}_{u}\rho^1_{pq}\exp({}^{k}_{u}\theta_{pq})$ be an odd Jacobi elliptic function with inverse $u = x+iy$. Then, x and y fulfill the conditions in Table 5.

Table 4
Formulas for the inverse functions of the even Jacobi elliptic functions

Function	Real and imaginary parts
cn^{-1}	$x = \frac{1}{2}\mathrm{cn}^{-1}\left(\frac{{}^{k}_{u}\rho^{2}_{cn} - \sqrt{(1-2\,{}^{k}_{u}\rho^{2}_{cn}\cos(2\,{}^{k}_{u}\theta_{cn}) + {}^{k}_{u}\rho^{4}_{cn})(k^{4}\,{}^{k}_{u}\rho^{4}_{cn} + 2k^{2}k'^{2}\,{}^{k}_{u}\rho^{2}_{cn}\cos(2\,{}^{k}_{u}\theta_{cn}) + k'^{4})}}{1-k^{2}(1-2\,{}^{k}_{u}\rho^{2}_{cn}\cos(2\,{}^{k}_{u}\theta_{cn}) + {}^{k}_{u}\rho^{4}_{cn})}, k\right)$
	$y = \frac{1}{2}\mathrm{cn}^{-1}\left(\frac{1-k^{2}\left(1-2\,{}^{k}_{u}\rho^{2}_{cn}\cos\left(2\,{}^{k}_{u}\theta_{cn}\right) + {}^{k}_{u}\rho^{4}_{cn}\right)}{{}^{k}_{u}\rho^{2}_{cn} + \sqrt{(1-2\,{}^{k}_{u}\rho^{2}_{cn}\cos(2\,{}^{k}_{u}\theta_{cn}) + {}^{k}_{u}\rho^{4}_{cn})(k^{4}\,{}^{k}_{u}\rho^{4}_{cn} + 2k^{2}k'^{2}\,{}^{k}_{u}\rho^{2}_{cn}\cos(2\,{}^{k}_{u}\theta_{cn}) + k'^{4})}}, k'\right)$
dn^{-1}	$x = \frac{1}{2}\mathrm{dn}^{-1}\left(\frac{{}^{k}_{u}\rho^{2}_{dn} - k^{-2}\sqrt{(1-2\,{}^{k}_{u}\rho^{2}_{dn}\cos(2\,{}^{k}_{u}\theta_{dn}) + {}^{k}_{u}\rho^{4}_{dn})({}^{k}_{u}\rho^{4}_{dn} - 2k'^{2}\,{}^{k}_{u}\rho^{2}_{dn}\cos(2\,{}^{k}_{u}\theta_{dn}) + k'^{4})}}{1-k^{-2}(1-2\,{}^{k}_{u}\rho^{2}_{dn}\cos(2\,{}^{k}_{u}\theta_{dn}) + {}^{k}_{u}\rho^{4}_{dn})}, k\right)$
	$y = \frac{1}{2}\mathrm{dc}^{-1}\left(\frac{{}^{k}_{u}\rho^{2}_{dn} + k^{-2}\sqrt{(1-2\,{}^{k}_{u}\rho^{2}_{dn}\cos(2\,{}^{k}_{u}\theta_{dn}) + {}^{k}_{u}\rho^{4}_{dn})({}^{k}_{u}\rho^{4}_{dn} - 2k'^{2}\,{}^{k}_{u}\rho^{2}_{dn}\cos(2\,{}^{k}_{u}\theta_{dn}) + k'^{4})}}{1-k^{-2}(1-2\,{}^{k}_{u}\rho^{2}_{dn}\cos(2\,{}^{k}_{u}\theta_{dn}) + {}^{k}_{u}\rho^{4}_{dn})}, k'\right)$
cd^{-1}	$x = \frac{1}{2}\mathrm{cd}^{-1}\left(\frac{{}^{k}_{u}\rho^{2}_{cd} - \sqrt{\frac{1-2\,{}^{k}_{u}\rho^{2}_{cd}\cos\left(2\,{}^{k}_{u}\theta_{cd}\right) + {}^{k}_{u}\rho^{4}_{cd}}{1-2k^{2}\,{}^{k}_{u}\rho^{2}_{cd}\cos(2\,{}^{k}_{u}\theta_{cd}) + k^{4}\,{}^{k}_{u}\rho^{4}_{cd}}}}{1-k^{2}\,{}^{k}_{u}\rho^{2}_{cd}\sqrt{\frac{1-2\,{}^{k}_{u}\rho^{2}_{cd}\cos\left(2\,{}^{k}_{u}\theta_{cd}\right) + {}^{k}_{u}\rho^{4}_{cd}}{1-2k^{2}\,{}^{k}_{u}\rho^{2}_{cd}\cos(2\,{}^{k}_{u}\theta_{cd}) + k^{4}\,{}^{k}_{u}\rho^{4}_{cd}}}}, k\right)$
	$y = \frac{1}{2}\mathrm{nd}^{-1}\left(\frac{{}^{k}_{u}\rho^{2}_{cd} + \sqrt{\frac{1-2\,{}^{k}_{u}\rho^{2}_{cd}\cos\left(2\,{}^{k}_{u}\theta_{cd}\right) + {}^{k}_{u}\rho^{4}_{cd}}{1-2k^{2}\,{}^{k}_{u}\rho^{2}_{cd}\cos(2\,{}^{k}_{u}\theta_{cd}) + k^{4}\,{}^{k}_{u}\rho^{4}_{cd}}}}{1+k^{2}\,{}^{k}_{u}\rho^{2}_{cd}\sqrt{\frac{1-2\,{}^{k}_{u}\rho^{2}_{cd}\cos\left(2\,{}^{k}_{u}\theta_{cd}\right) + {}^{k}_{u}\rho^{4}_{cd}}{1-2k^{2}\,{}^{k}_{u}\rho^{2}_{cd}\cos(2\,{}^{k}_{u}\theta_{cd}) + k^{4}\,{}^{k}_{u}\rho^{4}_{cd}}}}, k'\right)$

Table 5
Odd functions: dependence of x, y on the complex moduli of some elliptic functions

Function	Condition on x	Condition on y
sn^{-1}	$\mathrm{cd}(2(x-K),k) = \frac{{}^{k}_{u-K}\rho^{2}_{cd} - {}^{k}_{u-K}\rho^{2}_{sn}}{1-k^{2}\,{}^{k}_{u-K}\rho^{2}_{sn}\,{}^{k}_{u-K}\rho^{2}_{cd}}$	$\mathrm{nd}(2y,k') = \frac{{}^{k}_{u-K}\rho^{2}_{cd} + {}^{k}_{u-K}\rho^{2}_{sn}}{1+k^{2}\,{}^{k}_{u-K}\rho^{2}_{sn}\,{}^{k}_{u-K}\rho^{2}_{cd}}$
$(i\mathrm{sc})^{-1}$	$\mathrm{nd}(2x,(1-k)') = \frac{{}^{1-k}_{iu-K}\rho^{2}_{cd} + {}^{1-k}_{iu-K}\rho^{2}_{sn}}{1+(1-k)^{2}\,{}^{1-k}_{iu-K}\rho^{2}_{sn}\,{}^{1-k}_{iu-K}\rho^{2}_{cd}}$	$\mathrm{cd}(2(y+K),1-k) = \frac{{}^{1-k}_{iu-K}\rho^{2}_{cd} - {}^{1-k}_{iu-K}\rho^{2}_{sn}}{1-(1-k)^{2}\,{}^{1-k}_{iu-K}\rho^{2}_{sn}\,{}^{1-k}_{iu-K}\rho^{2}_{cd}}$
$(-k'\mathrm{sd})^{-1}$	$\mathrm{cn}(2(x+K),k) = \frac{{}^{k}_{u+K}\rho^{2}_{cn} - {}^{k}_{u+K}\rho^{2}_{sn}\,{}^{k}_{u+K}\rho^{2}_{dn}}{1-k^{2}\,{}^{k}_{u+K}\rho^{4}_{sn}}$	$\mathrm{cn}(2y,k') = \frac{1-k^{2}\,{}^{k}_{u+K}\rho^{4}_{sn}}{{}^{k}_{u+K}\rho^{2}_{cn} + {}^{k}_{u+K}\rho^{2}_{sn}\,{}^{k}_{u+K}\rho^{2}_{dn}}$

Proof. The three inverses are respectivelly found by the following Jacobi transformations:

$$\mathrm{sn}(u,k) = \mathrm{cd}(u-K,k), \quad i\mathrm{sc}(u,k) = \mathrm{sn}(iu, 1-k) \quad \text{and} \quad -k'\mathrm{sd}(u,k) = \mathrm{cn}(u+K,k).$$

The inverse of sn is clear by the previous results on the even Jacobi functions. For the case of $(i\mathrm{sc})^{-1}$, we use $iu = ix - y$ in the formulas of sn^{-1}, *i.e.*,

$$\mathrm{cd}(2(-y-K),1-k) = \mathrm{cd}(2(y+K),1-k) = \frac{{}^{1-k}_{iu-K}\rho^{2}_{cd} - {}^{1-k}_{iu-K}\rho^{2}_{sn}}{1-(1-k)^{2}\,{}^{1-k}_{iu-K}\rho^{2}_{sn}\,{}^{1-k}_{iu-K}\rho^{2}_{cd}},$$

$$\mathrm{nd}(2x,(1-k)') = \frac{{}^{1-k}_{iu-K}\rho^{2}_{cd} + {}^{1-k}_{iu-K}\rho^{2}_{sn}}{1+(1-k)^{2}\,{}^{1-k}_{iu-K}\rho^{2}_{sn}\,{}^{1-k}_{iu-K}\rho^{2}_{cd}}.$$

The treatment of $(-k'\mathrm{sd})^{-1}$ is completely analogous. □

Table 6
Odd functions: the complex moduli in Table 5 are correctly related to the elliptic function

Function	Required moduli	Auxiliary moduli
sn^{-1}	${}^{\;k}_{u-K}\rho^2_{cd} = {}^k_u\rho^2_{sn}$	
	${}^{\;k}_{u-K}\rho^2_{sn} = {}^k_u\rho^2_{sn}\,{}^k_u\rho^2_{cn}\,{}^k_u\rho^{-2}_{dn}$, where	${}^k_u\rho^4_{cn} = 1 - 2\,{}^k_u\rho^2_{sn}\cos\left(2\,{}^k_u\theta_{sn}\right) + {}^k_u\rho^4_{sn}$
		${}^k_u\rho^4_{dn} = 1 - 2k^2\,{}^k_u\rho^2_{sn}\cos\left(2\,{}^k_u\theta_{sn}\right) + k^4\,{}^k_u\rho^4_{sn}$
sc^{-1}	${}^{\;1-k}_{iu-K}\rho^2_{cd} = {}^{1-k}_{\;iu}\rho^2_{sn} = {}^k_u\rho^2_{sc}$	
	${}^{\;1-k}_{iu-K}\rho^2_{sn} = {}^{1-k}_{\;iu}\rho^2_{sn}\,{}^{1-k}_{\;iu}\rho^2_{cn}\,{}^{1-k}_{\;iu}\rho^{-2}_{dn}$, where	${}^{1-k}_{\;iu}\rho^4_{cn} = 1 + 2\,{}^k_u\rho^2_{sc}\cos\left(2\,{}^k_u\theta_{sc}\right) + {}^k_u\rho^4_{sc}$
		${}^{1-k}_{\;iu}\rho^4_{dn} = 1 + 2(1-k)^2\,{}^k_u\rho^2_{sc}\cos\left(2\,{}^k_u\theta_{sc}\right)$
		$+ (1-k)^4\,{}^k_u\rho^4_{sc}$
sd^{-1}	${}^{\;k}_{u+K}\rho^2_{cn} = k'^2\,{}^k_u\rho^2_{sd}$	
	${}^{\;k}_{u+K}\rho^4_{sn} = 1 - 2k'^2\,{}^k_u\rho^2_{sd}\cos\left(2\,{}^k_u\theta_{sd}\right) + k'^4\,{}^k_u\rho^4_{sd}$	
	${}^{\;k}_{u+K}\rho^4_{dn} = k^4k'^4\,{}^k_u\rho^4_{sd} + 2k^2k'^4\,{}^k_u\rho^2_{sd}\cos\left(2\,{}^k_u\theta_{sd}\right) + k'^4$	

This time we have to be more careful to establish the right relations among the complex moduli of the elliptic functions.

Proposition 4. The complex moduli needed to compute the inverse functions $u = x + iy$ of the odd Jacobi elliptic functions depend on the complex modulus and argument of its respective Jacobi elliptic function. The formulas appear in Table 6.

Proof. We consider each of the functions.

sn^{-1}: On the one hand, the Jacobi transformation for sn in polar form provides

$${}^k_u\rho^1_{sn}\exp\left(i\,{}^k_u\theta_{sn}\right) = {}^{\;k}_{u-K}\rho^1_{cd}\exp\left(i\,{}^{\;k}_{u-K}\theta_{cd}\right).$$

On the other hand, the addition formula for sn together with the well-known values of the elliptic functions at $\pm K$ yield

$${}^{\;k}_{u-K}\rho^2_{sn} = {}^k_u\rho^2_{sn}\,{}^k_u\rho^2_{cn}\,{}^k_u\rho^{-2}_{dn}.$$

The auxiliary moduli ${}^k_u\rho^4_{cn}$, ${}^k_u\rho^4_{dn}$ are easily obtained with the help of the Pythagorean-like identities.

sc^{-1}: First, the imaginary Jacobi transformation for sc is written as

$$e^{i\frac{\pi}{2}}\,{}^k_u\rho^1_{sc}\exp\left(i\,{}^k_u\theta_{sc}\right) = {}^{1-k}_{\;iu}\rho^1_{sn}\exp\left(i\,{}^k_u\theta_{sn}\right) = {}^{\;1-k}_{iu-K}\rho^1_{cd}\exp\left(i\,{}^k_u\theta_{cd}\right).$$

Hence,

$${}^{1-k}_{\;iu}\rho^4_{cn} = 1 - 2\,{}^{1-k}_{\;iu}\rho^2_{sn}\cos\left(2\,{}^{1-k}_{\;iu}\theta_{sn}\right) + {}^{1-k}_{\;iu}\rho^4_{sn} = 1 - 2\,{}^k_u\rho^2_{sc}\cos\left(2\,{}^k_u\theta_{sc} + \pi\right) + {}^k_u\rho^4_{sc},$$

$$\begin{aligned}{}^{1-k}_{\;iu}\rho^4_{dn} &= 1 - 2(1-k)^2\,{}^{1-k}_{\;iu}\rho^2_{sn}\cos\left(2\,{}^{1-k}_{\;iu}\theta_{sn}\right) + (1-k)^4\,{}^{1-k}_{\;iu}\rho^4_{sn}\\ &= 1 + 2(1-k)^2\,{}^k_u\rho^2_{sc}\cos\left(2\,{}^k_u\theta_{sc}\right) + (1-k)^4\,{}^k_u\rho^4_{sc}.\end{aligned}$$

sd^{-1}: We start with

$$-k'\,{}^k_u\rho^1_{sd}\exp\left(i\,{}^k_u\theta_{sd}\right) = {}^{\;k}_{u+K}\rho^1_{cn}\exp\left(i\,{}^{\;k}_{u+K}\theta_{cn}\right)$$

and employ the standard Pythagorean-like identities.

□

Table 7
Formulas for the inverse functions of the odd Jacobi elliptic functions

Function	Real and imaginary parts
sn^{-1}	$x = \frac{1}{2}\mathrm{cd}^{-1}\left(\frac{{}^{k}_{u}\rho_{sn}^2\left(1-\frac{\sqrt{1-2\,{}^{k}_{u}\rho_{sn}^2\cos(2\,{}^{k}_{u}\theta_{sn})+\,{}^{k}_{u}\rho_{sn}^4}}{\sqrt{1-2k^2\,{}^{k}_{u}\rho_{sn}^2\cos(2\,{}^{k}_{u}\theta_{sn})+k^4\,{}^{k}_{u}\rho_{sn}^4}}\right)}{1-k^2\,{}^{k}_{u}\rho_{sn}^4\frac{\sqrt{1-2\,{}^{k}_{u}\rho_{sn}^2\cos(2\,{}^{k}_{u}\theta_{sn})+\,{}^{k}_{u}\rho_{sn}^4}}{\sqrt{1-2k^2\,{}^{k}_{u}\rho_{sn}^2\cos(2\,{}^{k}_{u}\theta_{sn})+k^4\,{}^{k}_{u}\rho_{sn}^4}}}, k\right)+K$
	$y = \frac{1}{2}\mathrm{nd}^{-1}\left(\frac{{}^{k}_{u}\rho_{sn}^2\left(1+\frac{\sqrt{1-2\,{}^{k}_{u}\rho_{sn}^2\cos(2\,{}^{k}_{u}\theta_{sn})+\,{}^{k}_{u}\rho_{sn}^4}}{\sqrt{1-2k^2\,{}^{k}_{u}\rho_{sn}^2\cos(2\,{}^{k}_{u}\theta_{sn})+k^4\,{}^{k}_{u}\rho_{sn}^4}}\right)}{1+k^2\,{}^{k}_{u}\rho_{sn}^4\frac{\sqrt{1-2\,{}^{k}_{u}\rho_{sn}^2\cos(2\,{}^{k}_{u}\theta_{sn})+\,{}^{k}_{u}\rho_{sn}^4}}{\sqrt{1-2k^2\,{}^{k}_{u}\rho_{sn}^2\cos(2\,{}^{k}_{u}\theta_{sn})+k^4\,{}^{k}_{u}\rho_{sn}^4}}}, k'\right)$
sc^{-1}	$x = \frac{1}{2}\mathrm{nd}^{-1}\left(\frac{{}^{k}_{u}\rho_{sc}^2\left(1+\frac{\sqrt{1+2\,{}^{k}_{u}\rho_{sc}^2\cos(2\,{}^{k}_{u}\theta_{sc})+\,{}^{k}_{u}\rho_{sc}^4}}{\sqrt{1+2(1-k)^2\,{}^{k}_{u}\rho_{sc}^4\cos(2\,{}^{k}_{u}\theta_{sc})+(1-k)^4\,{}^{k}_{u}\rho_{sc}^4}}\right)}{1+(1-k)^2\,{}^{k}_{u}\rho_{sc}^4\frac{\sqrt{1+2\,{}^{k}_{u}\rho_{sc}^2\cos(2\,{}^{k}_{u}\theta_{sc})+\,{}^{k}_{u}\rho_{sc}^2}}{\sqrt{1+2(1-k)^2\,{}^{k}_{u}\rho_{sc}^2\cos(2\,{}^{k}_{u}\theta_{sc})+(1-k)^4\,{}^{k}_{u}\rho_{sc}^4}}}, (1-k)'\right)$
	$y = \frac{1}{2}\mathrm{cd}^{-1}\left(\frac{{}^{k}_{u}\rho_{sc}^2\left(1-\frac{\sqrt{1+2\,{}^{k}_{u}\rho_{sc}^2\cos(2\,{}^{k}_{u}\theta_{sc})+\,{}^{k}_{u}\rho_{sc}^4}}{\sqrt{1+2(1-k)^2\,{}^{k}_{u}\rho_{sc}^4\cos(2\,{}^{k}_{u}\theta_{sc})+(1-k)^4\,{}^{k}_{u}\rho_{sc}^4}}\right)}{1-(1-k)^2\,{}^{k}_{u}\rho_{sc}^4\frac{\sqrt{1+2\,{}^{k}_{u}\rho_{sc}^2\cos(2\,{}^{k}_{u}\theta_{sc})+\,{}^{k}_{u}\rho_{sc}^2}}{\sqrt{1+2(1-k)^2\,{}^{k}_{u}\rho_{sc}^2\cos(2\,{}^{k}_{u}\theta_{sc})+(1-k)^4\,{}^{k}_{u}\rho_{sc}^4}}}, 1-k\right)-K$
sd^{-1}	$x = \frac{1}{2}\mathrm{cn}^{-1}\left(\frac{k'^2\,{}^{k}_{u}\rho_{sd}^2-\sqrt{\begin{array}{c}\left(1-2k'^2\,{}^{k}_{u}\rho_{sd}^2\cos\left(2\,{}^{k}_{u}\theta_{sd}\right)+k'^4\,{}^{k}_{u}\rho_{sd}^4\right)\\ \left(k^4k'^4\,{}^{k}_{u}\rho_{sd}^4+2k^2k'^4\,{}^{k}_{u}\rho_{sd}^2\cos\left(2\,{}^{k}_{u}\theta_{sd}\right)+k'^4\right)\end{array}}}{1-k^2\left(1-2k'^2\,{}^{k}_{u}\rho_{sd}^2\cos(2\,{}^{k}_{u}\theta_{sd})+k'^4\,{}^{k}_{u}\rho_{sd}^4\right)}, k\right)-K$
	$y = \frac{1}{2}\mathrm{cn}^{-1}\left(\frac{1-k^2\left(1-2k'^2\,{}^{k}_{u}\rho_{sd}^2\cos\left(2\,{}^{k}_{u}\theta_{sd}\right)+k'^4\,{}^{k}_{u}\rho_{sd}^4\right)}{k'^2\,{}^{k}_{u}\rho_{sd}^2+\sqrt{\begin{array}{c}\left(1-2k'^2\,{}^{k}_{u}\rho_{sd}^2\cos\left(2\,{}^{k}_{u}\theta_{sd}\right)+k'^4\,{}^{k}_{u}\rho_{sd}^4\right)\\ \left(k^4k'^4\,{}^{k}_{u}\rho_{sd}^4+2k^2k'^4\,{}^{k}_{u}\rho_{sd}^2\cos\left(2\,{}^{k}_{u}\theta_{sd}\right)+k'^4\right)\end{array}}}, k'\right)$

We recapitulate the results for the odd functions in one result.

Theorem 2. The real and imaginary parts of the inverses to the odd Jacobi elliptic functions are given by the formulas in Table 7.

3.3. *Calculation of the inverse elliptic functions*

For the sake of rendering usable the formulas en Tables 4 and 7, we must give procedures to compute the restrictions of the inverse Jacobi elliptic functions to the real line. There are at least two ways of accomplishing this purpose: either we can use the Jacobi original definitions or we can utilize certain integral formulas.

When we write Legendre elliptic integral of the first kind in the form $F(\varphi) = \int_0^{\varphi} d\phi/\sqrt{1-k^2\sin^2\phi}$, the amplitude is simply F^{-1} and we get at once

$$\mathrm{cn}^{-1}x = F(\arccos x).$$

Table 8
Two ways for computing the inverses of Jacobi elliptic functions

$\mathrm{pq}^{-1}x$	Jacobi definition	Direct integral formula	Domain
$\mathrm{cn}^{-1}x$	$F(\arccos x)$	$\int_x^1 \frac{dt}{(1-t^2)(k'^2+k^2t^2)}$	$-1 \leqslant x \leqslant 1$
$\mathrm{dn}^{-1}x$	$F\left(\arcsin\left(\sqrt{\frac{1-x^2}{k^2}}\right)\right)$	$\int_x^1 \frac{dt}{(1-t^2)(t^2-k'^2)}$	$k' \leqslant x \leqslant 1$
$\mathrm{nd}^{-1}x$	$F\left(\arcsin\left(\sqrt{\frac{x^2-1}{k^2x^2}}\right)\right)$	$\int_1^x \frac{dt}{(t^2-1)(1-k'^2t^2)}$	$1 \leqslant x \leqslant \frac{1}{k'}$
$\mathrm{cd}^{-1}x$	$F\left(\arcsin\left(\sqrt{\frac{1-x^2}{1-k^2x^2}}\right)\right)$	$\int_x^1 \frac{dt}{(1-t^2)(1-k^2t^2)}$	$-1 \leqslant x \leqslant 1$
$\mathrm{dc}^{-1}x$	$F\left(\arcsin\left(\sqrt{\frac{1-x^2}{k^2-x^2}}\right)\right)$	$\int_1^x \frac{dt}{(t^2-1)(t^2-k^2)}$	$1 \leqslant x < \infty$

With a little more effort, by using elliptic identities, we derive

$$\mathrm{dn}^{-1}x = F\left(\arcsin\left(\sqrt{\frac{1-x^2}{k^2}}\right)\right), \mathrm{cd}^{-1}x = F\left(\arcsin\left(\sqrt{\frac{1-x^2}{1-k^2x^2}}\right)\right) \text{ and}$$

$$\mathrm{dc}^{-1}x = F\left(\arcsin\left(\sqrt{\frac{1-x^2}{k^2-x^2}}\right)\right).$$

Many of the most popular programming languages for Mathematics come with a routine to approximate function F and the usual trigonometric inverses. This fact makes this approach easy to implement.

Still and all, there are also direct integral formulas to calculate these inverse functions, *v.* [18]. They are presented, together with the former formulas, in Table 8. As long as the integral formulas in Table 8 are less known, their implementation may demand some extra work.

3.4. Stereographic projection

With regard to the conformal structure for an open hemisphere $\mathbb{H}$ of the Riemann sphere $\mathbb{S}$, we need to introduce the stereographic projection in our formulas. Let $N \in \mathbb{H}$ denote the "North pole". Once we parametrize the sphere by means of a geographic system, *i.e.*, by local coordinates $\lambda \in (-\pi/2, \pi/2)$ (latitude) and $\theta \in (0, 2\pi)$ (longitude or azimuth), the translation $\mu = \lambda + \frac{\pi}{2} \in (0, \pi)$ permits to write the restricted stereographic projection $\mathbb{H}\backslash\{N\} \to \mathbb{C}$, $(\mu, \theta) \mapsto z = x + iy$, under the suitable form (*cf.* [15,21])

$$z = \tan\left(\frac{\mu}{2}\right)e^{i\theta}.$$

3.5. Conformality and singular points

For a particular Jacobi elliptic function pq, we have $z = \mathrm{pq}(u,k) = {}^k_u\rho^1_{pq} \exp({}^k_u\theta_{pq})$. Thus, in each of the preceding formulas in Tables 4 and 7 we must substitute

$${}^k_u\rho^1_{pq} = \tan\left(\frac{\mu}{2}\right) \quad \text{and} \quad {}^k_u\theta_{pq} = \theta$$

to get the correct values. In other words, we define the map projection associated with a specific Jacobi elliptic function as the composite function $u = \mathrm{pq}^{-1} \circ z$ of the inverse Jacobi elliptic function following the stereographic projection z. In our setting, these map projections are bijections applying a punctured open hemisphere onto a puncture open parallelogram where a Jacobi elliptic function is injective. Due

Table 9
Magnification reciprocals of the map projections

pq^{-1}	Reciprocal of $m(\mu,\, {}^{k}_{u}\theta_{pq},\, {}^{k}_{u}\rho^{1}_{pq}),\, {}^{k}_{u}\rho^{1}_{pq} = \tan\left(\frac{\mu}{2}\right)$
cn^{-1}	$2\cos^2\left(\frac{\mu}{2}\right)\sqrt[4]{(1-2\,{}^{k}_{u}\rho^{2}_{cn}\cos(2\,{}^{k}_{u}\theta_{cn})+\,{}^{k}_{u}\rho^{4}_{cn})\,(k^4\,{}^{k}_{u}\rho^{4}_{cn}+2k^2k'^2\,{}^{k}_{u}\rho^{2}_{cn}\cos(2\,{}^{k}_{u}\theta_{cn})+k'^4)}$
dn^{-1}	$2\cos^2\left(\frac{\mu}{2}\right)\sqrt[4]{(1-2\,{}^{k}_{u}\rho^{2}_{dn}\cos(2\,{}^{k}_{u}\theta_{dn})+\,{}^{k}_{u}\rho^{4}_{dn})\,({}^{k}_{u}\rho^{4}_{dn}-2k'^2\,{}^{k}_{u}\rho^{2}_{dn}\cos(2\,{}^{k}_{u}\theta_{dn})+k'^4)}$
cd^{-1}	$2\cos^2\left(\frac{\mu}{2}\right)\sqrt[4]{(1-2\,{}^{k}_{u}\rho^{2}_{cd}\cos(2\,{}^{k}_{u}\theta_{cd})+\,{}^{k}_{u}\rho^{4}_{cd})\,(1-2k^2\,{}^{k}_{u}\rho^{2}_{cd}\cos(2\,{}^{k}_{u}\theta_{cd})+k^4\,{}^{k}_{u}\rho^{4}_{cd})}$
sn^{-1}	$2\cos^2\left(\frac{\mu}{2}\right)\sqrt[4]{(1-2\,{}^{k}_{u}\rho^{2}_{sn}\cos(2\,{}^{k}_{u}\theta_{sn})+\,{}^{k}_{u}\rho^{4}_{sn})\,({}^{k}_{u}\rho^{4}_{dn}-2k'^2\,{}^{k}_{u}\rho^{2}_{dn}\cos(2\,{}^{k}_{u}\theta_{dn})+k'^4)}$
sc^{-1}	$2\cos^2\left(\frac{\mu}{2}\right)\sqrt[4]{(1+2k'^2\,{}^{k}_{u}\rho^{2}_{sc}\cos(2\,{}^{k}_{u}\theta_{sc})+k'^4\,{}^{k}_{u}\rho^{4}_{sc})\,(1+2\,{}^{k}_{u}\rho^{2}_{sc}\cos(2\,{}^{k}_{u}\theta_{sc})+\,{}^{k}_{u}\rho^{4}_{sc})}$
sd^{-1}	$2\cos^2\left(\frac{\mu}{2}\right)\sqrt[4]{(1-2k'^2\,{}^{k}_{u}\rho^{2}_{sd}\cos(2\,{}^{k}_{u}\theta_{sd})+k'^4\,{}^{k}_{u}\rho^{4}_{sd})\,(1+2k^2\,{}^{k}_{u}\rho^{2}_{sd}\cos(2\,{}^{k}_{u}\theta_{sd})+k^4\,{}^{k}_{u}\rho^{4}_{sd})}$

to the azimuthal symmetry of the stereographic projection, the magnification m of u is given almost everywhere as

$$m(\mu, \mathrm{pq}) = \left|\frac{du}{dz}\right|\left|\frac{dz}{d\mu}\right| = \left|\frac{d\mathrm{pq}^{-1}z}{dz}\right|\left|\frac{dz}{d\mu}\right| = \left|\frac{1}{d\mathrm{pq}u/du}\right|\left|\frac{dz}{d\mu}\right|.$$

Without trouble,

$$\left|\frac{dz}{d\mu}\right| = \frac{1}{2\cos^2(\mu/2)}$$

and, happily, the derivatives of the basic Jacobi elliptic functions are widely known:

$$\frac{d\mathrm{cn}u}{du} = -\mathrm{sn}u\mathrm{dn}u, \quad \frac{d\mathrm{sn}u}{du} = \mathrm{cn}u\mathrm{dn}u, \quad \frac{d\mathrm{dn}u}{du} = -k^2\mathrm{sn}u\mathrm{cn}u.$$

As a result,

$$\frac{d\mathrm{cd}u}{du} = -k'^2\mathrm{sd}u\mathrm{nd}u, \quad \frac{d\mathrm{sc}u}{du} = \mathrm{dc}u\mathrm{nc}u, \quad \frac{d\mathrm{sd}u}{du} = \mathrm{cd}u\mathrm{nd}u.$$

Theorem 3. Table 9 displays the magnification reciprocals of the map projections associated with Jacobi elliptic functions.

Proof. The expressions for the basic elliptic functions cn, sn and dn follow forthrightly from Tables 3 and 6. So, we address the remaining functions.

cd^{-1}: We begin with the Pythagorean-like identities

$$\mathrm{sd}^2u = \frac{1-\mathrm{cd}^2u}{k'^2}, \quad \mathrm{nd}^2u = \frac{1-k^2\mathrm{cd}^2u}{k'^2}.$$

Now, we rewrite these expressions in polar form, conjugate them and multiply each of them by its complex conjugate to obtain

$${}^{k}_{u}\rho^{4}_{sd} = \frac{1}{k'^4}\left(1-2\,{}^{k}_{u}\rho^{2}_{cd}\cos\left(2\,{}^{k}_{u}\theta_{cd}\right)+\,{}^{k}_{u}\rho^{4}_{cd}\right),$$

$${}^{k}_{u}\rho^{4}_{nd} = \frac{1}{k'^4}\left(1-2k^2\,{}^{k}_{u}\rho^{2}_{cd}\cos\left(2\,{}^{k}_{u}\theta_{cd}\right)+k^4\,{}^{k}_{u}\rho^{4}_{cd}\right).$$

sc^{-1}: This time the starting point is

$$\mathrm{dc}^2u = 1+k'^2\mathrm{sc}^2u, \quad \mathrm{nc}^2u = 1+\mathrm{sc}^2u.$$

Therefore,

$${}^{k}_{u}\rho^{4}_{dc} = 1+2k'^2\,{}^{k}_{u}\rho^{2}_{sc}\cos\left(2\,{}^{k}_{u}\theta_{sc}\right)+k'^4\,{}^{k}_{u}\rho^{4}_{sc}, \quad {}^{k}_{u}\rho^{4}_{nc} = 1+2\,{}^{k}_{u}\rho^{2}_{sc}\cos\left(2\,{}^{k}_{u}\theta_{sc}\right)+\,{}^{k}_{u}\rho^{4}_{sc}.$$

Table 10
Singularities of the map projections on the "Equator" ($\mu = \pi/2$)

pq^{-1}	Singularities independent of k	Feasible singularities depending on k
cn^{-1}	$\theta = 0, \pi$	$\theta = \pi/2, 3\pi/2$; $k = k' = 1/\sqrt{2}$
dn^{-1}	$\theta = 0, \pi$	None
cd^{-1}	$\theta = 0, \pi$	None
sn^{-1}	$\theta = 0, \pi$	None
sc^{-1}	$\theta = \pi/2, 3\pi/2$	None
sd^{-1}	none	None

sd^{-1}: The relations

$$\mathrm{cd}^2 u = 1 - k'^2 \mathrm{sd}^2 u \quad \text{and} \quad \mathrm{nd}^2 u = 1 + k^2 \mathrm{sd}^2 u$$

produce

$$ {}^{k}_{u}\rho^4_{cd} = 1 - 2k'^2\, {}^{k}_{u}\rho^2_{sd} \cos\left(2\, {}^{k}_{u}\theta_{sd}\right) + k'^4\, {}^{k}_{u}\rho^4_{sd},$$

$$ {}^{k}_{u}\rho^4_{nd} = 1 + 2k^2\, {}^{k}_{u}\rho^2_{sd} \cos\left(2\, {}^{k}_{u}\theta_{sd}\right) + k^4\, {}^{k}_{u}\rho^4_{sd}.$$

We notice the use of the formulas for the derivatives of Jacobi elliptic functions. □

One of the remarkable features of the map projections involved is the way they extend to the boundary of the punctured hemisphere. In the process, only a finite number of singularities is introduced.

Corollary 1. The extension of each map projection – associated with a Jacobi elliptic function – to its domain boundary is conformal almost everywhere and its singularities are located according to the following rules

- The "North pole" is mapped to the center of the parallelogram.
- In the rest of the boundary of the hemisphere, *i.e.* on the "Equator", an even number (0, 2 or 4) of singularities may appear. These singularities have order 2, where they do exist.

Proof. The image of the "North pole" is evident. On the "Equator", $\mu = \pi/2$ and so, ${}^{k}_{u}\rho^1_{pq} = \tan(\pi/4) = 1$. As the magnifications of map projections associated to the inverse elliptic functions $\mathrm{cn}^{-1}, \mathrm{dn}^{-1}, \mathrm{cd}^{-1}$ and sn^{-1} have a factor of the form $(2 - 2\cos 2\, {}^{k}_{u}\theta_{pq})^{-1/4}$, they blow up at $\theta = {}^{k}_{u}\theta_{pq} = 0, \pi$. For the function sc^{-1}, the corresponding factor is $(2 + 2\cos 2\theta)^{-1/4}$ and the map projection blows up at $\theta = \pi/2, 3\pi/2$. Now we seek other singularities for each pq^{-1}:

cn^{-1}: The magnification has a factor $(k^4 + 2k^2k'^2 \cos 2\theta + k'^4)^{-1/4}$. Hence, we have singularities at $\theta = \pi/2, 3\pi/2$ precisely when $k = k'$.

There are no other possible singular points. We summarize these results in Table 10. □

The functions obtained from Jacobi elliptic functions evaluated at the limiting elliptic moduli $k = 0, 1$ (whenever they exist and make sense) can be called improper. Jacobi elliptic functions defined in Section 2 above are all proper. By admitting improper Jacobi elliptic functions, we find new singularities:

dn^{-1}: The factor $(1 - 2k'^2 \cos 2\theta + k'^4)^{-1/4}$ with $\theta = 0, \pi$ leads to a singularity precisely when $k' = 1, k = 0$.

cd^{-1}: The factor $1 - 2k^2 \cos 2\theta + k^4$ vanishes at $\theta = 0, \pi$ if and only if $k = 1, k' = 0$.

sn^{-1}: $1 - 2k'^2 \cos 2\theta + k'^4 = 0$ holds when $\theta = 0, \pi$ together with $k' = 1, k = 0$.

In what follows, we shall not refer to these improper cases.

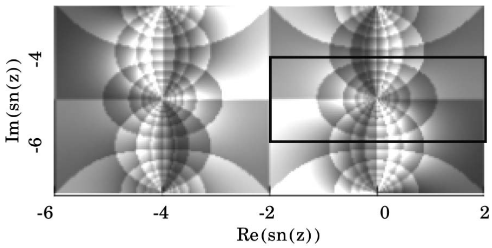

Fig. 1. Fundamental rectangle of sn(z) with $k = 0.8$.

4. n-uncial maps

The foregoing findings can be put together into one all-embracing notion.

Definition 1. Let n denote a positive integer. A map $f : \mathbb{S} \to \mathbb{C}$ on the Riemann sphere into the complex plane is n-uncial if it satisfies the following conditions:

- The restriction of $\hat{f}$ of f to a punctured open hemisphere $\mathbb{H}\backslash\{N\}$, N being its pole, is everywhere conformal onto an open parallelogram $\mathbb{P}\backslash\{C\}$, punctured at its center C.
- The extension $\bar{f}$ of $\hat{f}$ to the boundary $\partial(\mathbb{H}\backslash\{N\}) = \partial\mathbb{H} \cup \{N\}$ of its domain is conformal except at n points. N is always a singular point. Each singularity has order 2.

When $n = 1$, an n-uncial map is called uncial; when $n = 3$, teruncial; and when $n = 5$, quincuncial.

Corollary 1 implies at once the following consequence.

Theorem 4. The valid map projections associated to the inverse functions of the Jacobi elliptic functions are n-uncial with $n = 1, 3, 5$.

At long last, we can classify the maps projections associated to the proper inverse Jacobi elliptic functions according to the number of singularities:

$n = 1$ or uncial. sd^{-1}, $k \neq 0, 1$.
$n = 3$ or teruncial. cn^{-1}, $k \neq k'$; dn^{-1}, $k \neq 0$; cd^{-1}, $k \neq 1$; sn^{-1}, $k \neq 0$; sc^{-1}, $k \neq 0$, and sd^{-1}, $0 < k < 1$.
$n = 5$ or quincuncial. *Peirce quincuncial projection:* cn^{-1}, $k = k'$.

Corollary 2. Peirce quincuncial projection is the only quincuncial map projection obtained from a proper Jacobi elliptic function.

5. Concluding remarks

The practical purpose of devising map projections based on Jacobi elliptic functions leads naturally to an important theoretical notion, namely, that of n-uncial map. This is a property may be used to characterize (Jacobi) elliptic functions. Indeed, to what extend does this n-uncial property determine a (Jacobi) elliptic function? Apropos this question, how can this approach to (Jacobi) elliptic functions be of use to describe the Riemann surfaces of the inverse (Jacobi) elliptic functions? We believe these two questions are not too challenging and can be answered within the framework of the theory of functions of a complex variable. Further, the formulas for $x(\rho, \theta), y(\rho, \theta)$ in Tables 4 and 7 can easily be simplified to describe the images of the parallels of latitude and the meridians of the sphere.

More attractively for applications, does every map projection having a n-uncial property yield a tessellation of the plane similar to Peirce's tessellation in [15]? A preliminary analysis reveals that the answer is positive. In Fig. 1[1] we have sketched the fundamental rectangle for *sinus amplitudinis* with elliptic modulus $k = 0.8$. The image of a hemisphere correspond to the area framed by the small rectangle (the half part of a region where the function is one-to-one). A domain coloring technique with contour lines of the modulus and the argument exposes the images of the lines of constant latitude and longitude on this rectangle. Then, a map of a hemisphere can be drawn in this small rectangle and consequently we obtain a tessellation similar to Peirce's. It may be noted in passing that this map is teruncial, as it has been explained above.

Some other interesting questions arise from these results. They will be reported at a later date.

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Conflict of interest

The authors declare no conflict of interest. The founding sponsors had no role in the design of the study; in the analyses; in the writing of the manuscript, and in the decision to publish the results.

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[1]This figure has been adapted (cutting, changing colors and drawing a small rectangle) from the file Sn-k-08.png in Wikimedia Commons. It is described as the "elliptic Jacobi function, sn, corresponding to $k = 0.8$, generated using a version of the domain coloring method". It is published here under the Creative Commons Attribution-Share Alike 4.0 International license.

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