Experimental Results

Update

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CINVESTAV

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Karbach Algorithm: Confidence Belts

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Assumptions:

- Gaussian PDF $\mathcal{G}(x \mid \mu)$
- No nuisance parameters
- With constraints
- Single measure x_0 treatment

Algorithm

$$\int_{x_1}^{x_2} P(x|\mu)\,\mathrm{d}x = \alpha \; . \qquad R(x,\mu) = \frac{P(x|\mu)}{P(x|\mu_{\mathrm{best}})} \; . \qquad R(x_1) = R(x_2)$$

1. At the considered value of the true parameter μ_0, generate a toy experiment by drawing a value x_{toy} from the p.d.f. That is, draw from the unit Gaussian $G(\sigma = 1, \mu = \mu_0)$.

2. Compute $\Delta\chi^2_{\mathrm{toy}}$ for the toy experiment, following Eq. 6. For $\chi^2(x,\mu)$, x is set to x_{toy} and μ is set to μ_0. For $\chi^2(x,\mu_{\mathrm{best}})$, $x = x_{\mathrm{toy}}$ and $\mu_{\mathrm{best}} = \max(0, x_{\mathrm{toy}})$ implementing the boundary at zero.

3. Find the value $\Delta\chi^2_c$, such that α of the toy experiments have $\Delta\chi^2_{\mathrm{toy}} < \Delta\chi^2_c$.

4. The interval $[x_1, x_2]$ is given by all values of x such that $\Delta\chi^2(x,\mu_0) < \Delta\chi^2_c$.

Paper results

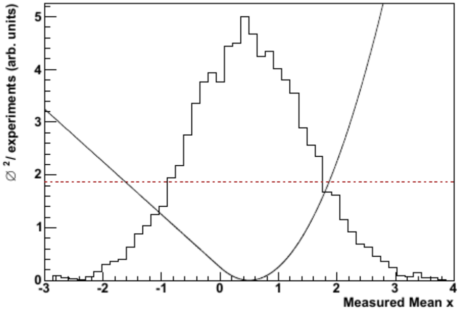


FIG. 1: Example set of 5000 toy experiments drawn from $G(\mu = 0.5, \sigma = 1)$ (histogram), together with the likelihood ratio following Eq. 6 computed in presence of a boundary $\mu > 0$ (solid line), and the value $\Delta\chi^2_c$ giving $\alpha = 90\%$ (dashed line).

Paper results

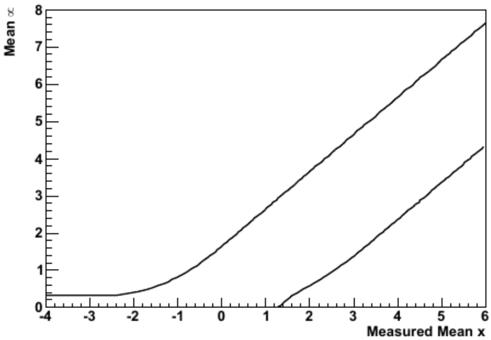


FIG. 2: Confidence belt for a unit Gaussian with a mean bound to be positive, $\alpha = 90\%$.

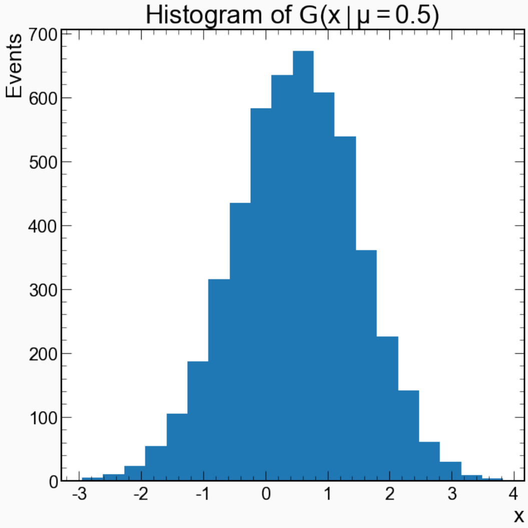
Implementation

- Step 1: Define a Gaussian PDF $\mathcal{G}(x \mid \mu)$
- Step 2: Generate a toy MC with n events

$$\mathcal{G}(x \mid \mu) \to \{x_0, ..., x_n\}$$

Implementation

- Step 3: Plot histogram



Implementation

- Step 4: Compute R or $\Delta\chi^2_{toys}$

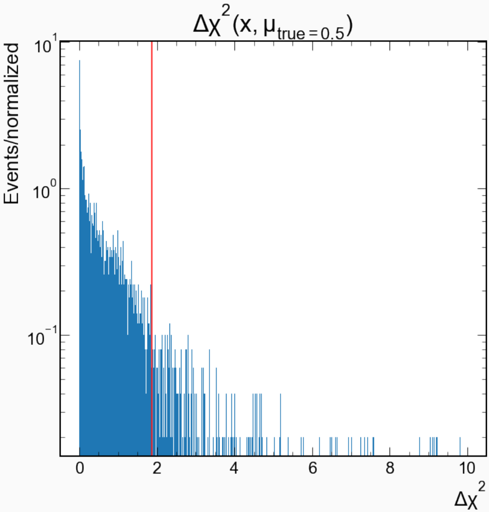
$$R = \frac{\mathcal{L}(\mu \mid x)}{\mathcal{L}(\hat{\mu} \mid x)}, \qquad \Delta\chi^2_{toys} = -2\ln R \tag{1}$$

Observations:

- x is a scalar quantity not a vector
- $\hat{\mu} = \mu_{best} = \max(0, x)$

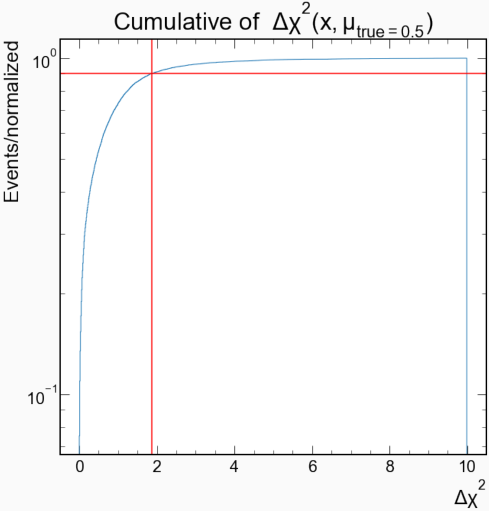
Implementation

- Step 5: Plot $\Delta\chi^2_{toys}$ histogram and apply np.percetile at desired confidence level = $\Delta\chi^2_c$



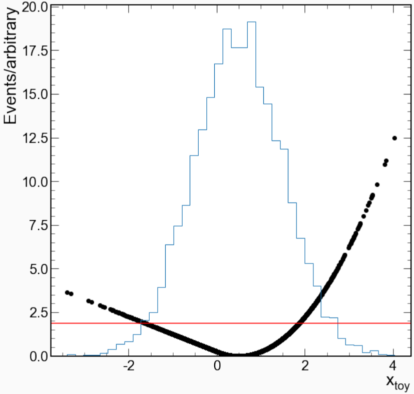
Implementation

Step 5 from CDF point of view



Implementation

- Step 6: Plot $\mathcal{G}(x \mid \mu)$ histogram, $\Delta\chi^2_{toys}$ and $\Delta\chi^2_c$

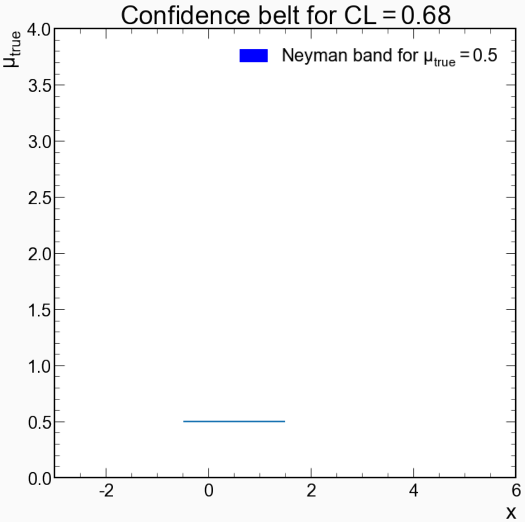


The acceptance interval $[x_1, x_2]$ is given by all values of x such that

$$\Delta\chi^2_{toys} < \Delta\chi^2_c \tag{2}$$

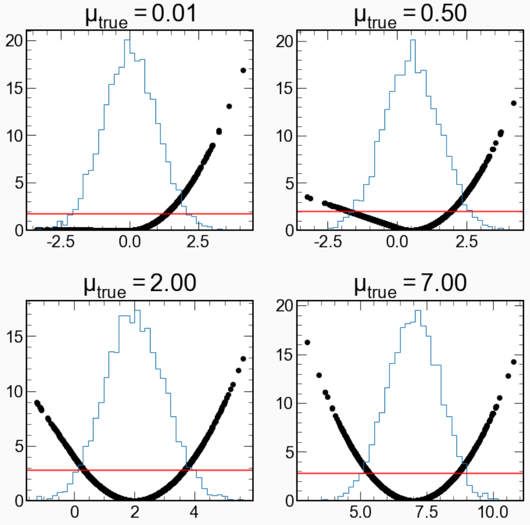
Implementation

- Step 7: Plot acceptance interval = Neymann band for $\mu_{true} = 0.5$

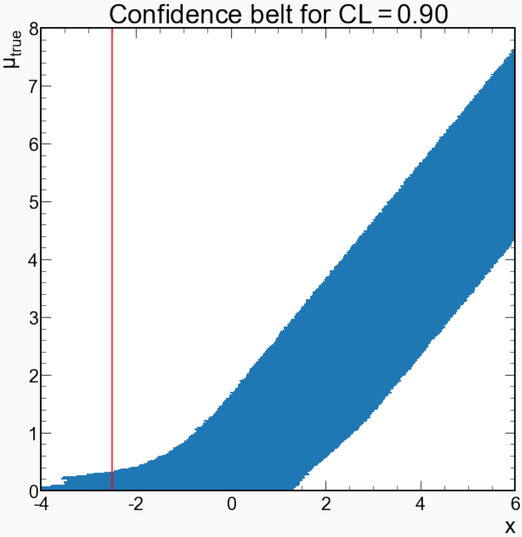


Implementation

- Step 8: Iterate for an interval of μ_{true} values $\rightarrow [\mu_0, \mu_\eta]$

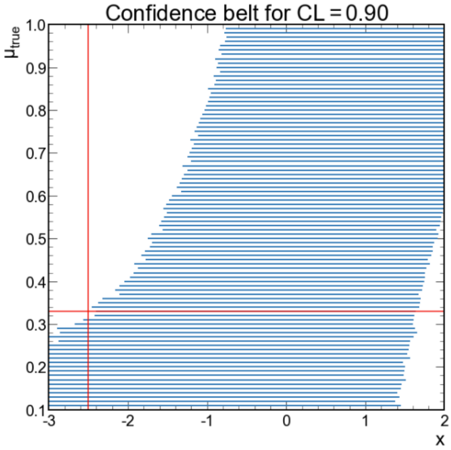


- Step 9: Plot confidence belt



Implementation

- Step 10: For a $x_0 = x_{obs}$ measure obtain confidence interval $[\mu_{inf}, \mu_{sup}]$



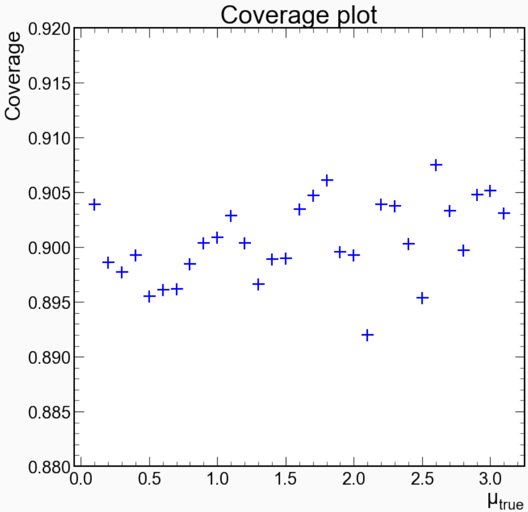
for $x_{obs} = -2.5 \Rightarrow [\mu_1 = 0, \mu_2 = 0.33]$

Implementation

- Step 11: Compare with FC table

−2.7	*0.00, 0.04*	*0.00, 0.29*
−2.6	*0.00, 0.05*	*0.00, 0.30*
−2.5	*0.00, 0.05*	*0.00, 0.32*
−2.4	*0.00, 0.05*	*0.00, 0.33*
−2.3	0.00, 0.05	0.00, 0.34
−2.2	0.00, 0.06	0.00, 0.36

- Step 12: Coverage test



Karbach Algorithm:
1-CL plots

Karbach Algorithm: Confidence Belts

Assumptions:

- Gaussian PDF $\mathcal{G}(x \mid \mu)$
- No nuisance parameters
- With constraints
- Single measure x_0 treatment

Algorithm

1. At the considered value of the true parameter μ_0, generate a toy experiment.
2. Compute $\Delta\chi^2_{\text{toy}} \equiv \Delta\chi^2(x_{\text{toy}}, \mu_0)$ for the toy experiment.
3. Compute $\Delta\chi^2_{\text{data}} \equiv \Delta\chi^2(x_0, \mu_0)$ for the measured value.
4. The $1 - \text{CL}$ ($\equiv 1 - \alpha(\mu_0)$) value is given by the fraction of toy experiments that have a larger $\Delta\chi^2$ than the measured value:

$$1 - \alpha(\mu_0) = \frac{N(\Delta\chi^2_{\text{data}} < \Delta\chi^2_{\text{toy}})}{N_{\text{toy}}} .$$

Paper Results

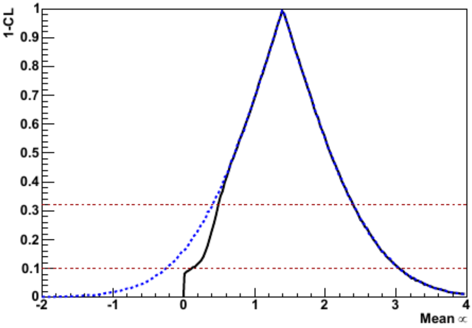


FIG. 4: Plot of $1 - \mathrm{CL}$ of a unit Gaussian for a measured mean of $x = 1.4$, both without boundaries (dashed) and for a boundary at $\mu > 0$ (solid). The horizontal lines indicate $\alpha = 68\%$ and $\alpha = 90\%$.

Implementation

- Step 1, 2, 3, 4 are the same

$$R = \frac{\mathcal{L}(\mu \mid x)}{\mathcal{L}(\hat{\mu} \mid x)}, \qquad \Delta\chi^2_{toys} = -2 \ln R \tag{3}$$

Observations:

- x is a scalar quantity not a vector
- $\hat{\mu} = \mu_{best} = \max(0, x)$

Implementation

- Step 5: Compute $\Delta\chi^2_{data} = \Delta\chi^2(x_0, \mu)$ for a measured mean $x_0 = 1.4$

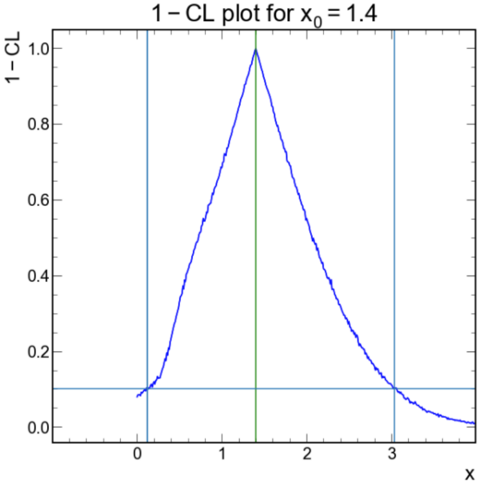
$$\Delta\chi^2_{data}(x_0 = 1.4, \mu) = -2\ln\left[\frac{\mathcal{L}(\mu \mid x_0)}{\mathcal{L}(\hat{\mu} \mid x_0)}\right] \tag{4}$$

Observations:

- x_0 is a scalar quantity
- $\hat{\mu} = \mu_{best} = \max(0, x_0)$

Implementation

- Step 6: Iterate for all values of μ



for $x_0 = 1.4 \Rightarrow [0.12, 3.04]$

Implementation

- Step 7: Compare with FC table

1.2	0.35, 2.20	0.00, 2.84
1.3	0.42, 2.30	0.02, 2.94
1.4	0.49, 2.40	0.12, 3.04
1.5	0.56, 2.50	0.22, 3.14
1.6	0.64, 2.60	0.31, 3.24

Karbach Algortithm: Confidence Belts and 1-CL plots with nuisance parameters

Karbach paper and nuisance parameters

Note that the general case quickly gets more expensive than the sketched example. If there are nuisance parameters present, the likelihood needs to be minimized in their respect when calculating χ^2 and χ^2_{best}. Also, the

Implementation

- Step 1: Define $G(x \mid \mu, \sigma)$ with σ as nuisance parameter
- Step 2: Obtain n events from pdf $\{x_0, , , x_n\}$
- Step 3: Compute R or $\Delta\chi^2_{toys}$

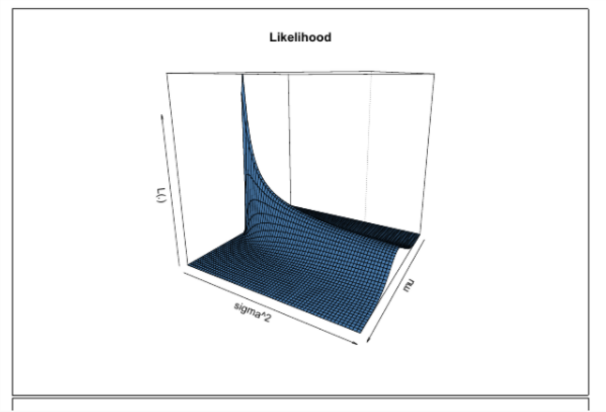
$$R = \frac{\mathcal{L}(\mu, \sigma \mid x)}{\mathcal{L}(\hat{\mu}, \hat{\sigma} \mid x)} \tag{5}$$

Observation: x is a scalar quantity.

But the denominator of (5) is undefined! same with profile likelihood version.

Conclusion

- If R is undefined hence we can not obtain confidence belts or 1-CL plots with this single (scalar) treatment of x values.



Extended Karbach Algorithm: 1-CL plots multi-values treatment failed case

Extended Karbach Algorithm

Assumptions:

- Gaussian PDF $\mathcal{G}(x \mid \mu, \sigma)$
- Nuisance parameters
- With constraints
- Multi measure $\vec{x}_0$ treatment

Implementation

- Step 1: Define $\mathcal{G}(x \mid \mu, \sigma)$ with σ as nuisance parameter
- Step 2: Obtain n events from PDF $\{x_0, ..., x_n\}$
- Step 3: Compute R or $\Delta\chi^2_{toys}$

$$R = \frac{\mathcal{L}(\mu, \hat{\hat{\sigma}}(\mu) \mid \{x_0, ..., x_n\})}{\mathcal{L}(\hat{\mu}, \hat{\sigma} \mid \{x_0, ..., x_n\}} \tag{6}$$

Implementation

- Step 4: Obtain $\Delta\chi^2_{data}$ with $\vec{x}_0 = \vec{x}_{obs} = \{x_0, ..., x_n\}_{obs}$

$$\Delta\chi^2_{data} = \frac{\mathcal{L}(\mu, \hat{\hat{\sigma}}(\mu) \mid \{x_0, ..., x_n\}_{obs})}{\mathcal{L}(\hat{\mu}, \hat{\sigma} \mid \{x_0, ..., x_n\}_{obs})} \tag{7}$$

- Step 5: Compute $1 - CL$ point

$$1 - CL = \frac{N(\Delta\chi^2_{data} < \Delta\chi^2_{toys})}{N_{toys}} \tag{8}$$

but in numerator of (8) we have only 1 or 0 so we canot obtain $1 - CL$ plots with this approach.

Extended Karbach Algorithm: 1-CL plots multi-values treatment

Extended Karbach Algorithm

Assumptions:

- Gaussian PDF $\mathcal{G}(x \mid \mu, \sigma)$
- Nuisance parameters
- With constraints
- Multi measure $\vec{x}_0$ treatment

Implementation

- Step 1: Define $\mathcal{G}(x \mid \mu, \sigma)$ with σ as nuisance parameter
- Step 2: Obtain N **toys MC samplers with** n **events** from PDF

$$\mathcal{G}(x \mid \mu, \sigma) \rightarrow [\{x_0, ..., x_n\}_0, ..., \{x_0, ..., x_n\}_N] \tag{9}$$

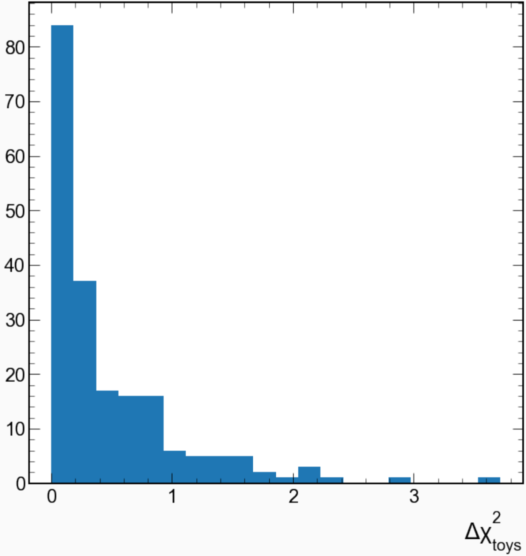
Implementation

- Step 3: Compute N values of R or $\Delta\chi^2_{toys}$

$$R_i = \frac{\mathcal{L}(\mu, \hat{\hat{\sigma}}(\mu) \mid \{x_0, ..., x_n\}_i)}{\mathcal{L}(\hat{\mu}, \hat{\sigma} \mid \{x_0, ..., x_n\}_i)} \tag{10}$$

Implementation

- Step 4: Plot distribution of R_i or $(\Delta\chi^2_{toys})_i$ in a histogram

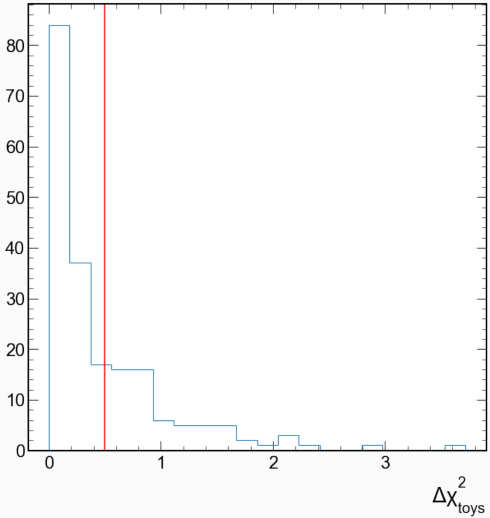


Implementation

- Step 5: Compute $\Delta\chi^2_{data}$ with "observed" data $\{x_0, ..., x_n\}_{obs}$

$$R = \frac{\mathcal{L}(\mu, \hat{\hat{\sigma}}(\mu) \mid \{x_0, ..., x_n\}_{obs})}{\mathcal{L}(\hat{\mu}, \hat{\sigma} \mid \{x_0, ..., x_n\}_{obs})} \tag{11}$$

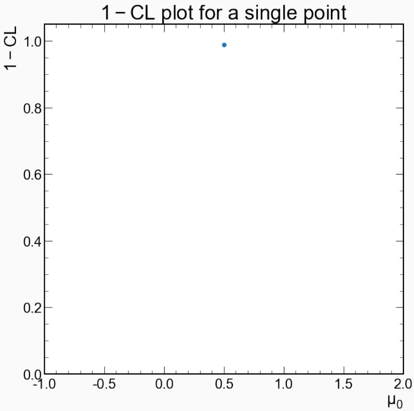
- Step 6: Plot distribution of $(\Delta\chi^2_{toys})$ values and the only value of $\Delta\chi^2_{data}$



Implementation

- Step 7: Compute 1-CL point

$$1 - CL = \frac{N((\Delta\chi^2_{data} < (\Delta\chi^2_{toys})_i)}{N_{toys}} \tag{12}$$

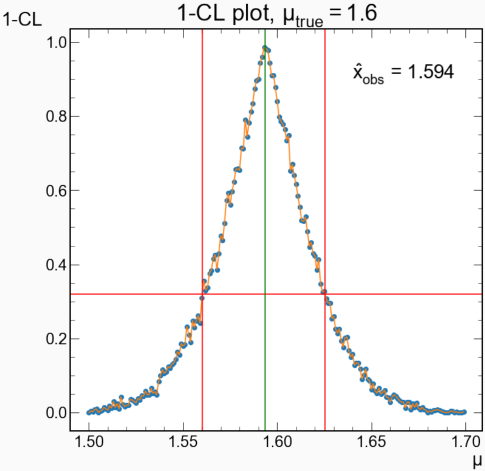


Implementation

$$1 - \mathrm{CL} = \int_{\Delta\chi^2_{\mathrm{data}}}^{\infty} P_{\chi^2}(t)\,\mathrm{d}t$$

Implementation

- Step 8: Iterate for all values of μ



Extended Karbach Algorithm: Confidence Belt Multi-value treatment

Algorithm

- Step 1: Define gaussian PDF $\mathcal{G}(x \mid \mu, \sigma)$
- Step 2: Obtain N **toys MC samplers with** n **events** from PDF

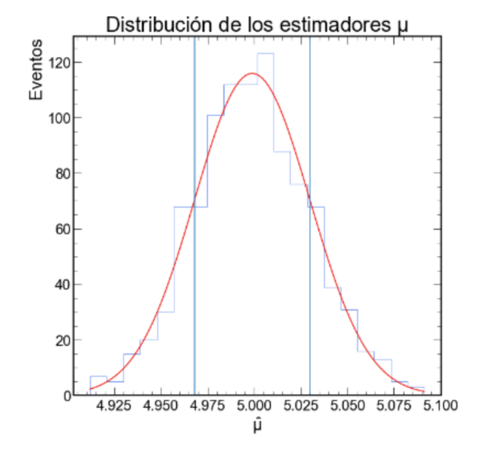
$$\mathcal{G}(x \mid \mu, \sigma) \rightarrow [\{x_0, ..., x_n\}_0, ..., \{x_0, ..., x_n\}_N] \tag{13}$$

- Step 3: Obtain maximum likelihood estimators $\hat{\mu}$ for each sampler $\{x_0, ..., x_N\}$

$$[\{x_0, ..., x_n\}_0, ..., \{x_0, ..., x_n\}_N] \rightarrow [\hat{\mu}_0, ..., \hat{\mu}_N] \tag{14}$$

Algorithm

- Step 4: Plot distribution of all $\hat{\mu}$ estimators



Apply original Karbach algorithm with $x = \hat{\mu}$

9.3 Confidence interval for a Gaussian distributed estimator

A simple and very important application of a confidence interval is when the distribution of $\hat{\theta}$ is Gaussian with mean θ and standard deviation $\sigma_{\hat{\theta}}$. That is, the cumulative distribution of $\hat{\theta}$ is

$$G(\hat{\theta}; \theta, \sigma_{\hat{\theta}}) = \int_{-\infty}^{\hat{\theta}} \frac{1}{\sqrt{2\pi\sigma_{\hat{\theta}}^2}} \exp\left(\frac{-(\hat{\theta}' - \theta)^2}{2\sigma_{\hat{\theta}}^2}\right) d\hat{\theta}'. \qquad (9.10)$$

This is a commonly occurring situation since, according to the central limit theorem, any estimator that is a linear function of a sum of random variables becomes Gaussian in the large sample limit. We will see that for this case, the

Algorithm

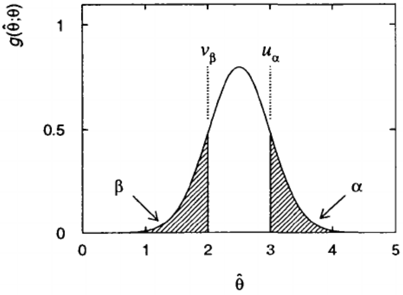


Fig. 9.1 A p.d.f. $g(\hat{\theta};\theta)$ for an estimator $\hat{\theta}$ for a given value of the true parameter θ. The two shaded regions indicate the values of $\hat{\theta} \le v_\beta$, which has a probability β, and $\hat{\theta} \ge u_\alpha$, which has a probability α.

Algorithm