Counter: CI - 140 - 149.

Outliers:

## MODULE - 3

### Binomial Distribution:

It is a discrete probability distribution.

A r.v X is said to follow binomial distribution with parameters n & p if its probability mass function is given by. $P(X = x) = P(r) = {}^{n}C_{r} \cdot p^{r} \cdot q^{n-r}$.

$r = 0, 1, 2, \ldots, n$.

Here $n$ = no. of trials.

$p$ = prob of success.

$q$ = prob. of failure.

$r$ = no. of successes out of $n$ trials.

Conditions:

1. No. of trials $n$ is finite and fixed.
2. All the trials are independent.
3. Each trials results in either success or failure (mutually exclusive outcomes).

4. The prob. of succes p remains the same in every trial then $q = 1 - p$.

5. X is a discrete r.v.

X ~~repeats~~ represents no. of successes & can take integer from 0 to n.

<u>Note</u>: $P(X = r) = {}^nC_r \cdot p^r \cdot q^{n-r}$

This can be used to find prob r successes out of n trials.

<u>Mean</u> (Expectation): of X.

$$\mu = E(X) = \sum_{r=0}^{n} r\, P(X = r)$$

$$= \sum_{r=0}^{n} r \cdot {}^nC_r \cdot p^r \cdot q^{n-r}$$

$$* \boxed{\mu = np} = E(X)$$

Sly $E(X^2) = \sum_{r=0}^{n} r^2 P(X = r)$

$$= \sum r^2 \cdot {}^nC_r \cdot p^r \cdot q^{n-r}$$

$$* E(X^2) = np + n(n-1)p^2.$$

<u>Variance</u>:

$$\sigma^2 = E(X^2) - [E(X)]^2$$

$$= np + n(n-1)p^2 - [np]^2$$

$$= np + \cancel{n^2p^2} - np^2 - \cancel{n^2p^2}$$

$$= np - np^2$$

$$= np(1-p)$$

$$* \boxed{\sigma^2 = npq}$$

Prob-1: A fair coin is tossed 6 times find the prob of getting 4 heads.

→ by binomial distribution, probability of getting r successes getting is

Sol: Given, $n = 6$ ↓ no. of trials.

$P(X=r) = {}^nC_r \cdot p^r \cdot q^{n-r}$

$p$ = prob. of success = prob. of getting head $= \frac{1}{2}$

$q$ = prob. of failure = prob. of getting tail $= 1 - \frac{1}{2} = \frac{1}{2}$

$r$ = prob. of getting heads $= 4$.

$$P(X=r) = {}^6C_4 \cdot \left[\frac{1}{2}\right]^4 \cdot \left[\frac{1}{2}\right]^2$$

$$= \frac{6!}{2! \cdot 4!} \cdot \left[\frac{1}{2}\right]^6$$

$$= \frac{6 \times 5}{2} \cdot \left[\frac{1}{2}\right]^6$$

$$= \left[\frac{1}{2}\right]^7 30$$

$$= \frac{15}{2^6}$$

$$= \frac{15}{64}$$

Prob-2: 20% of items produced from the factory are defective. Find the prob that in a sample of 5 choosen at random:

i) none is defective.

ii) 1 is defective.

iii) $P(1 < X < 4)$.

Sol: Given $n = 5$.

$$p = 1 - \frac{1}{5} = \frac{4}{5}.$$

$$q = \frac{20}{100} = \frac{1}{5}.$$

i) none is defective (r=0)

$$P(X=0) = {}^5C_0 \cdot \left[\frac{4}{5}\right]^0 \left[\frac{1}{5}\right]^5 = \frac{5!}{5! \cdot 0!} \cdot \left[\frac{1}{5}\right]^5$$

ii) 1 is defective ($r=1$)

$$P(X=1) = {}^5C_1 \cdot \left[\frac{4}{5}\right]^4 \left[\frac{1}{5}\right]$$

$$= \frac{5!}{4!\cdot 1!} \cdot \frac{4^4}{5^2}$$

$$= \frac{5 \times 4^4}{5^2}$$

$$= \frac{4^4}{5}$$

$$5 \times 0.2 \times (0.8)^4$$

$$= 0.4096$$

iii) $P(1 < X < 4) = P(X=2) + P(X=3)$

$$= {}^5C_2 \cdot \left[\frac{4}{5}\right]^2 \left[\frac{1}{5}\right]^3 + {}^5C_3 \left[\frac{4}{5}\right]^3 \left[\frac{1}{5}\right]^2$$

$$= \frac{5!}{3!\cdot 2!} \cdot \frac{4^2}{5^5} + \frac{5!}{2!\cdot 3!} \cdot \frac{4^3}{5^4}$$

$$= \frac{5 \times 4^2}{2} \cdot \frac{4^2}{5^5} + \frac{5 \times 4^2}{2} \cdot \frac{4^3}{5^4}$$

$$= \frac{2 \times 4^2}{5^4} + \frac{2 \times 4^3}{5^3}$$

$$= \frac{2 \times 4^2}{5^3}\left[\frac{1}{5} + 4\right]$$

$$= \frac{32}{125}\left[\frac{21}{5}\right]$$

$$= \frac{672}{625}$$

$$\begin{array}{r} 21 \\ \times\ 32 \\ \hline 42 \\ 63\phantom{0} \\ \hline 672 \end{array}$$

$$= {}^5C_2 \cdot (0.2)^2 (0.8)^{5-2}$$

$$= \frac{5!}{3!\cdot 2!} \cdot (0.2)^2 \cdot (0.8)^3$$

$$= \frac{5 \times 4^2}{2} (0.2)^2 (0.8)^3$$

$$= 10 \times 0.04 \times 0.512$$

$$= 5.12 \times 0.04$$

$$= 0.2048$$

Eg: The mean and variance of a binomial variate (r.v X) parameters n and p are 16 and 8 then find

i) $P(X \geq 1)$ ii) $P(X > 2)$.

Sol: given that,

$$\text{Mean} = 16$$

$$\Rightarrow np = 16 \rightarrow (1)$$

$$\text{Variance} = 8$$

$$\Rightarrow npq = 8 \rightarrow (2)$$

(1) in (2) $\Rightarrow 16(q) = 8$

$$q = \frac{1}{2}$$

$$p = 1 - q = \frac{1}{2}$$

(1) $\Rightarrow n\left[\frac{1}{2}\right] = 16$

$$\boxed{n = 32}$$

i) $P(X \geq 1) = P(X=1) + P(X=2) + \dots + P(X=32)$

$$= {}^{32}C_1 \cdot \left[\frac{1}{2}\right]\left[\frac{1}{2}\right]^{31} + {}^{32}C_2 \left[\frac{1}{2}\right]^2\left[\frac{1}{2}\right]^{30} + \dots + {}^{32}C_{32}\left[\frac{1}{2}\right]^{32}\left[\frac{1}{2}\right]^0$$

(or)

$$= 1 - P(X=0)$$

$$= 1 - {}^{32}C_0 \cdot \left[\frac{1}{2}\right]^0\left[\frac{1}{2}\right]^{32}$$

$$= 1 - {}^{32}C_0 \cdot \left[\frac{1}{2}\right]^{32}$$

$$= 1 - \frac{32!}{32! \cdot 0!} \cdot \left[\frac{1}{2}\right]^{32}$$

$$= 1 - \left[\frac{1}{2^{32}}\right].$$

ii) $P(X > 2) = P(X=3) + P(X=4) + \dots + P(X=32)$

$$= 1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$= 1 - \left[{}^{32}C_0 \cdot \left[\frac{1}{2}\right]^0\left[\frac{1}{2}\right]^{32} + {}^{32}C_1\left[\frac{1}{2}\right]\left[\frac{1}{2}\right]^{31} + {}^{32}C_2\left[\frac{1}{2}\right]^2\left[\frac{1}{2}\right]^{30}\right.$$

$$= 1 - \left[ \left(\frac{1}{2}\right)^{32} + \frac{32!}{31!\cdot 1!}\left(\frac{1}{2}\right)^{32} + \frac{32!}{30!\cdot 2!}\left(\frac{1}{2}\right)^{32} \right]$$

$$= 1 - \left(\frac{1}{2}\right)^{32}\left[1 + 32 + 496\right]$$

$$= 1 - 529\left[\frac{1}{2}\right]^{32}$$

$$= 0.9999$$

## Fitting Binomial distribution:

WKT in a random experiment consisting on $n$ independent trials the propability of getting $r$ successes is given by

$$P(X=r) = {}^nC_r \cdot p^r \cdot q^{n-r} \quad , r = 0, 1, 2, \ldots, n.$$

If this random experiment is repeated $N$ no. of times then the theoritical / expected frequency of getting $r$ success out of $n$ trials is given by $N(P(X=r))$.

$$N \cdot P(X=r) = N \cdot {}^nC_r \cdot p^r \cdot q^{n-r}$$

The possible no. of success together with these theoritical frequencies consists binomial frequency distribution as the follows

| Number of Successes (r) | Theoritical / Expected frequency $N \cdot P(X=r)$ |
|---|---|
| 0 | $N \cdot P(X=0)$ |
| 1 | $N \cdot P(X=1)$ |
| 2 | $N \cdot P(X=2)$ |
| ⋮ | ⋮ |
| n | $N \cdot P(X=n)$. |

Eg: Fit a binomial distribution to the following frequency distribution.

| $x$ : | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $f$ : | 13 | 25 | 52 | 58 | 32 | 16 | 4. |

Sol: Given,

$n = 6.$

$$\text{Mean} = \Sigma x_i f_i / \Sigma f_i = \Sigma x_i f_i / N$$

$$= \frac{25 + 104 + 174 + 128 + 80 + 24}{200}$$

$$= \frac{535}{200} = 2.675 = E(X).$$

$$\text{Variance} = E(X^2) - [E(X)]^2$$

$$E(X^2) = \Sigma x_i^2 f_i / \Sigma f_i$$

$$= 25 + 208 +$$

In binomial distribution

Mean is $nP$

$$\Rightarrow nP = 2.675$$

$$\Rightarrow P = \frac{2.675}{6} = 0.4458.$$

$$\Rightarrow q = 1 - 0.4458 = 0.5542.$$

$r$. $N \cdot P(X = r)$

0 $N \cdot P(X = 0) = N \cdot {}^nC_0 \cdot P^0 \cdot q^n$

$= 200 \cdot {}^6C_0 \cdot (0.4458)^0 (0.5542)$

$= 5.7946$

1. $N \cdot P(X = 1) = N \cdot {}^nC_1 \cdot P^1 \cdot q^{n-1}$

$= 200 \cdot {}^6C_1 \cdot (0.4458)(0.5542)^5$

$= 200 \times 6 \times 0.4458 \times 0.0522 = 27.9875$

2 $N \cdot P(X = 2)$

$= 200 \cdot {}^6C_2 \cdot (0.4458)^2 (0.5542)^4$

$= 200 \times 15 \times 0.1987 \times 0.0943 = 56.232$

3

$$N \cdot P(X=3) = 200 \cdot {}^6C_3 \cdot (0.4458)^3 (0.5542)^3$$
$$= 200 \times 20 \times 0.0885 \times 0.1702 = \underline{60}$$

4.

$$N \cdot P(X=4) = 200 \cdot {}^6C_4 (0.4458)^4 (0.5542)^2$$
$$= 200 \times \frac{\cancel{6}^{3} \times 5}{\cancel{2}} \times 0.0394 \times 0.3071$$
$$= 36.303 = \underline{36}.$$

5.

$$N \cdot P(X=5) = 200 \cdot {}^6C_5 \cdot (0.4458)^5 \cdot (0.5542)$$
$$= \underline{11.7097} = 12.$$

6.

$$N \cdot P(X=6) = 200 \cdot {}^6C_6 \cdot (0.4458)^6$$
$$= 200 \times 0.0078$$
$$= \underline{1.569} = 2.$$

Expected freq: 6 28 56 60 36 12 2 ⇒ total = 200.

Eg: In 256 sets of 12 tosses of a coin, in how many cases one can expect 8 heads and 4 tails?

Sol: Here $n = 12$, $P$ = probability of getting head $= \frac{1}{2}$

$q$ = Probability of tail $= 1 - \frac{1}{2} = \frac{1}{2}$.

$$P(\text{getting 8 heads}) = P(X=8)$$
$$= {}^{12}C_8 \cdot \left[\frac{1}{2}\right]^8 \left[\frac{1}{2}\right]^4$$
$$= \frac{12!}{4! \cdot 8!} \left[\frac{1}{2}\right]^{12}$$
$$= \frac{\cancel{12} \times \cancel{11} \times \cancel{10}^{5} \times 9}{\cancel{4} \times \cancel{3} \times \cancel{2}} \left[\frac{1}{2}\right]^{12} = 495 \left[\frac{1}{2}\right]^{12}$$
$$= 0.12085$$

$$\begin{array}{r} 45 \\ 11 \\ \hline 45 \\ 450 \\ \hline 495 \end{array}$$

The expected frequency of such cases in 256 sets is given by

$$N \cdot P(X=8) = 256 \times 495 \times \left[\frac{1}{2}\right]^{12} = 30.9375$$
$$= 31.$$

## Poisson distribution:

It is a discrete probability distribution. It is a limiting case of binomial distribution. A discrete r.v X is said to be follow poisson distribution with parameter $\lambda$ if its probability mass function is given by,

$$P(X=r) = \frac{e^{-\lambda}\lambda^{r}}{r!}, \quad r = 0, 1, 2, \ldots, \infty$$

Conditions:

1]. The no. of trials $n$ is very large. (That is: $n \to \infty$)

2]. The probability of succes in a trial is very small (That is $p \to 0$)

3]. The product of $n$ and $p$ is constant. That is: $np = \lambda$, where $\lambda$ is a +ve real number.

4] Each trial results in two mutually exclusive outcomes, termed as success and failure.

Mean of poisson distribution:

$$\mu = E(X) = \sum_{r=0}^{\infty} r \cdot P(X=r)$$

$$= \sum_{r=0}^{\infty} r \cdot \frac{e^{-\lambda} \cdot \lambda^{r}}{r!}$$

$$= \lambda \quad \text{(by simplifying)}$$

$$E(X^2) = \sum_{r=0}^{\infty} r^2 P(X=r)$$

$$= \sum_{r=0}^{\infty} r^2 \cdot \frac{e^{-\lambda} \cdot \lambda^{r}}{r!}$$

$$= \lambda + \lambda^2.$$

Then variance is given by

$$Var(X) = \sigma^2 = E(X^2) - [E(X)]^2$$

$$= \lambda + \lambda^2 - \lambda^2 = \lambda.$$

* Note:

Mean = variance = $\lambda$ (for poisson distribution)

Fitting Poisson distribution:

| $r$ | $N \cdot P(X=r)$ |
|---|---|
| 0 | $N \cdot P(X=0) = N \cdot e^{-\lambda} \lambda^0 / 0!$ |
| 1 | $N \cdot P(X=1) = N \cdot e^{-\lambda} \lambda^1 / 1!$ |
| 2 | $N \cdot P(X=2) = N \cdot e^{-\lambda} \lambda^2 / 2!$ |
| 3 | $N \cdot P(X=3) = N \cdot e^{-\lambda} \lambda^3 / 3!$ |
| ⋮ | ⋮ |

Eg: The average no. of accidents on any day on a national highway is 1.8. determine the prob that the no. of accidents are:

i) Atleast 1.

ii) Atmost 1.

Sol: $\lambda$ = Avg no. of accidents = 1.8.

i) $P(X \geq 1) = 1 - P(X=0) = P(X=1) + P(X=2) + \dots$

$$= 1 - \frac{e^{-1.8} \cdot [1.8]^0}{0!}$$

$$= 1 - [e^{-1.8}] = 0.8347$$

ii) $P(X \leq 1) = P(X=0) + P(X=1)$ → 0.1652

$$= 0.8347 + \frac{e^{-1.8}(1.8)}{1!}$$

$$= 0.8347 + \cancel{0.6621}\ 0.2973$$

$$= \cancel{1.4968}\ 1.1322$$

Eg: If the probability of bad react^n from certain injection is 0.001. Then find the prob that out of 2000 individuals more than 2 will get bad reaction.

Sol: given, $N = 2000$, $\lambda = 0.001$

$$N \cdot P(X>2) = N[1 - P(X=0) - P(X=1)]$$

$$= 2000\left[1 - \frac{e^{-0.001} \cdot (0.001)^0}{0!} - e^{-0.001} \cdot (0.001)\right.$$

Sol: Here, $p = 0.001$

$n = 2000$

$\lambda = np = 2000 \times 0.001 = 2.$

$P(X > 2) = 1 - P(X = 0) - P(X = 1) - P(X = 2)$

$= 1 - \frac{e^{-2}(2)^0}{0!} - \frac{e^{-2}(2)^1}{1!} - \frac{e^{-2}(2)^2}{2!}$

$= 1 - \frac{1}{e^2} - \frac{1}{e^2} - \frac{2}{e^2} \Rightarrow 1 -$

$1 - \frac{1}{7.3890} - \frac{1}{2.7182} - \frac{2}{}$

$= 1 - 0.1353 - 0.3678$

$= 1 - 0.5031$

$= 0.4969$

$= 1 - \frac{5}{e^2} = 0.324$

Eg: Fit a poisson distribution to the following tab

| $x$: | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $f$: | 122 | 60 | 15 | 2 | 1 |
| | 121 | 61 | 15 | 3 | 0 |

observed freq. $e^{-0.5} = 0.6065$

$N = 200$

Theoritical freq

Sol:

| $r$ | $N \cdot P(X = r)$ |
|---|---|
| 0 | $N \cdot \frac{e^{-\lambda} \cdot \lambda^r}{r!} = 200 \times \frac{e^{-0.5} \times (0.5)^0}{0!} = 121.30$ |
| 1 | $200 \times \frac{e^{-0.5}(0.5)}{1} = 60.65 = 61$ |
| 2 | $200 \times \frac{e^{-0.5}(0.5)^2}{2} = 15.1625 = 15$ |
| 3 | $200 \times \frac{e^{-0.5}(0.5)^3}{3!} = 2.527 = 3.$ |
| 4. | $200 \times \frac{e^{-0.5}(0.5)^4}{4!} = 0.3159 = 0$ |

$\text{Mean} = \frac{\Sigma x_i f_i}{\Sigma f_i} = \frac{60 + 30 + 6 + 4}{200} = \frac{100}{200} = \frac{1}{2} = 0.5$

for poisson distribution, mean, $\lambda = 0.5$

In a certain factory turning out razor blades, there is a small chance of 0.002 for any blade to be defective. The blades are supplied in packets of 10. using poisson distribution cal the approx no. of packets containing no def, 1 def, 2 def plates respectively in a consignment of 10,000 packets.

Sol: $p = 0.002$.

$n = 10$.

$N = 10,000$.

$\lambda = np = 0.002 \times 10 = 0.02$.

i) $N \cdot P(X=0) = N \cdot \frac{e^{-\lambda} \cdot \lambda^r}{r!} = 10,000 \times \frac{e^{-0.02} \times (0.02)^0}{0!}$

$= 10,000 \times 0.9801986$

$= 9801.986 = 9802$.

ii) $N \cdot P(X=1) = N \cdot \frac{e^{-\lambda} \cdot \lambda^r}{r!} = 10,000 \times \frac{e^{-0.02} \times (0.02)^1}{1!}$

$= 196.0397 = 196$.

iii) $N \cdot P(X=2) = 10,000 \times \frac{e^{-0.02} \cdot (0.02)^2}{2!}$

$= 10,000 \times \frac{0.9801986 \times 0.02 \times 0.02}{2!}$

$= 1.9603 = 2$.

total $= 9802 + 196 + 2 = 10,000$.

<u>Normal distribution</u>: It is a continuous probability distribution also known as Gaussian distribution.

<u>def</u>: A continuous r.v X is said to follow normal distribution with parameters $\mu$ and $\sigma^2$ if its probability density function is given by.

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-1/2\left[\frac{x-\mu}{\sigma}\right]^2}$$ → this is power with $-\frac{1}{2}$

for $-\infty < x < \infty$.

* Mean is given by $(E(x)) = \int_{-\infty}^{\infty} x f(x)\, dx$

$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left[\frac{x-\mu}{\sigma}\right]^2} dx.$$

$$= \mu$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left[\frac{x-\mu}{\sigma}\right]^2} dx.$$

$$= \mu^2 + \sigma^2$$

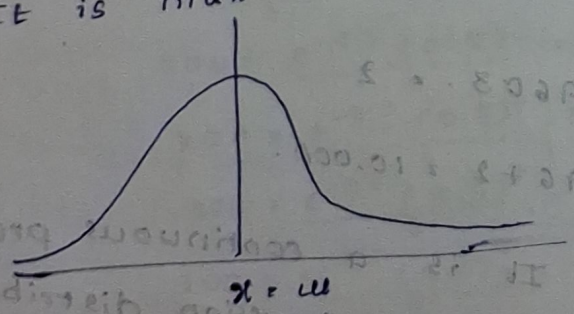
* variance is.

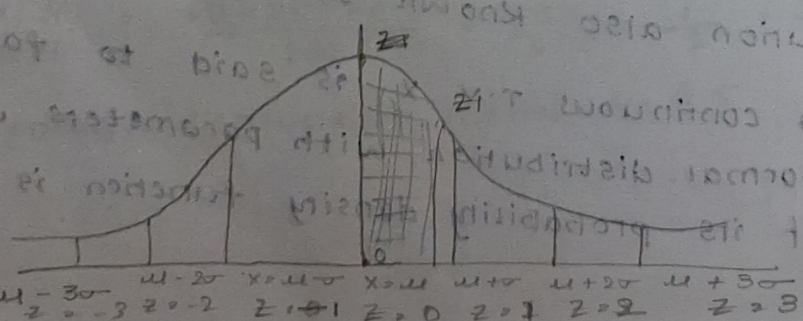
$$E(x^2) - [E(x)]^2 = \mu^2 + \sigma^2 - \mu^2 = \sigma^2.$$

<u>Properties</u>:

1. The graph of $f(x)$ is a bell shaped curve as shown in the figure. It is known as normal probability curve. (bell shaped)

2. The graph of $y = f(x)$ is symmetrical about ~~$x = y$~~ $x = \mu$. max value is $x = \mu$. It is max. value at $x = \mu$. max value is $\frac{1}{\sigma\sqrt{2\pi}}$.



$f(x)$ decreases rapidly as $x \to \pm\infty$. Here x-axis is an asymptote.

7. The graph is concave downward at $x = \mu$ & it is concave upward for large values of $x$.

8. Here concavity changes at $x = \mu \pm \sigma$ (These points are known as points of inflexion, which are calculated by solving $f''(x) = 0$)

→ If $\sigma$ is large, then the curve tend to be flat.

→ If $\sigma$ is small then the curve tend to be more peaked.

Normal probability integral (Area property).

$$P(\mu < X < x_1) = \int_{\mu}^{x_1} f(x)\,dx$$

$$= \int_{\mu}^{x_1} \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-1/2\left[\frac{x-\mu}{\sigma}\right]^2} dx.$$

put $\boxed{\frac{x-\mu}{\sigma} = z} \Rightarrow x = \mu + \sigma z.$

$dx = 0 + \sigma\,dz \Rightarrow dx = \sigma\,dz.$

Limits: $x = \mu \Rightarrow z = 0.$

$x = x_1 \Rightarrow z = \frac{x_1 - \mu}{\sigma} = z_1$ (say).

Using this in above integral, we got

$$P(0<z<z_1) = \int_{z=0}^{z_1} \frac{1}{\sigma\sqrt{2\pi}} e^{-z^2/2} \cdot \sigma \cdot dz$$

$$= \int_{0}^{z_1} \frac{1}{\sqrt{2\pi}} \cdot e^{-z^2/2}\, dz.$$

$z \to$ standard normal R.V.

$$= \int_{0}^{z_1} f(x)\,dz, \text{ where } f(x) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

↓

This is called pdf of standard normal r.v $z$.

This integral $P(0 < z < z_1) = \int_0^{z_1} f(x)\,dx$ represents area under the standard normal probability curve between $z = 0$ and $z = z_1$. All these areas values for different values of $z_1$ are available in normal distribution table.

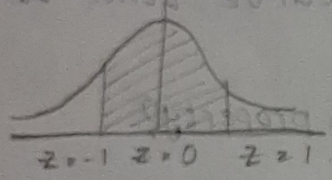
Eg-1: $P(\mu - \sigma < x < \sigma + \mu)$

using $\boxed{z = \frac{x - \mu}{\sigma}}$

$= P(-1 < z < 1)$

If $x = \mu - \sigma \Rightarrow z = -1$

If $x = \mu + \sigma \Rightarrow z = 1$




$= \int_{-1}^{1} f(z)\,dz$

$= 2\int_0^1 f(z)\,dz$. (by symmetry).

$= 2(0.3413)$

$= 0.6826 = 68.26\%$.

Eg-2: $P(\mu - 2\sigma < x < \mu + 2\sigma)$

$= P(-2 < x < 2)$

$= 2\int_0^2 f(z)\,dz$

$= 2(0.4772)$

$= 0.9544$

$= 95.44\%$.

Eg-3: $P(\mu - 3\sigma < X < \mu + 3\sigma)$

$= P(-3 < X < 3)$

$= 2\int_0^3 f(z)\,dz$

$= 2(0.4987)$

$= 0.9974$

$= 99.74\%$

Note:

$P(-\infty < X < \infty) = P(-\infty < Z < \infty)$

$= \int_{-\infty}^{\infty} f(z)\,dz$

$= 1$

Eg: If X is normal R.V with mean 30 and standard deviation 5, then find

i) $P(26 \le X \le 40)$

ii) $P(X \geq 45)$

iii) $P(|X-30| > 5)$

Sol: $X \sim N(\mu, \sigma^2)$

Given, $\mu = 30$, $\sigma = 5$.

using $Z = \frac{X-\mu}{\sigma}$

If $x = 26$, $Z = \frac{26-30}{5}$

$Z = -0.8$

If $x = 40$, $Z = \frac{40-30}{5}$

$= 2$

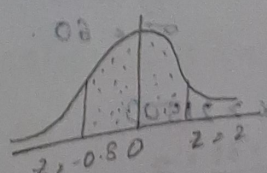
i) $P(26 \le X \le 40) = P(-0.8 \le Z \le 2)$

$= \int_{-0.8}^{0} f(z)\,dz + \int_0^2 f(z)\,dz$

$= P(0 \le Z \le 0.8) + 0.4772$

$= 0.2881 + 0.4772$

$= 0.7653$



ii) $P(X \geq 45)$.

$= P(Z \geq 3)$.

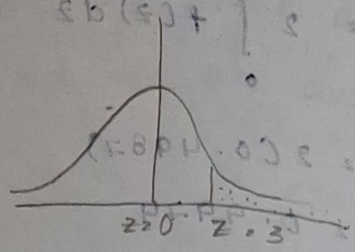
$= 0.5 - P(0 \leq Z \leq 3)$

$= 0.5 - 0.4987$

$= 0.0013$.

$Z = \frac{X-\mu}{\sigma}$

$Z = \frac{45-30}{5} = 3$.




iii) $P(|X-30| > 5)$

$= 1 - P(|X-30| \leq 5)$.

$= 1 - \int_{-1}^{1} f(z)\,dz$.

$= 1 - 2\int_{0}^{1} f(z)\,dz$

$= 1 - 2(0.3413)$

$= 1 - 0.6826$

$= 0.3174$.

WKT,

$|x| \leq a$

$\Rightarrow -a \leq x \leq a$.

Ely $|X-30| \leq 5$

$-5 \leq (X-30) \leq 5$

$25 \leq X \leq 35$.

$Z = \frac{25-30}{5} = -1$

$Z = \frac{35-30}{5} = 1$

Eg: In a test on 2000 electric bulbs, it was found that the life of a particular bulb was normally distributed with avg life of 2040 hrs & S.D of 60hrs. Estimate the no. of bulbs likely to burn for

i) more than 2150 hrs

ii) Less than 1950 hrs.

iii) more than 1920 hrs & Less than 2160 hrs.

i) $N \cdot P(X > 2150)$

ii) $N \cdot P(X > 1950)$

iii) $N \cdot P(1920 < X < 2160)$

Sol: Given, $N = 2000$

$\mu = 2040$

$\sigma = 60$.

$Z = \frac{X-\mu}{\sigma} = \frac{2150-2040}{60}$

$= \frac{110}{60} = 1.833$

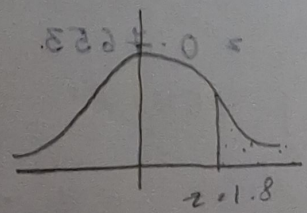
i) $P(X > 2150)$

$= P(Z > 1.8)$

$= 0.5 - P(0 < Z < 1.8)$

$= 0.5 - 0.4641$

$= 0.0359$




$N \cdot P(Z > 1.8) = 2000 \times 0.0359 = 71.8$.

ii) $P(X > 1950)$

$= P(Z > -1.5)$

$= 0.5 - P(-1.5 < Z < 0)$

$= 0.5 - P(0 < Z < 1.5)$

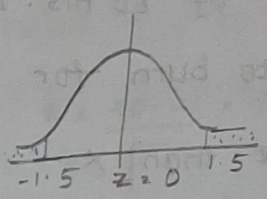
$= 0.5 - 0.4332$

$= 0.0668.$

$Z = \frac{X - \mu}{\sigma}$

$= \frac{1950 - 2040}{60}$

$= -1.5.$




N. $P(X > 1950) = 2000 \times 0.0668$

$= 133.6.$

iii) $P(1920 < X < 2160)$

$= P(-2 < Z < 2)$

$= 2P(0 < Z < 2)$

$= 2(0.4772)$

$= 0.9544$

$Z = \frac{1920 - 2040}{60}$

$= -2$

$Z = \frac{2160 - 2040}{60}$

$= 2$

Expected no. of bulbs.

N. $P(1920 < X < 2160) = 2000 \times 0.9544$

$= 1908.8.$

$\approx 1909$

Eg: If R.V $X \sim N(\mu, \sigma^2)$ with $\mu = 9$, $\sigma = 3$ then find

i) $P(X \geq 15)$, ii) $P(X \leq 15)$ iii) $P(0 \leq X \leq 9)$

iv) Find a S.T $P(X > a) = 0.16$.

Given $\mu = 9$, $\sigma = 3$.

i) $P(X \geq 15) = P(Z \geq 2)$

$= 0.5 - P(0 \leq Z \leq 2)$

$= 0.5 - 0.4772$

$= 0.0228.$

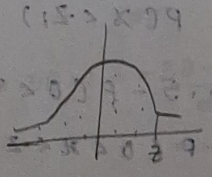
$Z = \frac{X - \mu}{\sigma}$

$= \frac{15 - 9}{3} = \frac{6}{3} = 2$

ii) $P(X \leq 15) = P(Z \leq 2)$

$= 0.5 - P(Z > 2)$

$Z = \frac{15 - 9}{3} = \frac{6}{3} = 2$



Eg: In a ~~code~~ test on 2000 electric bulbs, It was found that the Life of a particular bulb (make) was normaly distributed with an avg Life of and SD of 60 hrs. Estimate the no. of bulbs likely to burn for.

i) more than] x

Eg: In a distribution 10.03% of the items are under 25 kg wt and 89.97% of the items are under 70 kg wt. What are the mean & S.D of the distribution.

Sol: Let T.V x denotes weight of the items.

$x \sim N(\mu, \sigma^2)$.

Given,

$\cancel{P(X<20)}\ P(X<25) = 10.03\%$

$= \frac{10.03}{100} = 0.1003 \longrightarrow (1)$

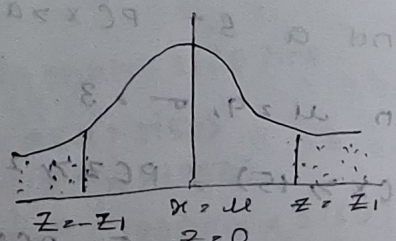
$P(X<70) = 89.97\%$

$= \frac{89.97}{100} = 0.8997 \longrightarrow (2)$

Now we need to find,

Mean $(\mu) = ?$

S.D $(\sigma^2) = ?$




from (1), we have.

$P(X<25) = 0.1003$.

$\Rightarrow P(Z<-Z_1) = 0.1003$

$\cancel{P(Z<}$

$P(X<Z_1) = 0.1003$

$0.5 - P(0<x<Z_1) = 0.1003$

$P(0<x<Z_1) = 0.5 - 0.1003$

$P(0<x<Z_1) = 0.3997$

$Z_1 = 1.28$.

$Z = \frac{x-\mu}{\sigma}$

$x = 25 \Rightarrow Z = \frac{25-\mu}{\sigma} = -Z_1$ (say) $\rightarrow (3)$

$x = 70 \Rightarrow Z = \frac{70-\mu}{\sigma}$

$z_1 = \frac{25-\mu}{\sigma} = 1.28$ (from - ③)

$25 - \mu = 1.28\sigma$

$\mu = 25 - 1.28\sigma \rightarrow$ ④

Illy from - ② we have

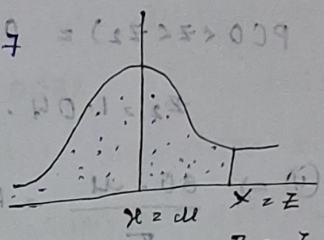
$P(X < 70) = 0.8997$

$z = \frac{70-\mu}{\sigma} = z_2$ (say) $\rightarrow$ ⑤

$P(z < z_2) = 0.8997$

$0.5 + P(0 < z < z_2) = 0.8997$

$P(0 < z < z_2) = 0.3997$




~~$z_2 = 0.3$~~

$z_2 = 1.28$

$\frac{70-\mu}{\sigma} = 1.28$

$70 - \mu = 1.28\sigma$

$\mu = 70 - 1.28\sigma \rightarrow$ ⑥

$70 - 1.28\sigma = 25 - 1.28\sigma$

solving ④ & ⑥

$\mu = 47.5; \quad \sigma = 17.578$

$\mu + 1.28\sigma = 25$

$\mu + 1.28\sigma = 70$

Eg: In a distribution which is exactly normal 12% items are under 30 & 85% of items are under 60. Find mean & S.D of the distribution.

Sol: Given,

$P(X < 30) = 12\% = 12/100 = 0.12 \rightarrow$ ①

$P(X < 60) = 85\% = 85/100 = 0.85 \rightarrow$ ②

$z = \frac{x-\mu}{\sigma}$

from ①,

$P(X < 30) = 0.12$

$x = 30 \Rightarrow z = \frac{30-\mu}{\sigma} = -z_1$ $\rightarrow$ ③

$P(z < -z_1) = 0.12$

$0.5 - P(0 < z < z_1) = 0.12 \Rightarrow P(0 < z < z_1) = 0.38$

$z_1 = 1.18$

$$\frac{30-\mu}{\sigma} = +1.18 \quad (\text{from} - ③)$$

$$30-\mu = +1.18\sigma$$

$$\mu = 30 + 1.18\sigma \rightarrow ⑤$$

from - ② we have,

$$P(X < 60) = 0.85$$

$$0.5 + P(0 < Z < Z_2) = 0.85$$

$$P(0 < Z < Z_2) = 0.35$$

$$Z_2 = 1.04$$

$x = 60$,

$$\frac{60-\mu}{\sigma} = Z_1 \text{ (say)} \rightarrow ④$$

from ④ $\Rightarrow \frac{60-\mu}{\sigma} = 1.04$

$$60-\mu = 1.04\sigma$$

$$\mu = 60 - 1.04\sigma$$

$$30 + 1.18\sigma = 60 - 1.04\sigma$$

$$0.14\sigma = 30$$

$$\sigma = 214.285$$

$$\sigma = 13.5655$$

$$\mu = 45.9395$$

Eg: The journey times of a bus service, 'T' min, are thought to be normally distributed with a mean of 62 and SD of 5. Determine the probability that the time of a random journey from this bus service will be b/w 50 & 60 min.

$$T \sim N(\mu = 62, \sigma = 5)$$

$$P(50 \le T \le 60)$$

$$Z = \frac{T-\mu}{\sigma}$$

L.L: $T = 50$:

$$Z_1 = \frac{50-62}{5} = -\frac{12}{5} = -2.4$$

U.L : $T = 60$

$$Z_2 = \frac{60-62}{5} = -\frac{2}{5} = -0.4$$

$P(50 \le T \le 60) = P(-2.4 \le Z \le -0.4)$

$Z_1 = 0.0082$.

$Z_2 = 0.3446$.

$P(-2.4 \le Z \le -0.4) = Z_2 - Z_1$

$= 0.3446 - 0.0082$

$= \underline{0.3364}$.

Ans.

## Geometric distribution:

It is a discrete probability distribution).

A discrete r.v X is said to follow geometric distribution if its probability mass function (PMF) is defined by $P(X = k)$ is given by, $P(X = k) = q^{k-1} p$, $k = 1, 2, 3, \ldots$

Here r.v X is defined as the trial number on which first success occur in a series of independent trial.

That means:

$X = 1$ means first success is on trial 1.

$X = 2$ means trial 1 is failure, first success is on trial 2.

$X = 3$ means trial 2 is failure, first success is on trial 3.

$X = K$ means trial $K-1$ is failure, first success is on trial $K$.

$X > K$ means first success after $K$ trials

$X \leq K$ means first success happens on or before $K^{th}$ trial.

Conditions for the applications of geometric distribution

1. All trials and independent.
2. Each trial has only 2 outcomes: Success & failure.
3. The probability of success $p$ remains same for each trial.
4. r.v $X$ represents the trial number on which first success happens.

Mean (or) expectation of $X$ is

$$E(X) = \sum_{K=1}^{\infty} K \cdot P(X = K)$$

$$= \sum_{K=1}^{\infty} K \cdot q^{K-1} \cdot p$$

$$\boxed{E(X) = \frac{1}{p}}$$

$$E(X^2) = \sum_{K=1}^{\infty} K^2 P(X = K)$$

$$= \sum_{K=1}^{\infty} K^2 \cdot q^{K-1} \cdot p$$

$$\boxed{E(X^2) = \frac{1+q}{p^2}}$$

variance of $X$:

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \frac{1+q}{p^2} - \frac{1}{p^2}$$

$$\boxed{Var(X) = \frac{q}{p^2}} \Rightarrow Var(X) = \frac{1-p}{p^2} \quad (\because q = 1-p).$$

Note: $P(X \le K) = \sum_{i=1}^{k} P(X = i)$

$\downarrow$

cummulative distribution function

$$= \sum_{i=1}^{k} q^{i-1} \cdot p = p \sum_{i=1}^{k} q^{i-1}$$

$$= p \cdot \sum_{i=1}^{k} q^{i}$$

$$= p\left[\frac{1-q^K}{1-q}\right] = \frac{1-q^K}{\cancel{1-q}}$$

$1 + q + q^2 + \dots + q^{K-1}$

$= 1\left[\frac{1-q^K}{1-q}\right]$

Note:

$P(X > K) = 1 - P(X \le K)$

$= 1 - [1 - q^K]$

total probability $= q^K$

Note: $P(X > m+n \mid X > m) = P(X > n)$

$L.H.S = P(X > m+n \mid X > m) = \frac{P((X > m+n) \cap (X > m))}{P(X > m)}$

memory less property.

$[a + ar + ar^2 + \dots + ar^{n-1} = a\left[\frac{1-r^n}{1-r}\right]$

$X > K$ means 1st success happens after $K$ trials

$$= \frac{P(X > m+n)}{P(X > m)}$$

$$= \frac{q^{m+n}}{q^m} = \frac{q^m \cdot q^n}{q^m}$$

$$= q^n$$

$$= P(X > n).$$

A communication link successfully transmits a packet on any attempt with probability $P = 0.3$ independently accross the attempts. If r.v $X$ is the no. of attempts until the 1st success, then find.

i) $P(X=4) = q^{4-1} P = q^3 \cdot P = (0.7)^3 \cdot (0.3) = 0.1029$

ii) $P(X>5) = q^5 = (0.7)^5 = 0.1680$

iii) $P(X \leq 3) = 1 - q^3 = 1 - (0.7)^3 = 0.657$

iv) $E(X), var(X)$ $\quad E(X) = \frac{1}{P} = \frac{1}{0.3} = 3.33$

Given $P = 0.3$ $\quad var(X) = \frac{q}{P^2} = \frac{0.7}{(0.3)^2}$

$q = 1 - P = 0.7$

Eg: If $P(X \geq 4) = 0.2401$ for geometric distribution, find the value of $P$

Sol: Given $P(X>4) = 0.2401$

$P(X > k) = q^k$

$P(X>4) = q^4 = 0.2401$

$q = (0.2401)^{1/4}$

Eg: Packets are delivered successfully with probability $P = 0.25$ what is the probability that first success occur on the 8th trial, given that 1st 5 trials are failed.

Sol: Given,

$P = 0.25$

$q = 1 - P = 0.75$

We need find,

$$P((X=8) / X>5) = \frac{P((X=8) \cap (X>5))}{P(X>5)}$$

$$= \frac{P(X=8)}{P(X>5)}$$

$$P(X=k) = q^{k-1} \cdot p$$

$$= \frac{q^{k-1} \cdot p}{P(X>5) \rightarrow q^k}$$

$$= \frac{(0.75)^{8-1} \cdot (0.25)}{(0.75)^8}$$

$$= (0.75)^2 \cdot (0.25)$$

$$= 0.1406$$

Eg: In a geometric distribution the variance is 12. Then find the value of P.

Sol: $\frac{q}{p^2} = 12$

$$q = 12p^2$$

$$\frac{1-p}{12} = p^2 \Rightarrow 12p^2 + p - 1 = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{1 - 4(12)(-1)}}{2(12)}$$

$$= \frac{-1 \pm \sqrt{1+48}}{24}$$

$$= \frac{-1 \pm \sqrt{49}}{24}$$

$$= \frac{-1 \pm 7}{24}$$

$$p = -\frac{1}{3}, \frac{1}{4}$$

$$p = \frac{1}{4}.$$

Eg: A Surver completes a request successfully with probability $p = 0.25$ attempt. What is the probability that 1st success happens on or after 6th attempt.

SOL: Given,

$P = 0.25$

$q = 1 - P = 0.75$

We need to find,

$$P(X \geq 6) = P(X > 5)$$
$$= q^5$$
$$= (0.75)^5$$
$$= 0.2375.$$

Hypergeometric distribution:

[It is a discrete probability distribution].

A discrete random variable 'X' is said to follow hypergeometric distribution with parameters N, m, n if its probability mass function (pmf) is given by

$$* \; P(X = i) = \frac{{}^{m}C_{i} \cdot {}^{(N-m)}C_{(n-i)}}{{}^{N}C_{n}}$$

for $\text{Max}(0, M-(N-m)) \leq i \leq \text{Min}(n, m)$

Here

N = population size.

n = sample size.

m = no. of success in the population.

N-m = no. of failures in the population

X = no. of success in the sample.

Conditions for the applicability:

1). Population size N is finite and known.

2). The population is partitioned into two categories success to failures, with m successes and N-m failures in total.

3] A sample of size n is drawn from the population.

4] Sampling is done without replacement.

5] Random variable 'X' is number of successes in the sample.

Mean (or) Expected value [E(X)] of X is:

$$E(X) = \sum_i i P(X = i)$$

$$= \sum_i i \left[ \frac{{}^{m}C_i \cdot {}^{(N-m)}C_{(n-i)}}{{}^{N}C_n} \right].$$

$$\boxed{E(X) = n \cdot \frac{m}{N}}$$

$$E(X^2) = \sum_i i^2 P(X = i)$$

$$= \sum_i i^2 \left[ \frac{{}^{m}C_i \cdot {}^{(N-m)}C_{(n-i)}}{{}^{N}C_n} \right]$$

$$= n\frac{M}{n} + \frac{n(n-1)m(m-1)}{N(N-1)}$$

Then

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \left[ n\frac{M}{N} + \frac{n(n-1)m(m-1)}{N(n-1)} \right] - \left[ n\frac{m}{N} \right]^2$$

$$\boxed{Var(X) = n \cdot \frac{M}{n} \left[ 1 - \frac{m}{N} \right] \left[ \frac{N-n}{N-1} \right]}$$

BLANK

Prob: A production batch has 100 items, of which 15 are defective. An infinite draws 10 items at random without replacement. Find i) $P(X=0)$ ii) $P(X=1)$ iii) $P(X=2)$, iv) $P(X \le 2)$ v) $E(X)$, $Var(X)$.

Sol: Given,

$N = 100$, $m = 15$ (defective items)

$N-m = 85$ (non-defective items)

$n = 10$.

Now we need to find,

$$P(X=0) = \frac{{}^{15}C_0 \cdot {}^{85}C_{10}}{{}^{100}C_{10}}$$

using ${}^{n}C_r = \dfrac{n!}{(n-r)! \cdot r!}$

$$= \frac{1\left[\dfrac{85!}{75! \cdot 10!}\right]}{\left[\dfrac{100!}{90! \cdot 10!}\right]}$$

$$= 0.1808.$$

$$P(X=1) = \frac{{}^{15}C_1 \cdot {}^{85}C_9}{{}^{100}C_{10}}$$

$$= \frac{15\left[\dfrac{85!}{76! \cdot 9!}\right]}{\left[\dfrac{100!}{90! \cdot 10!}\right]}$$

$$= 0.3568$$

$$P(X=2) = \frac{{}^{15}C_2 \cdot {}^{85}C_8}{{}^{100}C_{10}} = \frac{\left[\dfrac{15!}{13! \cdot 2!}\right]\left[\dfrac{85!}{77! \cdot 8!}\right]}{\left[\dfrac{100!}{90! \cdot 10!}\right]}$$

$$= 0.2919.$$

$P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$

$= 0.1808 + 0.3568 + 0.2919$

$= 0.8295$.

$E(X) = n \cdot \frac{m}{N} = 10\left[\frac{15}{100}\right] = 1.5$

$var(X) = n \cdot \frac{m}{N}\left(1 - \frac{m}{N}\right)\left(\frac{N-n}{N-1}\right)$

$= 10\left[\frac{15}{100}\right]\left[1 - \frac{15}{100}\right]\left[\frac{100-10}{100-1}\right]$

$= 10\left[\frac{15}{100}\right]\left[\frac{85}{100}\right]\left[\frac{90}{99}\right]$

$= 1.1590$

Prob: A small box contains 12 items in which 5 are faulty. Four items are selected without replacement. Given that atleast one faulty is found, find the prob that exactly one faulty is found.

Sol: Given:

$N = 12,\ m = 5,\ N - m = 12 - 5 = 7$

$n = 4$

Now we need,

$P(X=1 \mid X \ge 1) = \frac{P[(X=1) \cap (X \ge 1)]}{P(X \ge 1)}$ [∵ using conditional]

$= \frac{P(X=1)}{P(X \ge 1)} = \frac{P(X=1)}{1 - P(X=0)}$

$= \frac{\left[{}^5C_1 \cdot {}^7C_3 / {}^{12}C_4\right]}{1 - \left[{}^5C_0 \cdot {}^7C_4 / {}^{12}C_4\right]}$

$\frac{\frac{5!}{4!\,1!} \cdot \frac{7!}{4!\,3!} \Big/ \frac{12!}{8!\,4!}}{1 - \frac{\frac{5!}{5!\,0!} \cdot \frac{7!}{3!\,4!}}{\frac{12!}{8!\,4!}}} = \frac{\frac{5 \times 7 \times 6 \times 5}{3 \times 2}}{1 - \left[\frac{7 \times 5 \times 6}{3 \times 2}\right]} = \frac{175}{495 - 35} = \frac{175}{460} = 0.3804$

Prob: A lot contains 300 items of which 18 are defective. A sample of 25 is inspected w/o replacement. The lot is accepted if at most 1 defective is found. Find the probability of acceptance.

Sol: Given,

$N = 300$.

$m = 18$ (defective)

$N - m = 300 - 18 = 282$ (non-defective)

$n = 25$.

$$P(X \le 1) = P(X=0) + P(X=1)$$

$$= \frac{{}^{18}C_0 \cdot {}^{282}C_{25}}{{}^{300}C_{25}} + \frac{{}^{18}C_1 \cdot {}^{282}C_{24}}{{}^{300}C_{25}}$$

$$= \frac{\left[\frac{282!}{257! \cdot 25!}\right]}{\left[\frac{300!}{275! \cdot 25!}\right]} + \frac{18\left[\frac{282!}{258! \cdot 24!}\right]}{\left[\frac{300!}{275! \cdot 25!}\right]}$$

$$= \frac{6.65 \times 10^{34} + 140.76 \times 10^{34}}{41.9 \times 10^{34}}$$

$$= \frac{147.41}{41.9}$$

Ans: $0.1990 + 0.3471 = 0.5461$.

Prob: A fair coin tossed repeatedly until a head happens. If $X$ is the no of tosses required to get first head, then find,

a). $E(X)$ b) $Var(X) = ?$ c). $P(X \le 4) = ?$

Sol: $p$ = success $= \frac{1}{2}$

$q$ = failure $= 1 - \frac{1}{2} = \frac{1}{2}$.

a) $E(X) = \frac{1}{p} = \frac{1}{1/2} = 2$.

b) $Var(X) = \frac{q}{p^2} = \frac{1/2}{\left[\frac{1}{2}\right]^2} = 2$

c). $P(X \le 4) = 1 - q^x = 1 - \left[\frac{1}{2}\right]^4$

$= 1 - \frac{1}{16}$

$= \frac{15}{16} = 0.9375$

* Exponential Distribution:

(It is a continuous Probability distribution)

A continuous r.v $X$ is said to follow exponential distribution with parameter $\lambda\ (>0)$ if its probability density function (pdf) is given by.

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x \ge 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Here $\lambda$ is rate parameter.

RV $X$ represents non-negative waiting time.

Mean (or) expected value of $X$:

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{-\infty}^{0} 0 \cdot x\, dx + \int_{0}^{\infty} x \lambda e^{-\lambda x} \cdot dx$$

$$= \lambda \int_{0}^{\infty} x e^{-\lambda x}\, dx.$$

$\int f_1 f_2\, dx = f_1 \int f_2\, dx - \int \frac{d}{dx}(f_1) \int f_2\, dx$

$$= \cancel{\lambda}\left[\frac{x e^{-\lambda x}}{-\cancel{\lambda}} - \int \left[\frac{1 e^{-\lambda x}}{-\cancel{\lambda}}\right] dx\right]$$

$$= -x e^{-\lambda x} + \int e^{-\lambda x}\, dx \Rightarrow \left[\underbrace{-x e^{-\lambda x}}_{0} + \frac{e^{-\lambda x}}{-\lambda}\right]_0^{\infty}$$

$$\cancel{= \left[0 - \frac{e^{-\infty}}{-\lambda}\right] - [}$$

$$= \left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^{\infty}$$

$$= -\frac{1}{\lambda}[0-1] = \frac{1}{\lambda}$$

$$\therefore \boxed{E(x) = \frac{1}{\lambda}}$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$

$$= \int_{-\infty}^{0} x^2 (0)\,dx + \int_0^{\infty} x^2 \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_0^{\infty} x^2 e^{-\lambda x}\,dx$$

$$= x^2 \int e^{-\lambda x}\,dx - \int\left(\frac{d}{dx}(x^2) \int e^{-\lambda x}\,dx\right) dx$$

$$= \lambda\left[\left(\frac{x^2 e^{-\lambda x}}{-\cancel{\lambda}}\right) - \int\left(\frac{2x e^{-\lambda x}}{-\cancel{\lambda}}\right) dx\right]$$

$$= \left[-x^2 e^{-\lambda x}\right]_0^{\infty} + \int (2x e^{-\lambda x})\,dx$$

$$= 2\left[x \int e^{-\lambda x}\,dx - \int\left(\frac{d}{dx}(x) \int e^{-\lambda x}\,dx\right) dx\right]$$

$$= 2\left[\frac{x e^{-\lambda x}}{-\lambda} - \int \frac{e^{-\lambda x}}{-\lambda}\,dx\right]$$

$$= 2\left[\frac{x e^{-\lambda x}}{-\lambda} + \frac{1}{\lambda}\,\frac{e^{-\lambda x}}{-\lambda}\right] = 2\left[\frac{x e^{-\lambda x}}{-\lambda} - \frac{e^{-\lambda x}}{\lambda^2}\right]$$

$$= \left[\underbrace{-x^2 e^{-\lambda x}}_{0} + \underbrace{\frac{2x e^{-\lambda x}}{-\lambda}}_{0} - \frac{2e^{-\lambda x}}{\lambda^2}\right]$$

$$= -\frac{2}{\lambda^2}\left[e^{-\lambda x}\right]_0^{\infty} \Rightarrow \left[-\frac{2}{\lambda^2}\left[e^{-\infty} - e^{0}\right]\right]$$

$$\Rightarrow -\frac{2}{\lambda^2}[0-1] = \frac{2}{\lambda^2}$$

$$E(x^2) = \frac{2}{\lambda^2}.$$

$$var(x) = E(x^2) - [E(x)]^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$= \frac{1}{\lambda^2}$$

$$\boxed{Var(x) = \frac{1}{\lambda^2}}$$

<u>Cummulative</u> <u>distribution</u> <u>function</u> [F(x)] :

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\,dx.$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{x} \lambda e^{-\lambda x}\,dx.$$

$$= \lambda \int_{0}^{x} e^{-\lambda x}\,dx$$

$$= \left[\frac{\cancel{\lambda} e^{-\lambda x}}{-\cancel{\lambda}}\right]_0^x$$

$$= -[e^{-\lambda x} - e^0] = 1 - e^{-x\lambda}.$$

$$F(x) = P(X \le x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-\lambda x} & \text{for } x \ge 0 \end{cases}$$

* CDF

<u>Note</u>: $P(a < x < b) = \int_a^b f(x)\,dx$

$$= \int_a^b F'(x)\,dx$$

$$= F(x)\Big]_a^b$$

$$= F(b) - F(a) *$$

* $P(a < x < b) = P(a \le x \le b) = P(a < x \le b) = P(a \le x < b)$

Note:

$$P(X \le a) = \int_{-\infty}^{a} f(x)\,dx$$

$$P(X \geq a) = \int_{a}^{\infty} f(x)\,dx.$$

Prob: The length of a phone call X (in minutes) is an exponential r.v with parameter $\lambda = \frac{1}{10}$. If someone arrives immediately ahead of you at public telephone booth, find the probability that you will have to wait.

i) more than 10 mins.

ii) between 10 & 20 mins.

Sol: $\lambda = \frac{1}{10}$, $X$ = length of call; $X \sim Exp(\lambda)$.

i) $P(X > 10) = P(\text{more than 10 min}) = \int_{10}^{\infty} f(x)\,dx$

$$= \int_{10}^{\infty} \lambda e^{-\lambda x}\,dx.$$

$$= \lambda \int_{10}^{\infty} e^{-\lambda x}\,dx.$$

$$= \cancel{\lambda}\left[\frac{e^{-\lambda x}}{-\cancel{\lambda}}\right]_{10}^{\infty} = -\left[e^{-\lambda x}\right]_{10}^{\infty}$$

$$= -\left[e^{-\infty} - e^{-10\lambda}\right]$$

$$= -\left[0 - e^{-\cancel{10}\cdot\frac{1}{\cancel{10}}}\right]$$

$$= e^{-1} = \frac{1}{e} = 0.3678.$$

ii) $P(10 < X < 20) = \int_{10}^{20} f(x)\,dx$

$$= \int_{10}^{20} \lambda e^{-\lambda x}\,dx = \lambda \int_{10}^{20} e^{-\lambda x}\,dx$$

$$= -\left[e^{-\lambda x}\right]_{10}^{20}$$

$$= -\left[e^{-20\lambda} - e^{-10\lambda}\right]$$

$$= e^{-10\lambda} - e^{-20\lambda} = e^{-1} - e^{-2}$$

$= \frac{1}{e} - \frac{1}{e^2}$

$= 0.3678 - 0.1353.$

$= 0.2325.$

Prob: A Sensors life time X (hours) is exponential with mean 200 hrs Find

i) The failure rate x

ii) The probability that the sensor lasts at 50 hrs.

iii) The probability that its lifetime is b/w 50 & 100 hrs.

Sol: Here,

$X \sim Exp(\lambda)$

Given:

Mean is 200 $\Rightarrow \frac{1}{\lambda} = 200$.

Now we find:

a) The value of $\lambda$

Since $\frac{1}{\lambda} = 200$ then $\Rightarrow \lambda = \frac{1}{200} = 0.005$

b) $P(X \geq 50) = \int_{50}^{\infty} f(x)\,dx$

$= \int_{50}^{\infty} \lambda e^{-\lambda x}\,dx$

$= \lambda \int_{50}^{\infty} e^{-\lambda x}\,dx = \cancel{\lambda}\left[\frac{e^{-\lambda x}}{-\cancel{\lambda}}\right]_{50}^{\infty}$

$= -[e^{-\infty} - e^{-50\lambda}]$

$= e^{-50\lambda} = e^{-50(0.005)}$

$= e^{-0.25} = 0.7788.$

(or)

Method-II: $P(X \geq 50) = P(X > 50)$

$= 1 - P(X \leq 50)$

$= 1 - F(50)$

$= 1 - [1 - e^{-\lambda(50)}]$

$= e^{-50\lambda}$

$= e^{-50(0.005)}$

$= 0.7788$.

c). $P(50 < x < 100) = \int_{50}^{100} f(x)\,dx$.

$= \int_{50}^{100} \lambda e^{-\lambda x}\,dx$

$= \frac{\cancel{\lambda}[e^{-\lambda x}]_{50}^{100}}{-\cancel{\lambda}} = -[e^{-100\lambda} - e^{-50\lambda}]$

$= -[e^{-100(0.005)} - e^{-50(0.005)}]$

$= -[e^{-0.5} - e^{-0.25}]$

$= -0.6065 + 0.7788$

$= 0.1723$.

(or)

Method II:

$P(50 < x < 100) = F(100) - F(50)$

$= [1 - e^{-\lambda(100)}] - [1 - e^{-\lambda(50)}]$

$= [-e^{-(0.005)(100)} + e^{-(0.005)(50)}]$

$= -0.6065 + 0.7788$

$= 0.1723$.

Prob: For an exponential lifetime x, the probability item lasts more than 5 time units is 0.6 determine

a) The rate $\lambda$, b) mean life c) probability that the life time atmost 4 time units

Sol: Here

$X \sim EXP(\lambda)$

Given

$P(X \geq 5) = 0.6$

$$\int_5^\infty f(x)\,dx = 0.6$$

$$\int_5^\infty \lambda e^{-\lambda x}\,dx = 0.6 \Rightarrow \lambda \int_5^\infty e^{-\lambda x}\,dx = 0.6$$

$$\Rightarrow \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_5^\infty = 0.6$$

$$\Rightarrow \left[-e^{-\lambda x}\right]_5^\infty = 0.6$$

$$\Rightarrow -\left[e^{-\infty} - e^{-5\lambda}\right] = 0.6$$

$$\Rightarrow 0 + e^{-5\lambda} = 0.6$$

take log on both sides $\Rightarrow -5\lambda = \log(0.6)$

$$\Rightarrow -5\lambda = -0.2218$$

$$\boxed{\lambda = 0.1022}$$

b) mean is $\frac{1}{\lambda} = \frac{1}{0.1022}$

$= 9.7847$

$P(X \leq 4) = F(4)$

$= 1 - e^{-\lambda(4)}$

$= 1 - e^{-4(0.1022)}$

$= 1 - e^{-0.4088}$

$= 1 - 0.6644$

$= 0.3356$