GITAM (Deemed to be University)

Department of Mathematics and Statistics

PARTIAL DERIVATIVES

Lecture Notes Prepared by

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Contents

1. Partial Derivatives

* If $z = f(x,y)$ then the first order partial derivatives are $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
* The second-order partial derivatives are $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial y^2}$,$\frac{\partial^2 z}{\partial x \partial y}$ and $\frac{\partial^2 z}{\partial y \partial x}$.

Problem 1: If $z = x^3 + 2x^2y^2 + y^3$ then find first and second order partial derivatives.

Solution: $a)$ Given $z = x^3 + 2x^2y^2 + y^3$

differentiating with respect to $'x'$ partially.

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \frac{\partial}{\partial x}x^3 + \frac{\partial}{\partial x}2x^2y^2 + \frac{\partial}{\partial x}y^3 \\
&= 3x^2 + 2y^2(\frac{\partial}{\partial x}x^2) + 0 \\
&= 3x^2 + 2y^2(2x) \\
&= 3x^2 + 4xy^2 \\
\therefore \frac{\partial z}{\partial x} &= 3x^2 + 4xy^2.
\end{aligned}$$

$b)$ Given $z = x^3 + 2x^2y^2 + y^3$

differentiating with respect to $'y'$ partially.

$$\begin{aligned}
\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y}x^3 + \frac{\partial}{\partial y}2x^2y^2 + \frac{\partial}{\partial y}y^3 \\
&= 0 + 2x^2(\frac{\partial}{\partial y}y^2) + 3y^2 \\
&= 2x^2(2y) + 3y^2 \\
&= 4x^2y + 3y^2 \\
\therefore \frac{\partial z}{\partial y} &= 4x^2y + 3y^2.
\end{aligned}$$

$c)$ Now we can find $\frac{\partial^2 z}{\partial x^2}$

$$
\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= \frac{\partial}{\partial x}(\frac{\partial z}{\partial x}) \\
&= \frac{\partial}{\partial x}(3x^2 + 4xy^2) \qquad\qquad (\because from(a)) \\
&= \frac{\partial}{\partial x}3x^2 + \frac{\partial}{\partial x}4xy^2 \\
&= 3\frac{\partial}{\partial x}x^2 + 4y^2\frac{\partial}{\partial x}x \\
&= 3(2x) + 4y^2 \\
\therefore \frac{\partial^2 z}{\partial x^2} &= 6x + 4y^2.
\end{aligned}
$$

$d)$ Now we can find $\frac{\partial^2 z}{\partial y^2}$

$$
\begin{aligned}
\frac{\partial^2 z}{\partial y^2} &= \frac{\partial}{\partial y}(\frac{\partial z}{\partial y}) \\
&= \frac{\partial}{\partial y}(4x^2y + 3y^2) \qquad\qquad (\because from(b)) \\
&= \frac{\partial}{\partial y}4x^2y + \frac{\partial}{\partial y}3y^2 \\
&= 4x^2\frac{\partial}{\partial y}y + 3\frac{\partial}{\partial y}y^2 \\
&= 4x^2 + 3(2y) \\
\therefore \frac{\partial^2 z}{\partial y^2} &= 4x^2 + 6y.
\end{aligned}
$$

$e)$ Now we can find $\frac{\partial^2 z}{\partial x \partial y}$

$$
\begin{aligned}
\frac{\partial^2 z}{\partial x \partial y} &= \frac{\partial}{\partial x}(\frac{\partial z}{\partial y}) \\
&= \frac{\partial}{\partial x}(4x^2y + 3y^2) \qquad\qquad (\because from(b)) \\
&= \frac{\partial}{\partial x}4x^2y + \frac{\partial}{\partial x}3y^2 \\
&= 4y\frac{\partial}{\partial x}x^2 + 3\frac{\partial}{\partial x}y^2 \\
&= 4y(2x) + 0 \\
\therefore \frac{\partial^2 z}{\partial x \partial y} &= 8xy.
\end{aligned}
$$

f) Now we can find $\frac{\partial^2 z}{\partial y \partial x}$

$$
\begin{aligned}
\frac{\partial^2 z}{\partial y \partial x} &= \frac{\partial}{\partial y}(\frac{\partial z}{\partial x}) \\
&= \frac{\partial}{\partial y}(3x^2 + 4xy^2) \qquad\qquad (\because from(a)) \\
&= \frac{\partial}{\partial y}3x^2 + \frac{\partial}{\partial y}4xy^2 \\
&= 3\frac{\partial}{\partial y}x^2 + 4x\frac{\partial}{\partial y}y^2 \\
&= 0 + 4x(2y) \\
\therefore \frac{\partial^2 z}{\partial y \partial x} &= 8xy.
\end{aligned}
$$

Problem 2: Find f_x and f_y for $f(x, y) = 2x^2 - 3y - 4$.

Solution:

$$f(x, y) = 2x^2 - 3y - 4$$

Step 1: Differentiate w.r.t. x:

$$f_x = \frac{\partial}{\partial x}(2x^2 - 3y - 4) = 4x$$

Step 2: Differentiate w.r.t. y:

$$f_y = \frac{\partial}{\partial y}(2x^2 - 3y - 4) = -3$$

Problem 3: Find f_x and f_y for $f(x, y) = x^2 - xy + y^2$.

Solution:

$$f(x, y) = x^2 - xy + y^2$$

Step 1: Differentiate w.r.t. x:

$$f_x = \frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial x}(y^2) = 2x - y + 0 = 2x - y$$

Step 2: Differentiate w.r.t. y:

$$f_y = \frac{\partial}{\partial y}(x^2) - \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial y}(y^2) = 0 - x + 2y = -x + 2y$$

Problem 4: Find f_x and f_y for $f(x, y) = 2x^2 + y^2$.

Solution:

$$f(x, y) = 2x^2 + y^2$$

Step 1: Differentiate w.r.t. x:

$$f_x = \frac{\partial}{\partial x}(2x^2 + y^2) = 4x$$

Step 2: Differentiate w.r.t. y:

$$f_y = \frac{\partial}{\partial y}(2x^2 + y^2) = 2y$$

Problem 5: Find f_x and f_y for $f(x, y) = \frac{x}{x^2+y^2}$.

Solution: Let $u = x$, $v = x^2 + y^2$, so $f = \frac{u}{v}$. We use the quotient rule:

$$\frac{\partial}{\partial x}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$$

Step 1: Differentiate w.r.t. x:

$$u' = 1, \quad v' = 2x$$

$$f_x = \frac{(1)(x^2 + y^2) - x(2x)}{(x^2 + y^2)^2} = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

Step 2: Differentiate w.r.t. y:

$$\frac{\partial}{\partial y}\left(\frac{x}{x^2 + y^2}\right) = \frac{0 \cdot (x^2 + y^2) - x(2y)}{(x^2 + y^2)^2} = \frac{-2xy}{(x^2 + y^2)^2}$$

Problem 6: Find f_x and f_y for $f(x, y) = \sin^2(x - 3y)$.

Solution: Let $u = x - 3y$, so $f = \sin^2 u$. Using the chain rule:

$$\frac{d}{dx}(\sin^2 u) = 2\sin u \cdot \cos u \cdot u'$$

Step 1: Differentiate w.r.t. x:

$$u_x = 1 \quad \Rightarrow \quad f_x = 2\sin(x-3y)\cos(x-3y) = \sin(2(x-3y))$$

Step 2: Differentiate w.r.t. y:

$$u_y = -3 \quad \Rightarrow \quad f_y = 2\sin(x-3y)\cos(x-3y)(-3) = -3\sin(2(x-3y))$$

Problem 7: Find f_x and f_y for $f(x, y) = \cos^2(3x - y^2)$.

Solution: Let $u = 3x - y^2$, so $f = \cos^2 u$. Using the chain rule:

$$\frac{d}{dx}(\cos^2 u) = 2\cos u \cdot (-\sin u) \cdot u'$$

Step 1: Differentiate w.r.t. x:

$$u_x = 3 \quad \Rightarrow \quad f_x = 2\cos u(-\sin u)(3) = -6\cos(3x-y^2)\sin(3x-y^2) = -3\sin(2(3x-y^2))$$

Step 2: Differentiate w.r.t. y:

$$u_y = -2y$$

$$\Rightarrow \quad f_y = 2\cos u(-\sin u)(-2y) = 4y\cos(3x-y^2)\sin(3x-y^2) = 2y\sin(2(3x-y^2))$$

Problem 8: Find f_x and f_y for $f(x, y) = \sqrt{2x^2 + y^2}$

Solution:

$$f(x, y) = \sqrt{2x^2 + y^2} = (2x^2 + y^2)^{1/2}$$

Partial derivative with respect to x:

$$f_x = \frac{d}{dx}\left((2x^2 + y^2)^{1/2}\right) = \frac{1}{2}(2x^2 + y^2)^{-1/2} \cdot \frac{d}{dx}(2x^2 + y^2)$$

$$= \frac{1}{2\sqrt{2x^2 + y^2}} \cdot (4x) = \frac{2x}{\sqrt{2x^2 + y^2}}$$

Partial derivative with respect to y:

$$f_y = \frac{d}{dy}\left((2x^2+y^2)^{1/2}\right) = \frac{1}{2}(2x^2+y^2)^{-1/2}\cdot\frac{d}{dy}(2x^2+y^2)$$

$$= \frac{1}{2\sqrt{2x^2+y^2}}\cdot(2y) = \frac{y}{\sqrt{2x^2+y^2}}$$

Final Answer:

$$\boxed{f_x = \frac{2x}{\sqrt{2x^2+y^2}}, \quad f_y = \frac{y}{\sqrt{2x^2+y^2}}}$$

2. Chain Rule

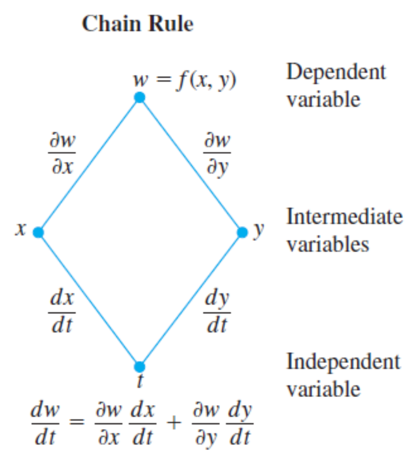
2.1 Chain Rule For Functions of One Independent Variable and Two Intermediate Variables

If $w = f(x, y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composite function $w = f(x(t), y(t))$ is a differentiable function of t, and

$$\frac{dw}{dt} = f_x(x(t), y(t))\cdot x'(t) + f_y(x(t), y(t))\cdot y'(t),$$

or using Leibniz notation,

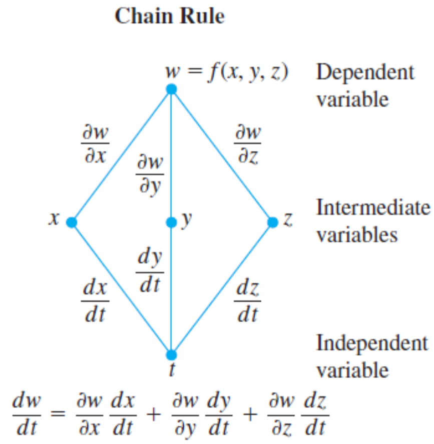
$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}.$$



2.2 Chain Rule for One Independent Variable and Three Intermediate Variables

If $w = f(x, y, z)$ is differentiable and $x = x(t)$, $y = y(t)$, $z = z(t)$ are differentiable functions of t, then w is a differentiable function of t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}.$$



Problem 1:

Use the Chain Rule to find the derivative of $w = xy$ with respect to t along the path $x = \cos t$, $y = \sin t$. What is the derivative's value at $t = \frac{\pi}{2}$?

Solution:

Step 1: Let $w = f(x, y) = xy$. We are given $x = \cos t$, $y = \sin t$.

Step 2: Use the Chain Rule:

$$\frac{dw}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

Step 3: Compute partial derivatives:

$$\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x$$

Step 4: Compute derivatives of $x(t)$ and $y(t)$:

$$\frac{dx}{dt} = \frac{d}{dt}(\cos t) = -\sin t, \quad \frac{dy}{dt} = \frac{d}{dt}(\sin t) = \cos t$$

Step 5: Substitute into the Chain Rule:

$$\frac{dw}{dt} = y \cdot (-\sin t) + x \cdot \cos t$$

Step 6: Plug in $x = \cos t$, $y = \sin t$:

$$\frac{dw}{dt} = \sin t(-\sin t) + \cos t \cdot \cos t = -\sin^2 t + \cos^2 t$$

Step 7: Evaluate at $t = \frac{\pi}{2}$:

$$\sin\left(\frac{\pi}{2}\right) = 1, \quad \cos\left(\frac{\pi}{2}\right) = 0$$

$$\frac{dw}{dt} = -1^2 + 0^2 = -1$$

Final Answer: $\boxed{\frac{dw}{dt} = -1 \text{ at } t = \frac{\pi}{2}}$

Problem 2: Find $\frac{du}{dt}$, if $u = x^3y^4$ where $x = t^3, \quad y = t^2$

Solution:

Given

$$\begin{aligned}
&u = x^3y^4 \quad (\because u = f(x,y)) \\
&x = t^3 \quad (\because x = \phi_1(t)) \\
&y = t^2 \quad (\because x = \phi_2(t))
\end{aligned}$$

we have

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} \tag{1}$$

$$\begin{aligned}
\text{Now}\frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}(x^3y^4) \\
&= y^4\frac{\partial}{\partial x}(x^3) \\
&= y^4(3x^2) \\
\frac{\partial u}{\partial x} &= 3x^2y^4
\end{aligned}$$

$$\begin{aligned}
\text{Now} \quad \frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}(x^3y^4) \\
&= x^3\frac{\partial}{\partial y}(y^4) \\
&= x^3(4y^3) \\
\frac{\partial u}{\partial y} &= 4x^3y^3
\end{aligned}$$

$$\frac{dx}{dt} = \frac{d}{dt}(t^3)$$
$$\frac{dx}{dt} = 3t^2$$
$$\frac{dy}{dt} = \frac{d}{dt}(t^2)$$
$$\frac{dy}{dt} = 2t$$

from the equation (1)

$$\begin{aligned}\frac{du}{dt} &= (3x^2y^4)(3t^2) + (4x^3y^3)(2t)\\ &= 3(t^3)^2(t^2)^4(3t^2) + 4(t^3)^3(t^2)^3(2t) \quad (\because x = t^3, y = t^2)\\ &= 9t^6t^8t^2 + 8t^9t^6t\\ &= 9t^{16} + +8t^{16}\end{aligned}$$
$$\therefore \frac{du}{dt} = 17t^{16}$$

Problem 3: Find $\frac{dw}{dt}$ if $w = xy + z$, where

$$x = \cos t, \quad y = \sin t, \quad z = t.$$

What is the derivative's value at $t = 0$?

Solution:

Step 1: Let $w = f(x, y, z) = xy + z$

Step 2: Use the Chain Rule for three variables:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

Step 3: Compute partial derivatives:

$$\frac{\partial w}{\partial x} = y, \quad \frac{\partial w}{\partial y} = x, \quad \frac{\partial w}{\partial z} = 1$$

Step 4: Compute derivatives of $x(t), y(t), z(t)$:

$$\frac{dx}{dt} = -\sin t, \quad \frac{dy}{dt} = \cos t, \quad \frac{dz}{dt} = 1$$

Step 5: Substitute all into the Chain Rule:

$$\frac{dw}{dt} = y(-\sin t) + x(\cos t) + 1$$

Step 6: Substitute $x = \cos t$, $y = \sin t$:

$$\frac{dw}{dt} = \sin t(-\sin t) + \cos t(\cos t) + 1 = -\sin^2 t + \cos^2 t + 1$$

Step 7: Evaluate at $t = 0$:

$$\sin(0) = 0, \quad \cos(0) = 1$$

$$\frac{dw}{dt} = -0 + 1^2 + 1 = 2$$

Final Answer: $\boxed{\frac{dw}{dt} = 2 \text{ at } t = 0}$

Problem 4: If $u = xy + yz + zx$ where $x = \frac{1}{t}, \quad y = e^t, \quad z = e^{-t}$ then Find $\frac{du}{dt}$.

Solution:

Given

$$\begin{aligned}
u &= xy + yz + zx \quad (\because u = f(x, y, z)) \\
x &= \frac{1}{t} \quad (\because x = \phi_1(t)) \\
y &= e^t \quad (\because y = \phi_2(t)) \\
z &= e^{-t} \quad (\because z = \phi_3(t))
\end{aligned}$$

we have

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt} \tag{2}$$

$$\begin{aligned}
\text{Now} \frac{\partial u}{\partial x} &= \frac{\partial}{\partial x}(xy + yz + zx) \\
&= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial x}(zx) \\
&= y + 0 + z \\
\frac{\partial u}{\partial x} &= y + z
\end{aligned}$$

$$\begin{aligned}
\frac{\partial u}{\partial y} &= \frac{\partial}{\partial y}(xy + yz + zx) \\
&= \frac{\partial}{\partial y}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial y}(zx) \\
&= x + z + 0 \\
\frac{\partial u}{\partial x} &= x + z
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial u}{\partial z} &= \frac{\partial}{\partial z}(xy + yz + zx) \\
&= \frac{\partial}{\partial z}(xy) + \frac{\partial}{\partial z}(yz) + \frac{\partial}{\partial z}(zx) \\
&= 0 + y + x \\
\frac{\partial u}{\partial x} &= y + x
\end{aligned}
$$

$$
\begin{aligned}
\frac{dx}{dt} &= \frac{d}{dt}\left(\frac{1}{t}\right) \\
&= -\frac{1}{t^2} \\
\frac{dy}{dt} &= \frac{d}{dt}(e^t) \\
&= e^t \\
\frac{dy}{dt} &= \frac{d}{dt}(e^{-t}) \\
&= -e^{-t}
\end{aligned}
$$

from the quation (2)

$$
\begin{aligned}
\frac{du}{dt} &= (y + z)\left(-\frac{1}{t^2}\right) + (x + z)e^t + (x + y)(-e^{-t}) \\
&= (e^t + e^{-t})\left(-\frac{1}{t^2}\right) + (\frac{1}{t} + e^{-t})e^t + (\frac{1}{t} + e^t)(-e^{-t}) \\
&= -\frac{(e^t + e^{-t})}{t^2} + \frac{(1 + te^{-t})e^t}{t} + \frac{(1 + te^t)(e^{-t}}{t} \\
&= \frac{-e^t - e^{-t} + (e^t + te^t e^{-t})t - (e^{-t} + te^t e^{-t})t}{t^2} \\
&= \frac{-e^t - e^{-t} + te^t + \cancel{t^2} - te^{-t} - \cancel{t^2}}{t^2} \\
&= \frac{-e^t - e^{-t} + te^t - te^{-t}}{t^2} \\
&= \frac{-e^t - e^{-t} + t(e^t + e^{-t})}{t^2} \\
&= \frac{-e^t - e^{-t}}{t^2} + \frac{\cancel{t}(e^t + e^{-t})}{t^{\cancel{2}}} \\
\therefore \frac{du}{dt} &= \frac{-e^t - e^{-t}}{t^2} + \frac{e^t + e^{-t}}{t}
\end{aligned}
$$

2.3 Multivariable Chain Rule: Several Intermediate and Independent Variables

Case 1: Two Independent Variables and One Intermediate Variable

$$w = f(x), \; x = g(r, s)$$
$$\frac{\partial w}{\partial r} = \frac{dw}{dx} \cdot \frac{\partial x}{\partial r}, \quad \frac{\partial w}{\partial s} = \frac{dw}{dx} \cdot \frac{\partial x}{\partial s}$$

Case 2: Two Independent Variables and Two Intermediate Variables

$$w = f(x, y), \; x = g(r, s), \; y = h(r, s)$$
$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r}, \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s}$$

Case 3: Two Independent Variables and Three Intermediate Variables

$$w = f(x, y, z), \; x = g(r, s), \; y = h(r, s), \; z = k(r, s)$$
$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}$$

Problem 1: Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if

$$w = x^2 + y^2, \quad x = r - s, \quad y = r + s.$$

Solution:

Step 1: Use the Chain Rule:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r}, \quad \frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s}$$

Step 2: Compute:

$$\frac{\partial w}{\partial x} = 2x, \quad \frac{\partial w}{\partial y} = 2y$$

$$\frac{\partial x}{\partial r} = 1, \quad \frac{\partial x}{\partial s} = -1, \quad \frac{\partial y}{\partial r} = 1, \quad \frac{\partial y}{\partial s} = 1$$

Step 3: Substitute:

$$\frac{\partial w}{\partial r} = 2x \cdot 1 + 2y \cdot 1 = 2x + 2y = 2(r-s) + 2(r+s) = 4r$$

$$\frac{\partial w}{\partial s} = 2x \cdot (-1) + 2y \cdot 1 = -2x + 2y = -2(r-s) + 2(r+s) = 4s$$

Final Answer: $\boxed{\frac{\partial w}{\partial r} = 4r, \quad \frac{\partial w}{\partial s} = 4s}$

Problem 2: Express $\frac{\partial w}{\partial r}$ and $\frac{\partial w}{\partial s}$ in terms of r and s if

$$w = x + 2y + z^2, \quad x = \frac{r}{s}, \quad y = r^2 + \ln s, \quad z = 2r.$$

Solution:

Step 1: Use the Chain Rule:

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial s}$$

Step 2: Compute:

$$\frac{\partial w}{\partial x} = 1, \quad \frac{\partial w}{\partial y} = 2, \quad \frac{\partial w}{\partial z} = 2z$$

$$\frac{\partial x}{\partial r} = \frac{1}{s}, \quad \frac{\partial x}{\partial s} = -\frac{r}{s^2}, \quad \frac{\partial y}{\partial r} = 2r, \quad \frac{\partial y}{\partial s} = \frac{1}{s}, \quad \frac{\partial z}{\partial r} = 2, \quad \frac{\partial z}{\partial s} = 0$$

Step 3: Substitute:

$$\frac{\partial w}{\partial r} = 1 \cdot \frac{1}{s} + 2 \cdot 2r + 2z \cdot 2 = \frac{1}{s} + 4r + 4(2r) = \frac{1}{s} + 12r$$

$$\frac{\partial w}{\partial s} = 1 \cdot \left(-\frac{r}{s^2}\right) + 2 \cdot \frac{1}{s} + 0 = -\frac{r}{s^2} + \frac{2}{s}$$

Final Answer: $\boxed{\frac{\partial w}{\partial r} = \frac{1}{s} + 12r, \quad \frac{\partial w}{\partial s} = -\frac{r}{s^2} + \frac{2}{s}}$

3. Differentiation for an Implicit functions

Let $f(x, y) = 0$ be a given implicit function of x and y then

$$\frac{dy}{dx} = -\frac{f_x}{f_y} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$$

$$\frac{d^2y}{dx^2} = -\left[\frac{f_{xx}(f_y)^2 - 2f_{xx}f_xf_y + f_{yy}(f_x)^2}{(f_y)^3}\right].$$

Note:

(1) $f_x = \frac{\partial f}{\partial x}, f_y = \frac{\partial f}{\partial y}, f_{xx} = \frac{\partial^2 f}{\partial x^2}, f_{yy} = \frac{\partial^2 f}{\partial y^2}, f_{xy} = \frac{\partial^2 f}{\partial x \partial y}, f_{yx} = \frac{\partial^2 f}{\partial y \partial x}.$

(2) If $f(x, z) = 0$ then $\frac{dz}{dx} = -\frac{f_x}{f_z}$.

(3) If $f(y, z) = 0$ then $\frac{dz}{dy} = -\frac{f_y}{f_z}$.

Problem 1: Find $\frac{dy}{dx}$ if

$$y^2 - x^2 - \sin(xy) = 0$$

Solution:

Let $f(x, y) = y^2 - x^2 - \sin(xy)$. Use implicit differentiation:

$$\frac{dy}{dx} = -\frac{f_x}{f_y}$$

Compute partial derivatives:

$$f_x = -2x - \cos(xy) \cdot y$$

$$f_y = 2y - \cos(xy) \cdot x$$

Thus,

$$\boxed{\frac{dy}{dx} = \frac{2x + y\cos(xy)}{2y - x\cos(xy)}}$$

Problem 2: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at $(0, 0, 0)$, given:

$$x^3 + z^2 + ye^{xz} + z\cos y = 0$$

Solution:

Let $F(x, y, z) = x^3 + z^2 + ye^{xz} + z\cos y$. Differentiate implicitly using:

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$$

Compute partial derivatives:

Step 1:

$$F_x = 3x^2 + y \cdot e^{xz} \cdot z \quad \text{(product rule on } ye^{xz}\text{)}$$

Step 2:

$$F_y = e^{xz} + z(-\sin y)$$

Step 3:

$$F_z = 2z + ye^{xz} \cdot x + \cos y$$

Now evaluate at $(x, y, z) = (0, 0, 0)$:

$$F_x = 0 + 0 = 0, \quad F_y = 1 + 0 = 1, \quad F_z = 0 + 0 + 1 = 1$$

So,

$$\boxed{\frac{\partial z}{\partial x} = -\frac{0}{1} = 0, \quad \frac{\partial z}{\partial y} = -\frac{1}{1} = -1}$$

Problem 3: Find $\frac{dy}{dx}$ in the following cases.

a) $x\sin(x - y) - (x + y) = 0$.

b) $x^y = y^x$.

c) $(\cos x)^y - (\sin y)^x = 0$.

d) $(\tan x)^y + y^{\cot x} = a$.

e) $y^{x^y} = \sin x$.

Solution:

The implicit differentiation is given by

$$\boxed{\frac{dy}{dx} = -\frac{f_x}{f_y}}. \tag{3}$$

a) Given $x\sin(x-y)-(x+y)=0 \quad (\because f(x,y)=0)$

differentiating with respect to $'x'$ partially

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x\sin(x-y)-(x+y)) \\
&= x\frac{\partial}{\partial x}\sin(x-y)+\sin(x-y)\frac{\partial}{\partial x}x-\frac{\partial}{\partial x}(x+y) \\
&= x\cos(x-y)\frac{\partial}{\partial x}(x-y)+\sin(x-y)-1 \\
\frac{\partial f}{\partial x} &= x\cos(x-y)+\sin(x-y)-1
\end{aligned}$$

again $f(x,y)=x\sin(x-y)-(x+y)$

differentiating with respect to $'y'$ partially

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x\sin(x-y)-(x+y)) \\
&= x\frac{\partial}{\partial y}\sin(x-y)-\frac{\partial}{\partial y}(x+y) \\
&= x\cos(x-y)\frac{\partial}{\partial y}(x-y)-1 \\
\frac{\partial f}{\partial y} &= -x\cos(x-y)-1
\end{aligned}$$

Now from equation (3)

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \\
&= -\frac{x\cos(x-y)+\sin(x-y)-1}{-x\cos(x-y)-1} \\
&= \frac{x\cos(x-y)+\sin(x-y)-1}{x\cos(x-y)+1} \\
(\because x\sin(x-y)=x+y &\implies \sin(x-y)=\frac{x+y}{x}) \\
&= \frac{x\cos(x-y)+\frac{x+y}{x}-1}{x\cos(x-y)+1} \\
&= \frac{x^2\cos(x-y)+x+y-x}{x(x\cos(x-y)+1)} \\
\therefore \frac{dy}{dx} &= \frac{x^2\cos(x-y)+y}{x^2\cos(x-y)+x}.
\end{aligned}$$

b) Given $x^y=y^x \implies x^y-y^x=0 \quad (\because f(x,y)=0)$

differentiating with respect to $'x'$ partially

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x^y - y^x)\\ &= y\ x^{y-1} - y^x \log y\\ &= y\ x^y x^{-1} - y^x \log y \quad (\because x^y = y^x)\\ &= y\ y^x x^{-1} - y^x \log y\\ \frac{\partial f}{\partial x} &= y^x \left(\frac{y}{x} - \log y\right)\end{aligned}$$

again $f(x, y) = x^y - y^x$

differentiating with respect to $'y'$ partially

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x^y - y^x)\\ &= x^y \log x - x\ y^{x-1}\\ &= y^x \log x - x\ y^x\ y^{-1} \quad (\because x^y = y^x)\\ \frac{\partial f}{\partial y} &= y^x \left(\log x - \frac{x}{y}\right)\end{aligned}$$

Now from equation (3)

$$\begin{aligned}\frac{dy}{dx} &= -\frac{y^x \left(\frac{y}{x} - \log y\right)}{y^x \left(\log x - \frac{x}{y}\right)}\\ &= -\frac{\left(\frac{y - x\log y}{x}\right)}{\left(\frac{y\log x - x}{y}\right)}\\ \therefore \frac{dy}{dx} &= \frac{y(y - x\log y)}{x(x - y\log x)}\end{aligned}$$

Altern:

Given $x^y = y^x$

taking 'log' on both sides

$$\begin{aligned}\log x^y &= \log y^x\\ y\log x &= x\log y\\ y\log x - x\log y &= 0 \quad (\because f(x, y) = 0)\end{aligned}$$

differentiating with respect to $'x'$ partially

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(y\log x - x\log y)\\ &= y\frac{\partial}{\partial x}\log x - \log y\frac{\partial}{\partial x}x\\ &= y\left(\frac{1}{x}\right) - \log y\\ &= \frac{y}{x} - \log y\\ \frac{\partial f}{\partial x} &= \frac{y - x\log y}{x}\end{aligned}$$

again $f(x,y) = y\log x - x\log y$

differentiating with respect to $'y'$ partially

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{\partial}{\partial y}(y\log x - x\log y)\\ &= \log x\frac{\partial}{\partial y}y - x\frac{\partial}{\partial y}\log y\\ &= \log x - x\left(\frac{1}{y}\right)\\ &= \log x - \frac{x}{y}\\ \frac{\partial f}{\partial x} &= \frac{y\log x - x}{y}\end{aligned}$$

Now from equation (3)

$$\begin{aligned}\frac{dy}{dx} &= -\frac{\frac{y - x\log y}{x}}{\frac{y\log x - x}{y}}\\ &= -\frac{\left(\frac{y-x\log y}{x}\right)}{\left(\frac{y\log x - x}{y}\right)}\\ \therefore \frac{dy}{dx} &= \frac{y(y - x\log y)}{x(x - y\log x)}\end{aligned}$$

c) Given $(\cos x)^y - (\sin y)^x = 0$

$\implies (\cos x)^y = (\sin y)^x$

taking $'\log'$ on the both sides

$$\begin{aligned}\log(\cos x)^y &= \log(\sin y)^x\\ y\log(\cos x) &= x\log(\sin y)\\ y\log(\cos x) - x\log(\sin y) &= 0 \quad (\because f(x,y) = 0)\end{aligned}$$

differentiating with respect to $'x'$ partially

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(y\log(\cos x) - x\log(\sin y)) \\
&= y\frac{\partial}{\partial x}\log(\cos x) - \log(\sin y)\frac{\partial}{\partial x}x \\
&= y\left(\frac{-\sin x}{\cos x}\right) - \log(\sin y) \\
\frac{\partial f}{\partial x} &= \frac{-y\sin x - \cos x\log(\sin y)}{\cos x}
\end{aligned}$$

again $f(x,y) = y\log(\cos x) - x\log(\sin y) = 0$

differentiating with respect to $'y'$ partially

$$\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(y\log(\cos x) - x\log(\sin y)) \\
&= \log(\cos x)\frac{\partial}{\partial y}y - x\frac{\partial}{\partial y}\log(\sin y) \\
&= \log(\cos x) - x\left(\frac{\cos y}{\sin y}\right) \\
\frac{\partial f}{\partial y} &= \frac{\sin y\log(\cos x) - x\cos y}{\sin y}
\end{aligned}$$

Now from equation (3)

$$\begin{aligned}
\frac{dy}{dx} &= -\frac{\frac{-y\sin x - \cos x\log(\sin y)}{\cos x}}{\frac{\sin y\log(\cos x) - x\cos y}{\sin y}} \\
\therefore \frac{dy}{dx} &= \frac{\sin y(y\sin x + \cos x\log(\sin y))}{\cos x(\sin y\log(\cos x) - x\cos y)}
\end{aligned}$$

d) Given $(\tan x)^y + y^{\cot x} = a \quad (\because f(x,y) = 0)$

differentiating with respect to $'x'$ partially

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(\tan x)^y + y^{\cot x} - a) \\
&= y(\tan x)^{y-1}\frac{\partial}{\partial x}(\tan x) + y^{\cot x}\log y\frac{\partial}{\partial x}(\cot x) - 0 \\
&= y(\tan x)^{y-1}(\sec^2 x) + y^{\cot x}\log y(-\csc^2 x) \\
\frac{\partial f}{\partial x} &= y(\tan x)^{y-1}(\sec^2 x) - y^{\cot x}\log y(\csc^2 x)
\end{aligned}$$

again $f(x, y) = (\tan x)^y + y^{\cot x} - a$

differentiating with respect to $'y'$ partially

$$
\begin{aligned}
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(\tan x)^y + y^{\cot x} - a) \\
&= (\tan x)^y \log(\tan x) + \cot x \; y^{\cot x - 1}\frac{\partial}{\partial y}(y) - 0 \\
\frac{\partial f}{\partial y} &= (\tan x)^y \log(\tan x) + \cot x y^{\cot x - 1}
\end{aligned}
$$

Now from equation (3)

$$
\frac{dy}{dx} = -\frac{y(\tan x)^{y-1}(\sec^2 x) - y^{\cot x}\log y(\csc^2 x)}{(\tan x)^y \log(\tan x) + \cot x \; y^{\cot x - 1}}
$$

e) We start from

$$
y^{x^y} = \sin x.
$$

Take the natural logarithm of both sides (valid for $y > 0$ and $\sin x > 0$):

$$
x^y \ln y = \ln(\sin x).
$$

Step 1: Differentiate both sides

We note that

$$
x^y = e^{y \ln x} \quad \Rightarrow \quad \frac{d}{dx}x^y = x^y \left(y' \ln x + \frac{y}{x}\right).
$$

Differentiating the left-hand side of $x^y \ln y$ using the product rule:

$$
\frac{d}{dx}\left(x^y \ln y\right) = \left[x^y \left(y' \ln x + \frac{y}{x}\right)\right] \ln y + x^y \cdot \frac{y'}{y}.
$$

Differentiating the right-hand side:

$$
\frac{d}{dx}\left(\ln(\sin x)\right) = \cot x.
$$

Thus we have

$$
x^y \left(y' \ln x + \frac{y}{x}\right) \ln y + x^y \cdot \frac{y'}{y} = \cot x.
$$

Step 2: Collect y' terms

Factor out y' from the appropriate terms:

$$x^y\left[y'\left(\ln y\,\ln x + \frac{1}{y}\right) + \frac{y}{x}\ln y\right] = \cot x.$$

$$y'\left(\ln y\,\ln x + \frac{1}{y}\right)x^y + \frac{y\ln y}{x}x^y = \cot x.$$

Step 3: Solve for y'

$$y'\left(\ln y\,\ln x + \frac{1}{y}\right)x^y = \cot x - \frac{y\ln y}{x}x^y.$$

$$y' = \frac{\frac{\cot x}{x^y} - \frac{y\ln y}{x}}{\ln y\,\ln x + \frac{1}{y}}.$$

Step 4: Simplify using the original equation

From $x^y\ln y = \ln(\sin x)$ we have

$$\frac{1}{x^y} = \frac{\ln y}{\ln(\sin x)}.$$

Substituting into the expression for y' gives

$$\boxed{\frac{dy}{dx} = \frac{\ln y\left(\frac{\cot x}{\ln(\sin x)} - \frac{y}{x}\right)}{\ln y\,\ln x + \frac{1}{y}}}$$

(valid where $\ln y$, $\ln(\sin x)$, and $\ln x$ are defined).

Exercises Problems on Multivariable Chain Rule and Implicit Differentiation

Exercise 1:

Question:

Let $w = x^2 + y^2$, where $x = \cos t$, $y = \sin t$, and $t = \pi$. Find $\dfrac{dw}{dt}$.

Hint: Use the chain rule for functions of one variable: $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial w}{\partial y} \cdot \dfrac{dy}{dt}$.

Exercise 2:

Question: Let $w = x^2 + y^2$, where $x = \cos t + \sin t$, $y = \cos t - \sin t$, $t = 0$. Find $\dfrac{dw}{dt}$.

Hint: Differentiate $x(t)$ and $y(t)$ separately, then apply the chain rule.

Exercise 3:

Question: $w = \dfrac{x}{z} + \dfrac{y}{z}$, $x = \cos^2 t$, $y = \sin^2 t$, $z = 1/t$, $t = 3$. Find $\dfrac{dw}{dt}$.

Hint: Use quotient rule and chain rule together.

Exercise 4:

Question: $w = \ln(x^2 + y^2 + z^2)$, where $x = \cos t$, $y = \sin t$, $z = 4\sqrt{t}$, $t = 3$. Find $\dfrac{dw}{dt}$.

Hint: Let $u = x^2 + y^2 + z^2$, then $\dfrac{dw}{dt} = \dfrac{1}{u} \cdot \dfrac{du}{dt}$.

Exercise 5:

Question: $w = 2ye^x - \ln z$, $x = \ln(t^2 + 1)$, $y = \tan^{-1} t$, $z = e^t$, $t = 1$. Find $\dfrac{dw}{dt}$.

Hint: Use the chain rule: $\dfrac{dw}{dt} = \dfrac{\partial w}{\partial x}x' + \dfrac{\partial w}{\partial y}y' + \dfrac{\partial w}{\partial z}z'$.

Exercise 6:

Question: $w = z - \sin(xy)$, $x = t$, $y = \ln t$, $z = e^{t-1}$, $t = 1$. Find $\dfrac{dw}{dt}$.

Exercise 7:

Question: $z = 4e^x \ln y$, $x = \ln(u \cos v)$, $y = u \sin v$ at $(u, v) = (2, \pi/4)$. Find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$.

Hint: Use the multivariable chain rule: $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u}$.

Exercise 9:

Question: $w = xy + yz + xz$, $x = u + v$, $y = u - v$, $z = uv$ at $(u, v) = (\frac{1}{2}, 1)$. Find $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$.

Hint: Compute all partial derivatives via chain rule with three paths.

Exercise 10:

Question: $w = \ln(x^2 + y^2 + z^2)$, $x = ue^v \sin u$, $y = ue^v \cos u$, $z = ue^v$ at $(u, v) = (-2, 0)$. Find $\dfrac{\partial w}{\partial u}$ and $\dfrac{\partial w}{\partial v}$.

Hint: Let $u = x^2 + y^2 + z^2$ and use $\dfrac{\partial w}{\partial u} = \dfrac{1}{u} \cdot \dfrac{du}{dv}$.

Exercise 11:

Question: $x^3 - 2y^2 + xy = 0$ at $(1, 1)$. Find $\dfrac{dy}{dx}$.

Hint: Use implicit differentiation: $3x^2 - 4y\dfrac{dy}{dx} + y + x\dfrac{dy}{dx} = 0$.

Exercise 12:

Question: $xy + y^2 - 3x - 3 = 0$ at $(-1, 1)$. Find $\dfrac{dy}{dx}$.

$\square$

Exercise 13:

Question: $x^2 + xy + y^2 - 7 = 0$ at $(1, 2)$. Find $\dfrac{dy}{dx}$.

$\square$

Exercise 14:

Question: $xe^y + \sin(xy) + y - \ln 2 = 0$ at $(0, \ln 2)$. Find $\dfrac{dy}{dx}$.

$\square$

Exercise 15:

Question: $\dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} = 1$ at $(2, 3, 6)$. Find $\dfrac{dz}{dx}$ and $\dfrac{dz}{dy}$.

$\square$

Exercise 16:

Question: $w = x^2 + \dfrac{y}{x}$, $x = u - 2v + 1$, $y = 2u + v - 2$, at $u = 0, v = 0$. Find $\dfrac{\partial w}{\partial v}$.

$\square$

Exercise 17:

Question: $w = f(s^3 + t^2)$, $f'(x) = e^x$. Find $\dfrac{\partial w}{\partial t}$ and $\dfrac{\partial w}{\partial s}$.

$\square$

Exercise 18:

Question: $w = f(ts^2, \frac{s}{t})$, where $\dfrac{\partial f}{\partial x} = xy$, $\dfrac{\partial f}{\partial y} = \dfrac{x^2}{2}$. Find $\dfrac{\partial w}{\partial t}$ and $\dfrac{\partial w}{\partial s}$.

$\square$

***

4. Extreme Values and Saddle Points

To find the **local extreme values** of a function of a single variable, we look for points where the graph has a **horizontal tangent line**. At such points, we then determine whether the function has:

- a **local maximum**,
- a **local minimum**, or
- a **point of inflection**.

For a function $f(x, y)$ of two variables, we look for points where the surface $z = f(x, y)$ has a **horizontal tangent plane**. These are critical points where the partial derivatives satisfy:

$$f_x(x, y) = 0 \quad \text{and} \quad f_y(x, y) = 0.$$

At such points, we classify the critical point as one of the following:

- a **local maximum**,
- a **local minimum**, or
- a **saddle point**.

We begin our study by formally defining **local maxima** and **local minima** (See Figure 1).

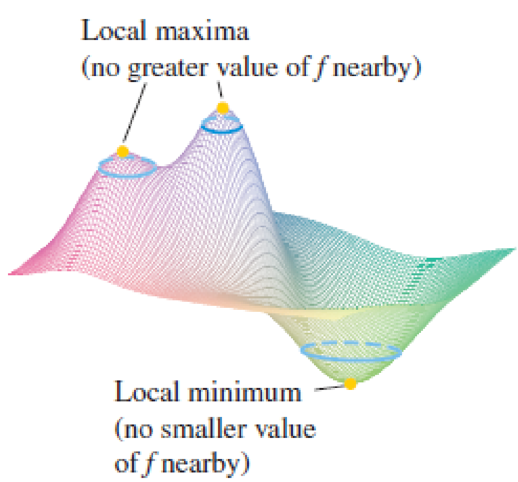


Figure 1: **Local extrema on a surface:** A local maximum occurs at a mountain peak (no greater value of f nearby), and a local minimum occurs at a valley low point (no smaller value of f nearby).

4.1 Second Derivative Test for Local Extreme Values

Let $f(x,y)$ be a given function. To find the maxima and minima values of $f(x,y)$ we follow the following rules:

Rule 1: Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and solve the equations

$$\frac{\partial f}{\partial x} = 0; \quad \frac{\partial f}{\partial y} = 0$$

to obtain critical points $(a_1, b_1), (a_2, b_2), \dots$.

Rule 2: Find the second-order partial derivatives:

$$r = \frac{\partial^2 f}{\partial x^2}, \quad s = \frac{\partial^2 f}{\partial x \partial y}, \quad t = \frac{\partial^2 f}{\partial y^2}.$$

Rule 3: Compute the discriminant:

$$D = rt - s^2.$$

Rule 4: If $D > 0$ and $r < 0$, then f has a **local maximum**.

Rule 5: If $D > 0$ and $r > 0$, then f has a **local minimum**.

Rule 6: If $D < 0$, then the critical point is a **saddle point**.

Rule 7: If $D = 0$, then **further investigation is required**.

The Discriminant (Hessian):

The expression

$$D = rt - s^2 = f_{xx} f_{yy} - (f_{xy})^2$$

is called the **discriminant** or the **Hessian** of f.

It is sometimes easier to remember it in **determinant form**: This is equivalent to the determinant:

$$D = \begin{vmatrix} r & s \\ s & t \end{vmatrix} = rt - s^2$$

or

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2$$

This value is used to classify critical points (a, b) of a function $f(x, y)$.

4.2 Saddle Points

Definition: A differentiable function $f(x, y)$ has a **saddle point** at a critical point (a, b) if, in every open disk centered at (a, b), there are domain points (x, y) such that:

$$f(x, y) > f(a, b) \quad \text{and} \quad f(x, y) < f(a, b).$$

The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a **saddle point** of the surface (Figure 2).

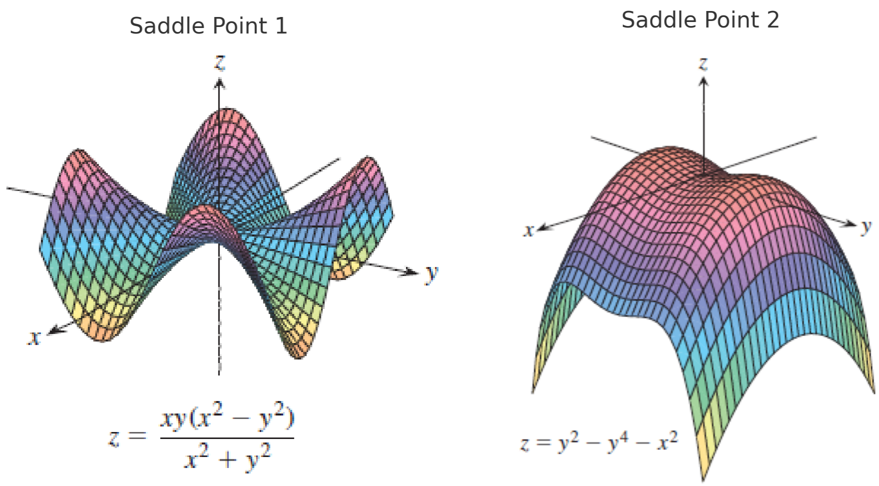


Figure 2: Saddle points at the origin..

Problem 1: Find the maximum and minimum value of $3x^2 - y^2 + x^3$.

Solution: Let $f(x, y) = 3x^2 - y^2 + x^3$

Step 1: Compute the first-order partial derivatives:

$$f_x = \frac{\partial f}{\partial x} = 6x + 3x^2, \quad f_y = \frac{\partial f}{\partial y} = -2y$$

Step 2: Set the partial derivatives equal to zero:

$$6x + 3x^2 = 0 \Rightarrow 3x(2 + x) = 0 \Rightarrow x = 0, \ -2$$

$$-2y = 0 \Rightarrow y = 0$$

So, the critical points are $(0, 0)$ and $(-2, 0)$.

Step 3: Compute the second-order partial derivatives:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = 6 + 6x, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2} = -2, \quad f_{xy} = 0$$

Step 4: Evaluate the discriminant $D = f_{xx}f_{yy} - (f_{xy})^2$
At $(0, 0)$:

$$f_{xx} = 6, \quad f_{yy} = -2, \quad D = (6)(-2) - 0 = -12 < 0 \Rightarrow \text{saddle point}$$

At $(-2, 0)$:

$$f_{xx} = 6+6(-2) = -6, \quad f_{yy} = -2, \quad D = (-6)(-2)-0 = 12 > 0 \Rightarrow \text{since } f_{xx} < 0, \text{ local maximum}$$

Step 5: Compute the value of $f(x, y)$ at critical points:

$$f(0, 0) = 0, \quad f(-2, 0) = 3(-2)^2 + (-2)^3 = 12 - 8 = 4$$

Conclusion: The function has a local maximum value $\boxed{4}$ at $(-2, 0)$ and a saddle point at $(0, 0)$.

Problem 2: Find the maximum and minimum value of $f(x, y) = x^3y^2(1 - x - y)$.

Solution:

Let $f(x, y) = x^3y^2(1 - x - y)$

Step 1: Identify the domain. Since the function involves x^3y^2, we consider the domain $x \geq 0$, $y \geq 0$, and $x+y \leq 1$, i.e., the triangle with vertices at $(0, 0), (1, 0), (0, 1)$.

Step 2: Compute the first-order partial derivatives.

$$f_x = \frac{\partial f}{\partial x} = 3x^2y^2(1-x-y)-x^3y^2 = x^2y^2(3(1-x-y)-x) = x^2y^2(3-3x-3y-x) = x^2y^2(3-4x-3y)$$

$$f_y = \frac{\partial f}{\partial y} = x^3{\cdot}2y(1-x-y)-x^3y^2 = x^3y(2(1-x-y)-y) = x^3y(2-2x-2y-y) = x^3y(2-2x-3y)$$

Step 3: Set the first-order partial derivatives equal to zero:

$$f_x = 0 \Rightarrow x^2y^2(3 - 4x - 3y) = 0 \Rightarrow x = 0 \text{ or } y = 0 \text{ or } 3 - 4x - 3y = 0 \tag{1}$$

$$f_y = 0 \Rightarrow x^3 y(2 - 2x - 3y) = 0 \Rightarrow x = 0 \text{ or } y = 0 \text{ or } 2 - 2x - 3y = 0 \tag{2}$$

From equations (1) and (2), solve:

$$3 - 4x - 3y = 0 \quad \text{and} \quad 2 - 2x - 3y = 0$$

Solving these two: Subtract:

$$(3 - 4x - 3y) - (2 - 2x - 3y) = 1 - 2x \Rightarrow x = \frac{1}{2}$$

Substitute into one:

$$2 - 2 \cdot \frac{1}{2} - 3y = 0 \Rightarrow 2 - 1 - 3y = 0 \Rightarrow y = \frac{1}{3}$$

So interior critical point is $\left(\frac{1}{2}, \frac{1}{3}\right)$. Check it is in domain:

$$x + y = \frac{5}{6} < 1 \quad \text{OK}$$

Compute:

$$f\left(\frac{1}{2}, \frac{1}{3}\right) = \left(\frac{1}{2}\right)^3 \left(\frac{1}{3}\right)^2 \left(1 - \frac{1}{2} - \frac{1}{3}\right) = \frac{1}{8} \cdot \frac{1}{9} \cdot \left(\frac{1}{6}\right) = \frac{1}{432}$$

Step 4: Check boundary points.
On $x = 0$: $f = 0$, on $y = 0$: $f = 0$
On boundary $x + y = 1 \Rightarrow y = 1 - x$:

$$f(x, 1 - x) = x^3(1 - x)^2(1 - x - (1 - x)) = x^3(1 - x)^2 \cdot 0 = 0$$

So boundary max is 0. Interior value is $\frac{1}{432}$
Conclusion:

$$\boxed{\text{Maximum value: } \frac{1}{432} \text{ at } \left(\frac{1}{2}, \frac{1}{3}\right), \quad \text{Minimum: 0 on boundary.}}$$

Problem 3: Find the maximum and minimum value of

$$f(x, y) = xy + \frac{a^3}{x} + \frac{a^3}{y}.$$

Solution:

Step 1: Compute the first-order partial derivatives.

$$f_x = \frac{\partial f}{\partial x} = y - \frac{a^3}{x^2}, \quad f_y = \frac{\partial f}{\partial y} = x - \frac{a^3}{y^2}$$

Step 2: Set the partial derivatives equal to zero to find critical points.

$$f_x = 0 \Rightarrow y = \frac{a^3}{x^2} \quad (1)$$

$$f_y = 0 \Rightarrow x = \frac{a^3}{y^2} \quad (2)$$

Substitute equation (1) into equation (2):

$$x = \frac{a^3}{\left(\frac{a^3}{x^2}\right)^2} = \frac{a^3}{\frac{a^6}{x^4}} = \frac{a^3x^4}{a^6} = \frac{x^4}{a^3} \Rightarrow x = \frac{x^4}{a^3} \Rightarrow x^4 - a^3x = 0 \Rightarrow x(x^3 - a^3) = 0$$

Step 3: Solve the above equation.

$$x = 0 \quad \text{or} \quad x^3 = a^3 \Rightarrow x = a$$

Since $x = 0$ is not in the domain of $f(x, y)$ (division by zero), we discard it.

So, $x = a$. Using equation (1),

$$y = \frac{a^3}{x^2} = \frac{a^3}{a^2} = a$$

Step 4: Find the function value at the critical point (a, a).

$$f(a, a) = a \cdot a + \frac{a^3}{a} + \frac{a^3}{a} = a^2 + a^2 + a^2 = 3a^2$$

Step 5: Second derivative test.

Compute second-order partials:

$$f_{xx} = \frac{2a^3}{x^3}, \quad f_{yy} = \frac{2a^3}{y^3}, \quad f_{xy} = 1$$

At (a, a):

$$f_{xx} = \frac{2a^3}{a^3} = 2, \quad f_{yy} = 2, \quad f_{xy} = 1$$

Calculate the discriminant:

$$D = f_{xx}f_{yy} - (f_{xy})^2 = (2)(2) - (1)^2 = 4 - 1 = 3 > 0$$

Since $D > 0$ and $f_{xx} > 0$, the function has a **local minimum** at (a, a).

Conclusion: The function has a minimum value $\boxed{3a^2}$ at $(x, y) = (a, a)$. There is no maximum since the function is unbounded as $x \to 0^+$ or $y \to 0^+$.

Problem 4: Find the local extreme values of the function

$$f(x, y) = xy - x^2 - y^2 - 2x - 2y + 4$$

Solution:

Step 1: Compute the first-order partial derivatives:

$$f_x = \frac{\partial f}{\partial x} = y - 2x - 2, \quad f_y = \frac{\partial f}{\partial y} = x - 2y - 2$$

Step 2: Set the partial derivatives equal to zero to find critical points:

$$f_x = 0 \Rightarrow y - 2x - 2 = 0 \tag{1}$$

$$f_y = 0 \Rightarrow x - 2y - 2 = 0 \tag{2}$$

From (1): $y = 2x + 2$

Substitute into (2):

$$x - 2(2x + 2) - 2 = 0 \Rightarrow x - 4x - 4 - 2 = 0 \Rightarrow -3x = 6 \Rightarrow x = -2$$

Then from (1): $y = 2(-2) + 2 = -2$

So the only critical point is $(-2, -2)$

Step 3: Compute the second-order partial derivatives:

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} = -2, \quad f_{yy} = \frac{\partial^2 f}{\partial y^2} = -2, \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y} = 1$$

Now compute the discriminant:

$$D = f_{xx}f_{yy} - (f_{xy})^2 = (-2)(-2) - 1^2 = 4 - 1 = 3 > 0$$

Since $D > 0$ and $f_{xx} < 0$, the function has a **local maximum** at $(-2, -2)$.

Step 4: Evaluate $f(-2, -2)$:

$$f(-2,-2) = (-2)(-2) - (-2)^2 - (-2)^2 - 2(-2) - 2(-2) + 4 = 4 - 4 - 4 + 4 + 4 + 4 = 8$$

$$\boxed{\text{Local maximum: } f(-2,-2) = 8}$$

There are no local minima or saddle points.

4.3 Absolute Maxima and Minima on a Closed Bounded Region

We organize the search for the absolute extrema of a continuous function $f(x, y)$ on a closed and bounded region R into three steps:

Step 1: Find and evaluate f at critical points in the interior of R.

Step 2: Evaluate f on the boundary of R.

Step 3: Compare all values found in Steps 1 and 2 to determine the absolute maximum and minimum.

Problem 1: Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$, and $y = 9 - x$.

Solution:

Step 1: Find critical points in the interior of the region.

Compute partial derivatives:

$$f_x = 2 - 2x, \quad f_y = 4 - 2y$$

Set them to zero:

$$2 - 2x = 0 \Rightarrow x = 1, \quad 4 - 2y = 0 \Rightarrow y = 2$$

So, the interior critical point is $(1, 2)$, and it lies inside the triangle. Evaluate:

$$f(1, 2) = 2 + 2(1) + 4(2) - 1^2 - 2^2 = 2 + 2 + 8 - 1 - 4 = \boxed{7}$$

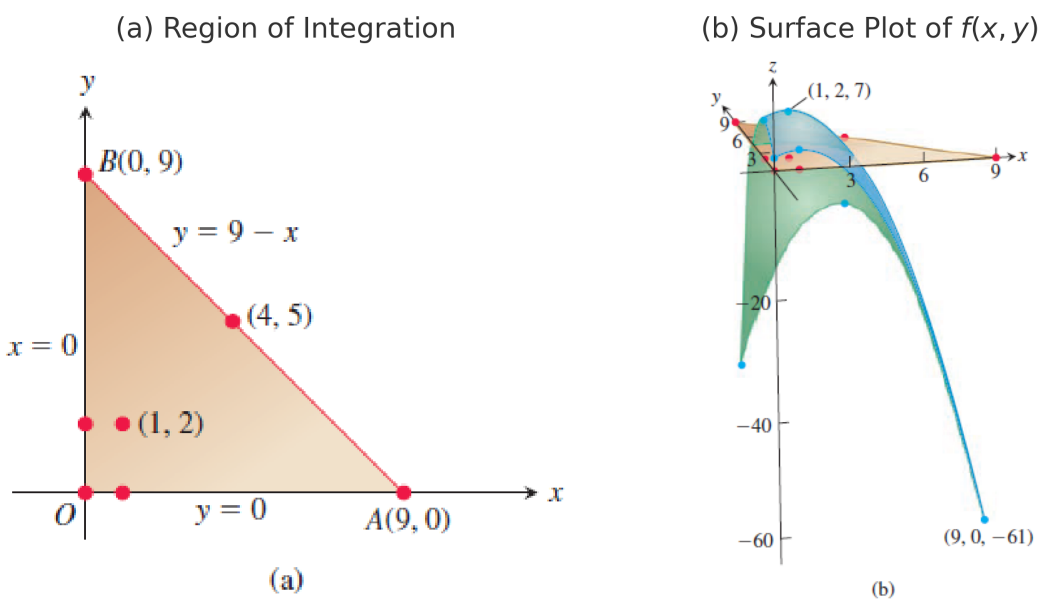


Figure 3: (a) Region bounded by $x = 0$, $y = 0$, and $y = 9 - x$; (b) Graph of $f(x, y) = 2 + 2x + 4y - x^2 - y^2$ over the triangle

Step 2: Evaluate f on the boundary of the triangle 3(a).

(i) Along $x = 0$, $y \in [0, 9]$:

$$f(0, y) = 2 + 0 + 4y - 0 - y^2 = 2 + 4y - y^2$$

Let $g(y) = 2 + 4y - y^2$, then $g'(y) = 4 - 2y = 0 \Rightarrow y = 2$

Evaluate:

$$f(0, 0) = 2, \quad f(0, 2) = 2 + 8 - 4 = 6, \quad f(0, 9) = 2 + 36 - 81 = -43$$

(ii) Along $y = 0$, $x \in [0, 9]$:

$$f(x, 0) = 2 + 2x + 0 - x^2 - 0 = 2 + 2x - x^2$$

Let $g(x) = 2 + 2x - x^2$, then $g'(x) = 2 - 2x = 0 \Rightarrow x = 1$

Evaluate:

$$f(0,0) = 2, \quad f(1,0) = 2 + 2 - 1 = 3, \quad f(9,0) = 2 + 18 - 81 = -61$$

(iii) Along $y = 9 - x$, $x \in [0,9]$: Substitute into f:

$$f(x, 9-x) = 2 + 2x + 4(9-x) - x^2 - (9-x)^2$$

$$= 2 + 2x + 36 - 4x - x^2 - (81 - 18x + x^2)$$
$$= 38 - 2x - x^2 - 81 + 18x - x^2 = -43 + 16x - 2x^2$$

Let $h(x) = -43 + 16x - 2x^2$, then $h'(x) = 16 - 4x = 0 \Rightarrow x = 4 \Rightarrow y = 5$

Evaluate:

$$f(0,9) = -43, \quad f(4,5) = 2 + 8 + 20 - 16 - 25 = \boxed{-11}, \quad f(9,0) = -61$$

Step 3: Compare all values.

- $f(1,2) = 7$
- $f(0,2) = 6$
- $f(1,0) = 3$
- $f(4,5) = -11$
- Others: $f(0,0) = 2$, $f(9,0) = -61$, $f(0,9) = -43$

$\boxed{\text{Absolute Maximum: } f(1,2) = 7}$, $\boxed{\text{Absolute Minimum: } f(9,0) = -61}$

Summary:
We list all the function value candidates: 7, 2, -61, 3, -43, 6, -11.
The **maximum** is $\boxed{7}$, which f assumes at the point $(1,2)$.
The **minimum** is $\boxed{-61}$, which f assumes at the point $(9,0)$.
See Figure 3(b) for visualization.

Problem 2: Find the absolute maxima and minima of the function

$$f(x,y) = 2x^2 - 4x + y^2 - 4y + 1$$

on the closed triangular region bounded by the lines $x = 0$, $y = 2$, and $y = 2x$ in the first quadrant.

Solution:

Step 1: Find critical points in the interior by setting partial derivatives to zero.

$$f_x = 4x - 4, \quad f_y = 2y - 4$$

Set $f_x = 0$ and $f_y = 0$:

$$4x - 4 = 0 \Rightarrow x = 1, \quad 2y - 4 = 0 \Rightarrow y = 2$$

So, critical point: $(1, 2)$

Check if this point lies in the triangular region: $x = 1 \geq 0$, $y = 2 \leq 2$, $y = 2 \geq 2x = 2$,

$$f(1, 2) = 2(1)^2 - 4(1) + (2)^2 - 4(2) + 1 = 2 - 4 + 4 - 8 + 1 = -5$$

Step 2: Evaluate $f(x, y)$ on the boundary.

<u>Edge 1: $x = 0$, $0 \leq y \leq 2$</u>

$$f(0, y) = 0 - 0 + y^2 - 4y + 1 = y^2 - 4y + 1$$

Critical point: $f'(y) = 2y - 4 = 0 \Rightarrow y = 2$

$$f(0, 0) = 1, \quad f(0, 2) = 4 - 8 + 1 = -3$$

<u>Edge 2: $y = 2$, $0 \leq x \leq 1$</u>

$$f(x, 2) = 2x^2 - 4x + 4 - 8 + 1 = 2x^2 - 4x - 3$$

Critical point: $f'(x) = 4x - 4 = 0 \Rightarrow x = 1$

$$f(0, 2) = -3, \quad f(1, 2) = -5$$

<u>Edge 3: $y = 2x$, $0 \leq x \leq 1$</u>

$$f(x, 2x) = 2x^2 - 4x + (2x)^2 - 4(2x) + 1 = 2x^2 - 4x + 4x^2 - 8x + 1 = 6x^2 - 12x + 1$$

Critical point:

$$f'(x) = 12x - 12 = 0 \Rightarrow x = 1 \Rightarrow f(1, 2) = -5, \quad f(0, 0) = 1$$

Step 3: Compare all candidate values:

$$f(0, 0) = 1, \quad f(0, 2) = -3, \quad f(1, 2) = -5$$

Answer: The **absolute maximum** is $\boxed{1}$ at $(0,0)$, and the **absolute minimum** is $\boxed{-5}$ at $(1,2)$.

Problem 3: Find the absolute maxima and minima of the function

$$f(x,y) = x^2 + y^2$$

on the closed triangular region bounded by the lines $x = 0$, $y = 0$, and $y + 2x = 2$ in the first quadrant.

Solution:

Step 1: Find interior critical points.

$$f_x = 2x, \quad f_y = 2y \Rightarrow f_x = 0 \Rightarrow x = 0, \quad f_y = 0 \Rightarrow y = 0$$

This point $(0,0)$ lies on the boundary. So, no interior extrema. Check boundary.

Step 2: Evaluate f on each boundary segment.

Edge 1: $x = 0$, $0 \le y \le 2$

$$f(0,y) = y^2, \quad \Rightarrow f(0,0) = 0, \quad f(0,2) = 4$$

Edge 2: $y = 0$, $0 \le x \le 1$

$$f(x,0) = x^2, \quad \Rightarrow f(0,0) = 0, \quad f(1,0) = 1$$

Edge 3: $y + 2x = 2 \Rightarrow y = 2 - 2x$, with $0 \le x \le 1$
Substitute into $f(x,y)$:

$$f(x, 2-2x) = x^2 + (2-2x)^2 = x^2 + 4 - 8x + 4x^2 = 5x^2 - 8x + 4$$

Differentiate:

$$f'(x) = 10x - 8 = 0 \Rightarrow x = \frac{4}{5}, \quad y = 2 - 2x = \frac{2}{5} \Rightarrow f\left(\frac{4}{5}, \frac{2}{5}\right) = \left(\frac{16}{25} + \frac{4}{25}\right) = \frac{20}{25} = \frac{4}{5}$$

Check endpoints:

$$f(0,2) = 4, \quad f(1,0) = 1$$

Step 3: Compare all values:

$$f(0,0) = 0, \quad f(0,2) = 4, \quad f(1,0) = 1, \quad f\left(\frac{4}{5}, \frac{2}{5}\right) = \frac{4}{5}$$

Answer: The **absolute minimum** is $\boxed{0}$ at $(0,0)$, and the **absolute maximum** is $\boxed{4}$ at $(0,2)$.

Problem 4: Find the absolute maxima and minima of the function

$$f(x,y) = (4x - x^2)\cos y$$

on the closed rectangle defined by

$$1 \le x \le 3, \quad -\frac{\pi}{4} \le y \le \frac{\pi}{4}.$$

Solution:

Step 1: Identify the critical points in the interior.

Let us compute the partial derivatives:

$$f_x = (4 - 2x)\cos y, \quad f_y = -(4x - x^2)\sin y$$

Set $f_x = 0 \Rightarrow 4 - 2x = 0 \Rightarrow x = 2$

Set $f_y = 0 \Rightarrow \sin y = 0 \Rightarrow y = 0$ (since $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$)

So, critical point in the interior: $(2,0)$

$$f(2,0) = (4 \cdot 2 - 2^2)\cos(0) = (8-4)(1) = 4$$

Step 2: Evaluate $f(x,y)$ on the boundary.

Boundary 1: $x = 1$, $-\frac{\pi}{4} \le y \le \frac{\pi}{4}$

$$f(1,y) = (4-1)\cos y = 3\cos y \Rightarrow \text{Max at } y = 0 : f = 3, \quad \text{Min at } y = \pm\frac{\pi}{4} : f = 3\cdot\frac{\sqrt{2}}{2}$$

Boundary 2: $x = 3$

$$f(3,y) = (12-9)\cos y = 3\cos y \Rightarrow \text{same as above}$$

<u>Boundary 3: $y = \frac{\pi}{4}$</u>

$$f(x, \frac{\pi}{4}) = (4x - x^2) \cdot \frac{\sqrt{2}}{2} \Rightarrow \text{Max occurs at } x = 2,\ f(2, \frac{\pi}{4}) = 4 \cdot \frac{\sqrt{2}}{2} = 2\sqrt{2}$$

<u>Boundary 4: $y = -\frac{\pi}{4}$</u>

$$f(x, -\frac{\pi}{4}) = (4x - x^2) \cdot \frac{\sqrt{2}}{2} \Rightarrow \text{same as above}$$

Step 3: Compare all values:

$$\begin{aligned}
&f(2, 0) = 4, \\
&f(2, \pm\frac{\pi}{4}) = 4 \cdot \frac{\sqrt{2}}{2} = 2\sqrt{2} \approx 2.828, \\
&f(1, 0) = 3, \quad f(3, 0) = 3, \\
&f(1, \pm\frac{\pi}{4}) = f(3, \pm\frac{\pi}{4}) = 3 \cdot \frac{\sqrt{2}}{2} \approx 2.121.
\end{aligned}$$

Final Answer:

- **Absolute Maximum:** $\boxed{4}$ at $(2, 0)$
- **Absolute Minimum:** $\boxed{-3}$ at $(1, -\frac{\pi}{4})$ and $(3, -\frac{\pi}{4})$

***

5. Taylor's Formula for $f(x, y)$ at the Point (a, b)

Suppose $f(x, y)$ and its partial derivatives through order $n + 1$ are continuous throughout an open rectangular region R centered at a point (a, b). Then, throughout R,

$$\begin{aligned} f(a+h, b+k) = f(a, b) &+ (hf_x + kf_y)|_{(a,b)} \\ &+ \frac{1}{2!}\left(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right)|_{(a,b)} \\ &+ \frac{1}{3!}\left(h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy}\right)|_{(a,b)} + \cdots \\ &+ \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f\Big|_{(a,b)} \\ &+ \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f\Big|_{(a+ch,b+ck)} \end{aligned} \tag{4}$$

for some $0 < c < 1$. The first n derivative terms are evaluated at (a, b). The last term is evaluated at some point (a + ch, b + ck) on the line segment joining (a, b) and (a + h, b + k).

5.1 Taylor's Formula for $f(x, y)$ at the Origin

If $(a, b) = (0, 0)$ and we treat h and k as independent variables (denoted now by x and y), then above Equation (4) becomes:

$$\begin{aligned} f(x, y) = f(0, 0) &+ xf_x + yf_y \\ &+ \frac{1}{2!}\left(x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy}\right) \\ &+ \frac{1}{3!}\left(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}\right) + \cdots \\ &+ \frac{1}{n!}\left(x^n \frac{\partial^n f}{\partial x^n} + nx^{n-1} y \frac{\partial^n f}{\partial x^{n-1}\partial y} + \cdots + y^n \frac{\partial^n f}{\partial y^n}\right) \\ &+ \frac{1}{(n+1)!}\left(x^{n+1}\frac{\partial^{n+1} f}{\partial x^{n+1}} + (n+1)x^n y \frac{\partial^{n+1} f}{\partial x^n \partial y} + \cdots + y^{n+1}\frac{\partial^{n+1} f}{\partial y^{n+1}}\right)\Big|_{(cx,cy)} \end{aligned}$$

Note:

- The first n derivative terms are evaluated at the point $(0, 0)$.
- The last term is evaluated at a point (cx, cy) that lies on the line segment connecting the origin and the point (x, y).

Note: **Taylor Series Expansion of** $f(x,y)$ **about the point** (a,b):

$$\begin{aligned} f(x,y) = f(a,b) &+ \frac{1}{1!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right] f(a,b) \\ &+ \frac{1}{2!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^2 f(a,b) \\ &+ \cdots + \frac{1}{(n-1)!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^{n-1} f(a,b) + R_n \end{aligned} \tag{5}$$

where the remainder term is given by:

$$R_n = \frac{1}{n!}\left[(x-a)\frac{\partial}{\partial x} + (y-b)\frac{\partial}{\partial y}\right]^n f(a+\theta(x-a), b+\theta(y-b)), \tag{6}$$

for some $\theta \in (0,1)$, meaning the remainder is evaluated at a point along the line segment joining (a,b) and (x,y).

5.2 The Error Formula for Linear Approximations

We want to show that the difference $E(x,y)$ between the values of a function $f(x,y)$ and its linearization $L(x,y)$ at (x_0,y_0) satisfies the inequality

$$|E(x,y)| \leq \frac{1}{2}M\left(|x-x_0|+|y-y_0|\right)^2.$$

Assumptions:

- The function f is assumed to have continuous second partial derivatives throughout an open set containing a closed rectangular region R centered at (x_0,y_0).
- The number M is an upper bound for $|f_{xx}|, |f_{yy}|$, and $|f_{xy}|$ on R.

Problem 1: Find a quadratic approximation to the function

$$f(x,y) = \sin x \sin y$$

near the origin. Estimate the accuracy of the approximation if $0 \leq x \leq 0.1$ and $0 \leq y \leq 0.1$.

Solution:

Step 1: Taylor Expansion up to Second Order near the Origin

We write the second-degree Taylor approximation around the origin $(0,0)$:

$$f(x,y) \approx f(0,0)+f_x(0,0)x+f_y(0,0)y+\frac{1}{2}\left(f_{xx}(0,0)x^2+2f_{xy}(0,0)xy+f_{yy}(0,0)y^2\right)$$

Step 2: Compute Partial Derivatives

Start with:

$$f(x,y) = \sin x \sin y$$

- $f(0,0) = \sin 0 \cdot \sin 0 = 0$
- $f_x = \cos x \sin y \Rightarrow f_x(0,0) = 1 \cdot 0 = 0$
- $f_y = \sin x \cos y \Rightarrow f_y(0,0) = 0 \cdot 1 = 0$
- $f_{xx} = -\sin x \sin y \Rightarrow f_{xx}(0,0) = 0$
- $f_{yy} = -\sin x \sin y \Rightarrow f_{yy}(0,0) = 0$
- $f_{xy} = \cos x \cos y \Rightarrow f_{xy}(0,0) = 1 \cdot 1 = 1$

Step 3: Quadratic Approximation

$$f(x,y) \approx 0+0+0+\frac{1}{2}(0+2\cdot 1\cdot xy+0) = xy$$

$$\therefore \boxed{f(x,y) \approx xy}$$

Step 3: Error Estimation

The third-order error term is given by:

$$E(x,y) = \frac{1}{6}\left(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}\right)\Big|_{(c_x,c_y)}$$

where (c_x, c_y) lies between $(0,0)$ and (x,y).

Since all third-order partials of f are bounded by 1 (as they are combinations of sine and cosine), we have:

$$|f_{xxx}|, |f_{xxy}|, |f_{xyy}|, |f_{yyy}| \leq 1$$

Now let $|x| \leq 0.1$, $|y| \leq 0.1$, then:

$$\begin{aligned} |E(x,y)| &\leq \frac{1}{6}\left((0.1)^3 + 3(0.1)^3 + 3(0.1)^3 + (0.1)^3\right) \\ &= \frac{8}{6}(0.1)^3 = \frac{8}{6} \cdot 0.001 = 0.00133\overline{3} \end{aligned}$$

Conclusion

The quadratic approximation of $f(x,y) = \sin x \sin y$ near the origin is:

$$\boxed{f(x,y) \approx xy}$$

The error in the approximation satisfies:

$$\boxed{|E(x,y)| \leq 0.00134 \quad \text{for } |x| \leq 0.1,\ |y| \leq 0.1}$$

5.3 Problems on Second-Order Taylor Polynomials Centered at (0,0)

General Form:

$$T_2(x,y) = f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}\left(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2\right)$$

Problem 1: $f(x,y) = x^2 + y^2$

Solution:

$$f_x = 2x, \quad f_y = 2y, \quad f_{xx} = 2, \quad f_{xy} = 0, \quad f_{yy} = 2$$

At $(0,0)$:

$$f(0,0) = 0, \quad f_x(0,0) = 0, \quad f_y(0,0) = 0, \quad f_{xx}(0,0) = 2, \quad f_{xy}(0,0) = 0, \quad f_{yy}(0,0) = 2$$

$$T_2(x,y) = \frac{1}{2}(2x^2 + 2y^2) = x^2 + y^2$$

Problem 2: $f(x,y) = xy$

Solution:

$$f_x = y, \quad f_y = x, \quad f_{xx} = 0, \quad f_{xy} = 1, \quad f_{yy} = 0$$

At $(0,0)$:

$$f(0,0) = 0, \quad f_x(0,0) = 0, \quad f_y(0,0) = 0, \quad f_{xx}(0,0) = 0, \quad f_{xy}(0,0) = 1, \quad f_{yy}(0,0) = 0$$

$$T_2(x,y) = \frac{1}{2}(2xy) = xy$$

Problem 3: $f(x,y) = x^2y^2$

Solution:

$$f_x = 2xy^2, \quad f_y = 2yx^2, \quad f_{xx} = 2y^2, \quad f_{xy} = 4xy, \quad f_{yy} = 2x^2$$

At $(0,0)$:

$$f(0,0) = 0, \quad f_x(0,0) = 0, \quad f_y(0,0) = 0, \quad f_{xx}(0,0) = 0, \quad f_{xy}(0,0) = 0, \quad f_{yy}(0,0) = 0$$

$$T_2(x,y) = 0$$

5.4 Problems on Second and Third Order Taylor Polynomials Centered at (0,0)

General Form:

$$\begin{aligned} f(x,y) \approx\ & f(0,0) + f_x(0,0)x + f_y(0,0)y \\ & + \frac{1}{2!}\left[f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2\right] \\ & + \frac{1}{3!}\left[f_{xxx}(0,0)x^3 + 3f_{xxy}(0,0)x^2y + 3f_{xyy}(0,0)xy^2 + f_{yyy}(0,0)y^3\right] \end{aligned}$$

Problem 1: $f(x,y) = xe^y$

Solution:

Quadratic Approximation:

$$f(x,y) \approx f(0,0)+f_x(0,0)x+f_y(0,0)y+\frac{1}{2}(f_{xx}(0,0)x^2+2f_{xy}(0,0)xy+f_{yy}(0,0)y^2)$$

$$f(0,0) = 0, \quad f_x = e^y, \quad f_y = xe^y \Rightarrow f_x(0,0) = 1,\ f_y(0,0) = 0$$

$$f_{xx} = 0, \quad f_{xy} = e^y \Rightarrow f_{xy}(0,0) = 1, \quad f_{yy} = xe^y \Rightarrow f_{yy}(0,0) = 0$$

$$\boxed{f(x,y) \approx x + xy}$$

Cubic Approximation:

$$f_{xxx} = 0, \quad f_{xxy} = 0, \quad f_{xyy} = e^y \Rightarrow f_{xyy}(0,0) = 1, \quad f_{yyy} = xe^y \Rightarrow f_{yyy}(0,0) = 0$$

$$\boxed{f(x,y) \approx x + xy + \frac{1}{2}xy^2}$$

Problem 2: $f(x,y) = \sin(x^2 + y^2)$

Solution:

Quadratic Approximation:

$$f(x,y) \approx f(0,0) + \frac{1}{2}(f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2)$$

$$f(0,0) = 0, \quad f_x = 2x\cos(x^2+y^2),\ f_y = 2y\cos(x^2+y^2) \Rightarrow f_x(0,0) = 0$$

$$f_{xx} = 2\cos(x^2+y^2) - 4x^2\sin(x^2+y^2) \Rightarrow f_{xx}(0,0) = 2$$

$$f_{yy}(0,0) = 2, \quad f_{xy}(0,0) = 0$$

$$\boxed{f(x,y) \approx x^2 + y^2}$$

Cubic Approximation:

$$f_{xxx}(0,0) = f_{xxy}(0,0) = f_{xyy}(0,0) = f_{yyy}(0,0) = 0$$

$$\boxed{f(x,y) \approx x^2 + y^2}$$

Problem 3: $f(x,y) = \frac{1}{1-x-y}$

Solution:

$$f(x,y) = (1-x-y)^{-1}$$

Quadratic Approximation:

$$f(0,0) = 1, \quad f_x = \frac{1}{(1-x-y)^2} \Rightarrow f_x(0,0) = 1$$

$$f_y(0,0) = 1, \quad f_{xx} = \frac{2}{(1-x-y)^3} \Rightarrow f_{xx}(0,0) = 2$$

$$f_{yy}(0,0) = 2, \quad f_{xy}(0,0) = 2$$

$$\boxed{f(x,y) \approx 1 + x + y + x^2 + 2xy + y^2}$$

Cubic Approximation:

$$f_{xxx}(0,0) = 6, \quad f_{xxy}(0,0) = 6, \quad f_{xyy}(0,0) = 6, \quad f_{yyy}(0,0) = 6$$

$$\boxed{f(x,y) \approx 1 + x + y + x^2 + 2xy + y^2 + x^3 + 3x^2y + 3xy^2 + y^3}$$

Problem 4: Use Taylor's formula to find a quadratic approximation of $f(x,y) = \cos x \cos y$ near the origin. Estimate the error in the approximation if $0 \leq x \leq 0.1$ and $0 \leq y \leq 0.1$.

Solution:

We want to find the quadratic approximation:

$$f(x,y) \approx f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}\left[f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2\right]$$

Step 1: Compute the function and its derivatives at the origin. Let $f(x,y) = \cos x \cos y$

$$\begin{aligned}
f(0,0) &= \cos 0 \cos 0 = 1 \\
f_x(x,y) &= -\sin x \cos y \Rightarrow f_x(0,0) = 0 \\
f_y(x,y) &= -\cos x \sin y \Rightarrow f_y(0,0) = 0 \\
f_{xx}(x,y) &= -\cos x \cos y \Rightarrow f_{xx}(0,0) = -1 \\
f_{xy}(x,y) &= \sin x \sin y \Rightarrow f_{xy}(0,0) = 0 \\
f_{yy}(x,y) &= -\cos x \cos y \Rightarrow f_{yy}(0,0) = -1
\end{aligned}$$

Step 2: Plug into the quadratic approximation:

$$
\begin{aligned}
f(x,y) &\approx 1 + 0\cdot x + 0\cdot y + \frac{1}{2}\left[-1\cdot x^2 + 0\cdot 2xy - 1\cdot y^2\right] \\
&= 1 - \frac{1}{2}x^2 - \frac{1}{2}y^2
\end{aligned}
$$

Quadratic Approximation:

$$\boxed{f(x,y) \approx 1 - \frac{1}{2}x^2 - \frac{1}{2}y^2}$$

Error Estimation: The third-order Taylor remainder term involves third derivatives. Since $\cos x \cos y$ has bounded third partials near the origin, the third-order term is:

$$R_3(x,y) = \frac{1}{6}\left(f_{xxx}(\xi,\eta)x^3 + 3f_{xxy}(\xi,\eta)x^2y + 3f_{xyy}(\xi,\eta)xy^2 + f_{yyy}(\xi,\eta)y^3\right)$$

for some (ξ,η) in the rectangle $[0,x]\times[0,y]$.

We estimate an upper bound of the error. Since all third derivatives of $\cos x \cos y$ are bounded in magnitude by 1 in $[0, 0.1]$,

$$
\begin{aligned}
|R_3(x,y)| &\le \frac{1}{6}(|x|^3+3|x|^2|y|+3|x||y|^2+|y|^3) \\
&\le \frac{1}{6}(0.001 + 3\cdot 0.01\cdot 0.1 + 3\cdot 0.1\cdot 0.01 + 0.001) = \frac{1}{6}(0.001 + 0.003 + 0.003 + 0.00 \\
&= \frac{1}{6}(0.008) = 0.00133
\end{aligned}
$$

Conclusion:

- Quadratic approximation: $f(x,y) \approx 1 - \frac{1}{2}x^2 - \frac{1}{2}y^2$
- Maximum error in domain $[0, 0.1]\times[0, 0.1]$ is approximately $\boxed{0.00133}$

Problem 5: Find a quadratic approximation of $f(x,y) = e^x \sin y$ at the Origin. Estimate the error in the approximation if $0 \le x \le 0.1$ and $0 \le y \le 0.1$

We are given the function:

$$f(x,y) = e^x \sin y$$

and asked to find a quadratic approximation centered at the origin (0,0).

Step 1: Evaluate $f(0,0)$

$$f(0,0) = e^0 \cdot \sin 0 = 1\cdot 0 = 0$$

Step 2: First Partial Derivatives

$$f_x = \frac{\partial}{\partial x}(e^x \sin y) = e^x \sin y,$$
$$f_y = \frac{\partial}{\partial y}(e^x \sin y) = e^x \cos y$$

Evaluate at (0,0):

$$f_x(0,0) = e^0 \cdot \sin 0 = 0,$$
$$f_y(0,0) = e^0 \cdot \cos 0 = 1$$

Step 3: Second Partial Derivatives

$$f_{xx} = \frac{\partial^2}{\partial x^2}(e^x \sin y) = e^x \sin y,$$
$$f_{yy} = \frac{\partial^2}{\partial y^2}(e^x \cos y) = -e^x \sin y,$$
$$f_{xy} = \frac{\partial^2}{\partial x \partial y}(e^x \sin y) = \frac{\partial}{\partial x}(e^x \cos y) = e^x \cos y$$

Evaluate at (0,0):

$$f_{xx}(0,0) = 0,$$
$$f_{yy}(0,0) = 0,$$
$$f_{xy}(0,0) = 1$$

Step 4: Quadratic Approximation The general second-order Taylor approximation centered at (0,0) is:

$$f(x,y) \approx f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}\left[f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2\right]$$

Substituting values:

$$f(x,y) \approx 0 + 0 + y + \frac{1}{2}(0 + 2xy + 0) = y + xy$$

Step 5: Error Estimate for $0 \leq x \leq 0.1$, $0 \leq y \leq 0.1$ The error in the quadratic approximation is bounded by the third-order term:

$$|R_2(x,y)| \leq \frac{1}{6} \max_D |f_{xxx}, f_{xxy}, f_{xyy}, f_{yyy}| \cdot (|x| + |y|)^3$$

On the square $0 \leq x \leq 0.1$, $0 \leq y \leq 0.1$, the third-order partials like $f_{xxy} = e^x \cos y$ are bounded by $e^{0.1} \leq 1.105$, and $\cos y \leq 1$. So:

$$|R_2| \leq \frac{1}{6} \cdot 1.105 \cdot (0.2)^3 \approx 0.0015$$

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Derivative Formulas

Exponential Functions

$$\frac{d}{dx}(e^x) = e^x$$
$$\frac{d}{dx}(a^x) = a^x \ln a$$
$$\frac{d}{dx}(e^{g(x)}) = e^{g(x)} g'(x)$$
$$\frac{d}{dx}(a^{g(x)}) = \ln(a)\, a^{g(x)} g'(x)$$

Logarithmic Functions

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad x > 0$$
$$\frac{d}{dx}(\ln(g(x))) = \frac{g'(x)}{g(x)}$$
$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}, \quad x > 0$$
$$\frac{d}{dx}(\log_a(g(x))) = \frac{g'(x)}{g(x) \ln a}$$

Trigonometric Functions

$$\frac{d}{dx}(\sin x) = \cos x$$
$$\frac{d}{dx}(\cos x) = -\sin x$$
$$\frac{d}{dx}(\tan x) = \sec^2 x$$
$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x$$
$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}, \quad |x| < 1$$
$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}, \quad |x| < 1$$
$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$
$$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$
$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x| > 1$$
$$\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}, \quad |x| > 1$$

Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x$$
$$\frac{d}{dx}(\cosh x) = \sinh x$$
$$\frac{d}{dx}(\tanh x) = \mathrm{sech}^2 x$$
$$\frac{d}{dx}(\mathrm{csch}\, x) = -\mathrm{csch}\, x \coth x$$
$$\frac{d}{dx}(\mathrm{sech}\, x) = -\mathrm{sech}\, x \tanh x$$
$$\frac{d}{dx}(\coth x) = -\mathrm{csch}^2 x$$

Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$$
$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}, \quad x > 1$$
$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}, \quad |x| < 1$$
$$\frac{d}{dx}(\operatorname{csch}^{-1} x) = \frac{-1}{|x|\sqrt{1+x^2}}$$
$$\frac{d}{dx}(\operatorname{sech}^{-1} x) = \frac{-1}{x\sqrt{1-x^2}}, \quad 0 < x < 1$$
$$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}, \quad |x| > 1$$

Common Integrals

$$\int k\,dx = kx + C$$
$$\int x^n dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$
$$\int x^{-1} dx = \ln|x| + C$$
$$\int \frac{1}{ax+b} dx = \frac{1}{a}\ln|ax+b| + C$$
$$\int \ln(x) dx = x\ln x - x + C$$
$$\int e^x dx = e^x + C$$
$$\int \cos x\,dx = \sin x + C$$
$$\int \sin x\,dx = -\cos x + C$$
$$\int \sec^2 x\,dx = \tan x + C$$
$$\int \sec x \tan x\,dx = \sec x + C$$
$$\int \csc x \cot x\,dx = -\csc x + C$$
$$\int \csc^2 x\,dx = -\cot x + C$$
$$\int \tan x\,dx = \ln|\sec x| + C$$
$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$
$$\int \frac{1}{a^2+u^2} du = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$
$$\int \frac{1}{\sqrt{a^2-u^2}} du = \sin^{-1}\left(\frac{u}{a}\right) + C$$

Definite Integral Definition

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x, \quad \Delta x = \frac{b-a}{n}, \quad x_k = a + k\Delta x$$

Fundamental Theorem of Calculus

$$\int_a^b f(x)\,dx = F(b) - F(a), \quad \text{where } F'(x) = f(x)$$

Integration Properties

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$$

$$\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$$

Integration by Parts

$$\int u\,dv = uv - \int v\,du \quad \text{or} \quad \int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx$$

Integration by Substitution

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du, \quad \text{where } u = g(x),\ du = g'(x)\,dx$$

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