Unit-III
Probability ~~p~~ distribution

Discrete probability distributions:

1. Binomial
2. ~~Geometric~~ Poisson
3. Geometric
4. Hyper geometric.

Continuous Prob. distribution:

1. Normal
2. Exponential

Note: * For discrete random variable, Probability distribution is known as probability mass function. (P.M.F)

* For continuous random variable, it is known as probability density function (pdf).

Definition

Any experiment / trial which has only two possible outcomes either success (S) or failure (F) is known as Bernoulli experiment.

Let $p \to$ probability of getting success (S)

$1-p \to$ " " getting failure (F).

Let X = no. of success.

Then X : 1 0 [since only 1 trial] (1 success or 0 success)

$p(x)$: p $1-p$

The above is the pmf of X, where X is a Bernoulli random variable.

ex. (1) Toss a coin.

success → getting head
failure → " Tail

X:	1	0
$P(X)$:	1/2	1/2

(2) Answer to a true/False question

Success → True
failure → False

X:	1	0
$P(X)$:	1/2	1/2

~~(3) Roll a die.~~
~~Success → getting~~ an even number
~~failure → getting~~ an odd number

(1)

Binomial distribution

→ There are n independent trials
→ Each trial having two outcomes S/F.
→ p = probability of success for each trial
$1-p$ = " " failure " " "

Let X = no. of successes

Then X is a binomial random variable with parameters (n, p).

Note Binomial random variable is a sequence of Bernoulli ranom variables.

PMF $P(X=i) = {}^nC_i \, p^i (1-p)^{n-i}$, $i = 0, 1, 2, \ldots, n$

↙
Probability of getting i number of successes

ex. ① Five coins are flipped. If the outcomes are assumed independent, find the pmf of the number of heads obtained.

Let X = no. of heads

Then X is a binomial random variable with parameter $(5, 1/2)$.

$$n = 5$$
$$p = 1/2$$
$$1-p = 1/2$$

So

$$P(X=0) = p(0) = {}^5C_0 (1/2)^0 (1/2)^5 = \frac{1}{32}$$

$$P(X=1) = p(1) = {}^5C_1 (1/2)^1 (1/2)^4 = \frac{5}{32}$$

$$P(X=2) = p(2) = {}^5C_2 (1/2)^2 (1/2)^3 = \frac{10}{32}$$

$$P(X=3) = p(3) = {}^5C_3 (1/2)^3 (1/2)^2 = \frac{10}{32}$$

$$P(X=4) = p(4) = {}^5C_4 (1/2)^4 (1/2) = \frac{5}{32}$$

$$P(X=5) = p(5) = {}^5C_5 (1/2)^5 (1/2)^0 = 1/32$$

② It is known that the screws produced by a certain company will be defective with probability 0.01 independently of one another. The company sells the screws in packages of 10 and offers a money-back guarantee that atmost 1 of the 10 screws is defective. What proportion of packages sold must the company replace?

Let X = no. of defective screws in a package

Then X is a binomial random variable with

$$(n, p) = (10, 0.01), \quad 1-p = 0.99$$

The company will replace a package if it contains more than 1 defective screws i.e $X \geq 2$.

So $P(X \geq 2) = 1 - P(X \leq 2)$

$$= 1 - [P(X=0) + P(X=1)]$$

$$= 1 - [{}^{10}C_0 (0.01)^0 (0.99)^{10} + {}^{10}C_1 (0.01)^1 (0.99)^9]$$

$$= 1 - [(0.99)^{10} + 10 (0.01)^1 (0.99)^9]$$

$$\simeq 0.004$$

So $0.004 \times 100 = 0.4\%$ of the packages to be replaced.

Q. (3) A player bets on one of the numbers 1 through 6. Three dice are rolled. If the number bet by the player appears i times, $i = 1, 2, 3$, then the player win i units. If the number bet by the player doesn't appear on any of the dice, then he/she loses 1 unit. Is this a fair game?

Fair game : $E(X) = 0$.

Let X = no. of units winning. $\quad n = 3$, $p = 1/6$

Then $P(X=-1) = p(-1) = {}^3C_0 (\frac{1}{6})^0 (\frac{5}{6})^3 = 125/216$

$P(X=1) = p(1) = {}^3C_1 (1/6)^1 (5/6)^2 = 75/216$

$P(X=2) = p(2) = {}^3C_2 (1/6)^2 (5/6)^1 = 15/216$

$P(X=3) = p(3) = {}^3C_3 (1/6)^3 (5/6)^0 = 1/216$

So

$$E(X) = \sum x\, p(x)$$

$$= (-1).\,125/216 + 1.(75/216) + 2.\left(\frac{15}{216}\right) + 3.\left(1/216\right)$$

$$= -17/216 \neq 0.$$

The game is not fair. The player will lose 17 units per every 216 games.

Expectation & variance for Binomial distribution.

$$\boxed{\begin{aligned} E(X) &= np \\ var(X) &= npq, \quad q = 1-p \\ &= np(1-p). \end{aligned}}$$

② Poisson ~~random~~ distributions

→ This distribution is an ~~exp~~ approximation for a binomial distribution, i.e. when n is very large and p is small, binomial will tends to poisson..

pmf $p(i) = P(X=i) = \dfrac{e^{-\lambda}\lambda^{i}}{i!}, \quad i = 0,1,2,\ldots$

Note: let ~~X~~ X is a binomial random variable with parameters (n,p). If we let $n \to \infty$ and $p \to 0$ such that $np = \mu > 0$, then X becomes a poisson random variable with parameter $\mu = np$.

Applications

1. No. of errors in a page of a book
2. No. of people in a community who survive to age 100.
3. No. of car accidents in a particular road per day.
4. No. of wrong numbers dialed in a day.

...

$$\boxed{\begin{aligned} E(X) &= \lambda = np \\ var(X) &= \lambda = np \end{aligned}}$$

ex. Suppose that no. of typographical errors on a single page of a book has a poisson distribution with parameter $\lambda = 1/2$. Find the prob. that there is atleast one error on the page.

Let X = no. of errors on the page.

Then $P(X \geq 1) = 1 - P(X < 1)$

$= 1 - P(X = 0)$

$= 1 - \frac{e^{-\lambda}\lambda^0}{0!} = 1 - e^{-1/2}$

$\simeq 0.393$

ex. Suppose that the probability of an item produced by a certain machine will be defective is 0.1. Find the probability that a sample of 10 items will contain atmost 1 defective item.

$n = 10, \quad p = 0.1$

$\lambda = np = 1$

$$P(X \le 1) = P(X=0) + P(X=1)$$

$$= \frac{e^{-\lambda}\lambda^0}{0!} + \frac{e^{-\lambda}\lambda^1}{1!}$$

$$= e^{-1} + e^{-1} = 2e^{-1} \simeq 0.7358.$$

If we use binomial :→

$$P(X \le 1) = P(X=0) + P(X=1)$$

$$= {}^{10}C_0 (0.1)^0 (0.9)^{10} + {}^{10}C_1 (0.1)^1 (0.9)^9$$

$$= 0.7361$$

(3) Geometric distribution

suppose independent trials, each having success with probability p and failure with probability $1-p$ are performed until a success appears;

Let X = no. of trials required.

Then $P(X=n) = P(\underbrace{FF\cdots F}_{n-1 \text{ times}}S)$

pmf : $\boxed{P(X=n) = (1-p)^{n-1} p, \quad n = 1, 2, \ldots}$

$$\boxed{\begin{aligned} E(X) &= \frac{1}{p} \\ \operatorname{var}(X) &= \frac{1-p}{p^2} = q/p^2 \end{aligned}}$$

ex. An urn contains 7 white & 8 black balls. Balls are randomly selected - one at a time, until a black is obtained. If we assume that each ball selected is replaced before the next one is drawn, what is the probability that

1. exactly 3 draws are needed?
2. atleast 5 draws are needed?

Total = 7W + 8B = 15 balls.

$P(W) = 7/15 \quad P(B) = 8/15$.

let x = no. of draws.

Then $P(x = 3) = P(WWB)$

$= (P(W))^2 (P(B))^1$

$= (7/15)^2 \cdot 8/15$.

$P(x \geq 5) = P(x=5) + P(x=6) + \cdots$

$= P(WWWWB) + P(WWWWWB) + \cdots$

$= (P(W))^4 \cdot P(B) + (P(W))^5 \cdot P(B) + \cdots$

$= (7/15)^4 \cdot \frac{8}{15} + (7/15)^5 \cdot \frac{8}{15} + \cdots$

$= (7/15)^4 \left[8/15 + 8/15 \cdot 7/15 + 8/15 \cdot (7/15)^2 + \cdots \right]$

$\left[a + ar + ar^2 + \cdots = \frac{a}{1-r} \right]$

$a = 8/15$

$r = 7/15$

$= (7/15)^4 \left[\frac{8/15}{1 - 7/15} \right] = (7/15)^4 \times \frac{\cancel{8/15}}{\cancel{8/15}}$

$= (7/15)^4$.

or

probability of atleast 5 draws are needed to get a black ball is equal to the probability that first 4 balls are white.

so $P(WWWW) = (P(W))^4 = (7/15)^4$.

ex. If white balls = N
black " = M. in previous example,

then probability of

① exactly n draws are needed

$$= P(X=n) = P(\underbrace{WW\cdots W}_{(n-1)\text{ times}} B)$$

$$= (P(W))^{n-1} P(B)$$

$$= \left(\frac{N}{N+M}\right)^{n-1} \frac{M}{N+M} = \frac{MN^{n-1}}{N+M}$$

② at least k draws are needed

$$= P(X \geq k) = P(\underbrace{WW\cdots W}_{k-1\text{ times}}) = (P(W))^{k-1}$$

$$= \left(\frac{N}{N+M}\right)^{k-1}.$$

Hyper geometric distribution

An urn contains N balls of which m are white and $N-m$ are black. If n balls are choosen randomly with out replacement, then probability of i number of white balls selected is

$$P(X=i) = \frac{{}^{m}C_i \; {}^{N-m}C_{n-i}}{{}^{N}C_n} \quad i = 0, 1, 2, \cdots n$$

Here X = no. of white balls selected. (success)

~~p (success) = p(white ball selected)~~

$= {}^{m}C_1$

Let. $N = 20 = \underset{m}{\underset{\parallel}{8W}} + \underset{N-m}{\underset{\parallel}{12B}}$

Let $n = 10$.

① Then probability of $i=3$ white balls selected is

$$P(X=3) = \frac{{}^{8}C_3 \times {}^{12}C_7}{{}^{20}C_{10}}.$$

② probability of atleast 6 white balls selected

$$= P(X \geq 6) = P(X=6) + P(X=7) + P(X=8)$$

$$= \frac{{}^{8}C_6 \times {}^{12}C_4}{{}^{20}C_{10}} + \frac{{}^{8}C_7 \times {}^{12}C_3}{{}^{20}C_{10}} + \frac{{}^{8}C_8 \times {}^{12}C_2}{{}^{20}C_{10}}$$

Q. A purchaser of electrical components buys them in lots of size 10. It is his policy to inspect 3 components randomly from a lot and to accept the lot only if all 3 are non defective. If 30% of the lots have 4 defective components and 70% have only 1 defective, what proportion of the lots does the purchaser reject?

Solⁿ

Each lot is of size 10.

A lot is accepted if 3 components (out of 10) are randomly choosen and found to be all 3 non defective.

Also given, there are 2 types of lots, (Type I) a lot having 4 defective and a lot having 1 defective (Type II).

Let E_1 = a lot having 4 defective

$$P(E_1) = 30\% = 3/10$$

& E_2 = a lot having 1 defective

$$= 70\% = 7/10.$$

If a lot is accepted $= P(A)$ then it is either from type I or from type II so

$$P(A) = P(A/E_1) \cdot P(E_1) + P(A/E_2) \cdot P(E_2).$$

$$= \frac{4C_0 \times 6C_3}{10C_3}\left(\frac{3}{10}\right) + \frac{1C_0 \times 9C_3}{10C_3}\left(7/10\right) = 54/100$$

So P(a lot is rejected)

$$= 1 - \frac{54}{100} = \frac{46}{100}$$

So 46% of the lots are rejected.

If $X \to$ hypergeometric random variable

$$E(X) = \frac{mn}{N}$$

$$Var(X) = np(1-p)\left(\frac{N-n}{N-1}\right).$$

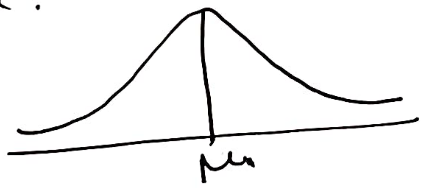
Continuous probability distributions

① **Normal distribution**

Pdf: $f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(x-\mu)^2/2\sigma^2}, \quad -\infty < x < \infty.$

$\mu \to$ mean

$\sigma^2 \to$ variance.



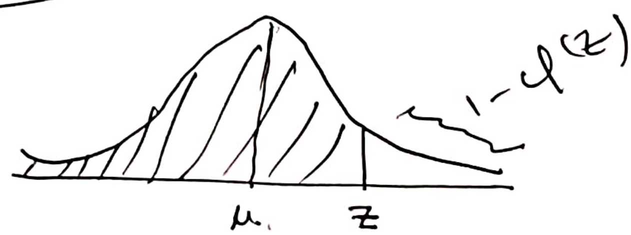
$* \int_{-\infty}^{\infty} f(x)\,dx = 1$

→ If X is normally distributed with parameters μ and σ^2, then $Z = \frac{X-\mu}{\sigma}$ is normally distributed with parameters 0 and 1.
And this random variable Z is called <u>standard normal random variable</u>.

→ If Z is standard normal r.v then

$$P(Z \le z) = \phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-y^2/2}\,dy.$$

→ $\boxed{\phi(-z) = 1 - \phi(z)}$



$P(Z \le z) = \phi(z) =$ Area under the above curve to the left of z

If X is a normal distribution with μ and σ^2

then ① $P(a \leq X \leq b) = P\left(\frac{a-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{b-\mu}{\sigma}\right)$

$$= P\left(\frac{a-\mu}{\sigma} \leq Z \leq \frac{b-\mu}{\sigma}\right)$$

$$= \phi\left(\frac{b-\mu}{\sigma}\right) - \phi\left(\frac{a-\mu}{\sigma}\right).$$

② $P(X \leq a) = P\left(\frac{X-\mu}{\sigma} \leq \frac{a-\mu}{\sigma}\right)$

$$= P\left(Z \leq \frac{a-\mu}{\sigma}\right) = \phi\left(\frac{a-\mu}{\sigma}\right)$$

③ $P(X > a) = 1 - P(X \leq a)$

$$= 1 - \phi\left(\frac{a-\mu}{\sigma}\right).$$

Note For cts. random variable.

$$P(a<X<b) = P(a \leq X \leq b) = P(a < X \leq b)$$
$$= P(a \leq X < b)$$

ex. If X is a normal random variable with $\mu = 3$ and $\sigma^2 = 9$, then

① $P(2<X<5) = P\left(\frac{2-3}{3} < \frac{X-3}{3} < \frac{5-3}{3}\right)$

$$= P\left(-\tfrac{1}{3} < Z < \tfrac{2}{3}\right)$$

$$= \phi(2/3) - \phi(-1/3)$$

$$= \phi(2/3) - [1 - \phi(1/3)]$$

$$= \phi(0.66) - 1 + \phi(0.33)$$

$$\simeq 0.3779$$ [find values from ϕ table]

② $P(X>0) = 1 - P(X \le 0)$

$= 1 - P\left(\frac{X-3}{3} \le \frac{0-3}{3}\right)$

$= 1 - P(Z \le -1)$

$= 1 - \phi(-1) = 1 - [1 - \phi(1)]$

$= \phi(1) \approx 0.8413$

③ $P(|X-3|>6) = 1 - P(|X-3| \le 6)$

$= 1 - P(-6 \le X-3 \le 6)$

$= 1 - P(-3 \le X \le 9)$

$= 1 - P\left(\frac{-3-3}{3} \le \frac{X-3}{3} \le \frac{9-3}{3}\right)$

$= 1 - P(-2 \le Z \le 2)$

$= 1 - [\phi(2) - \phi(-2)]$

$= 1 - \phi(2) + \phi(-2)$

$= 1 - \phi(2) + 1 - \phi(2)$

$= 2 - 2\phi(2) = 2[1 - \phi(2)]$

$\approx 0.0456.$

Note When n is Large, a binomial random variable with parameter n and p will have approximately the same distribution as a normal random variable with the same mean and variance of the binomial.

Q. In a particular branch of a bank, it is noticed that the duration time of the customers for being served by the teller is normally distributed with mean 5.5 minutes and standard deviation 0.6 minutes. Find the probability that a customer has to wait a) between 4.2 and 4.5 minutes.
b) for less than 5.2 minutes.
c) more than 6.8 minutes.

$\mu = 5.5$
$\sigma = 0.6$

Ans a) $P(4.2 < X < 4.5)$

$$= P\left(\frac{4.2-5.5}{0.6} < \frac{X-5.5}{0.6} < \frac{4.5-5.5}{0.6}\right)$$

$$= P(-2.16 < Z < -1.66)$$

$$= \phi(-1.66) - \phi(-2.16)$$

$$= 1 - \phi(1.66) - [1 - \phi(2.16)]$$

$$= 1 - \phi(1.66) - 1 + \phi(2.16)$$

$$= \phi(2.16) - \phi(1.66)$$

b) $P(X < 5.2) = P\left(\frac{X-5.5}{0.6} < \frac{5.2-5.5}{0.6}\right)$

$= P(Z < -0.5)$

$= \Phi(-0.5) = 1 - \Phi(0.5)$

c) $P(X > 6.8) = P\left(\frac{X-5.5}{0.6} > \frac{6.8-5.5}{0.6}\right)$

$= P(Z > 2.16)$

$= 1 - P(Z \leq 2.16)$

$= 1 - \Phi(2.16)$

Find Φ values from the table.

Exponential distribution

Pdf: $f(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0, \end{cases}$

$\lambda > 0$

Cdf: $F(a) = P(X \leq a)$

$$= \int_{-\infty}^{a} f(x)dx = \underbrace{\int_{-\infty}^{0} f(x)dx}_{=0} + \int_{0}^{a} f(x)dx$$

$$= \int_{0}^{a} f(x)dx = \int_{0}^{a} \lambda e^{-\lambda x}dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_{0}^{a} = \left[-e^{-\lambda x}\right]_{0}^{a}$$

$$= -e^{-\lambda a} + e^{0} = 1 - e^{-\lambda a}, \quad a \geq 0$$

$$\boxed{E(X) = \frac{1}{\lambda} \qquad var(X) = \frac{1}{\lambda^2}}$$

Q. Suppose that the length of a phone call in minutes is an exponential random variable with parameter $\lambda = 1/10$. If someone arrives immediately ahead of you at a public telephone booth, find the probability that you will have to wait

a). more than 10 minutes

b) between 10 and 20 minutes.

Ans $T = 1/10$

a) $P(X > 10) = 1 - P(X \leq 10) = 1 - F(10)$

$$= 1 - \left[1 - e^{-T\cancel{10}}\right]$$

$$= \cancel{1} - e^{-1/10(10)} = -e^{-1}$$

b) $P(10 < X < 20) = F(20) - F(10)$

$$= \cancel{1 - e^{-T20}}$$

$$= 1 - e^{-20T} - \left[1 - e^{-10T}\right]$$

$$= -e^{-20(1/10)} + e^{-10(1/10)}$$

$$= e^{-1} - e^{-2} \simeq 0.233$$

Q. The time (in hours) required to repair a machine is an exponentially random variable with parameter $T = 1/2$. What is the probability that a repair time exceeds 2 hours?

Ans. $T = 1/2$

$$P(X \geq 2) = 1 - P(X \leq 2)$$

$$= 1 - F(2) = 1 - \left[1 - e^{-2T}\right]$$

$$= e^{-2 \cdot 1/2} = e^{-1} \simeq 0.3679$$

Q. Suppose that the number of accidents occuring weekly on a particular stretch of a highway equals 3. Find the probability that there is at least one accident this week.

Q. Consider an experiment that consists of counting the number of α particles given off in a 1-second interval by 1 gram of radio active material. If we know from past experience that, on average, 3.2 such α-particles are given off, which is a good approximation to the probability that no more than 2 α-particles will appear?

Q. A sample of 3 items is selected at random from a box containing 20 items of which 4 are defective. Find the probability that the sample contain atleast 1 defective item.

Q. On a MCQ exam with 3 possible answers for each of the 5 questions, what is the probability that a student will get 4 or more correct answer just by guessing?

Q. If you buy a Lottery ticket in 50 Lotteries, in each of which your chance of winning a prize is $1/100$, what is the probability that you win a prize

a. atleast once ?

b. exactly once ?

c. atleast twice ?

Q. If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, then find

a) $P(X > 5)$ b) $P(4 < X < 16)$

c) $P(X < 8)$ d) $P(X < 20)$ e) $P(X > 16)$

Q. The annual rainfall (in inches) in a certain region is normally distributed with $\mu = 40$ and $\sigma = 4$. What is the probability that starting with this year, it will take more than 10 years before a year occurs having a rainfall of more than 50 inches ?

Q. Suppose that a batch of 100 items contains 6 that are defective and 94 that are non-defective. If X is the number of defective items in a randomly drawn sample of 10 items from the batch,
(1) Find $P(X=0)$. (2) $P(X>2)$.

Q. A purchaser of transistors buys them in lots of 20. It is is policy to randomly inspect 4 components from a lot and to accept the lot only if all 4 are nondefective. If each component in a lot is independently defective with probability 0.1, what proportion of lots is rejected?

Q. A call center receives an average of 10 calls per hour. What is the probability ~~that~~ of receiving exactly 5 calls in a given hour?

Q. If the probability of the defective bolt is 0.2, find the ~~pro~~ mean for the distribution of 400 bolts.