Rate Gyro Model

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1 Dynamics

The dynamics of a rate gyroscope takes the form[1]

$$I_a\ddot{\theta} + c\dot{\theta} + K_{tb}\theta = H\omega, \tag{1}$$

where θ is the gimbal angle, ω is the input angular velocity, H is the wheel angular momentum, I_a is the gimbal moment of inertia about the output axis (OA), c is the damping constant about the OA and K_{tb} is the torsion bar spring constant. The output of the gyro is

$$\hat{y} = \theta K_{po} + B, \tag{2}$$

where K_{po} is the pickoff sensitivity and B is the measurement bias. The idea of a rate gyroscope is that by measuring $\hat{y}$ one can get information about ω.

Assume that the value of the bias B is known and consider the debiased gyro output $y = \hat{y} - B$. Clearly, y satisfies a differential equation similar to θ:

$$\begin{aligned} y &= K_{po}\theta, \\ \dot{y} &= K_{po}\dot{\theta}, \\ \ddot{y} &= K_{po}\ddot{\theta} = \frac{K_{po}}{I_a}(H\omega - c\dot{\theta} - K_{tb}\theta) = \frac{K_{po}}{I_a}\left(H\omega - \frac{c}{K_{po}}\dot{y} - \frac{K_{tb}}{K_{po}}y\right), \end{aligned}$$

or

$$\ddot{y} + \frac{c}{I_a}\dot{y} + \frac{K_{tb}}{I_a}y = \frac{HK_{po}}{I_a}\omega\,. \tag{3}$$

If the gyroscope is well-designed, its dynamics will be critically damped with a gain close to 1. This will make y converge quickly to ω (more precisely, $y(t) - \omega(t) \to 0$, $t \to 0$ when ω changes slowly or is constant).

By introducing $a_1 = c/I_a$, $a_2 = K_{tb}/I_a$, $b_1 = H/(I_aK_{po})$, we can rewrite (3) in the “standard” form

$$\ddot{y} + a_1\dot{y} + a_2y = b_1\dot{u}\,. \tag{4}$$

Here we also introduced $\dot{u} = \omega$, which is meant to emphasize that the quantity to be estimated is the angular displacement u instead of the angular velocity (ω). Now, the right-hand side is critically damped with a gain of 1 if the following holds:

$$\text{gain} = \frac{b_1}{a_2} = 1 \quad , \quad D = a_1^2 - 4a_2 = 0. \tag{5}$$

Thus, if we are interested in estimating the parameters, we only need to estimate one parameter as the others can then be determined from (5).

[1] A. Lawrence, Modern Inertial Technoogy: Navigation, Guidance, and Control (2nd edition), Mechanical Engineering Series, 1998, Chapter 7, Rate Gyro Dynamics (page 100)

2 Simulation

The behavior of the gyro is given by the ODE described by (4) with the initial condition $y(t_0) = y_0$, $\dot{y}(t_0) = \dot{y}_0$, were $y_0, \dot{y}_0$ are the known initial conditions (usually, they would be $(0, 0)$, assuming a resting robot).

In order to simulate this ODE, we first transform it to a (multi-dimensional) first order ODE. Second, we will need to deal with the issue that the angular velocity ($\dot{u}$) is not available, just the angles (u).

To deal with this second issue, we integrate (4) respect to the time and get

$$\dot{y} + a_1 y + a_2 \int y = b_1 u. \tag{6}$$

Next, by introducing $w_1 = \int y$ and $w_2 = y$, we rewrite this system as:

$$\begin{aligned} \dot{w}_1 &= w_2 \,, \\ \dot{w}_2 &= -a_1 w_2 - a_2 w_1 + b_1 u \,, \end{aligned} \tag{7}$$

where $w_1(t_0) = w_2(t_0) = 0$. This can be further transformed into the

$$\dot{w} = Ew + Fu \tag{8}$$

first order ODE form, where $w = [w_1, w_2]^T$ and

$$E = \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \quad , \quad F = \begin{bmatrix} 0 \\ b_1 \end{bmatrix} . \tag{9}$$